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FROM THE MEDIAN TO THE GENERALIZED CENTER (*)

by Pierre HANSEN ([1]), Martine LABBÉ ([2]) and Jacques-François THISSE ([3])

Abstract. — *The purpose of this paper is twofold. First, we revisit the cent-dian location problem developed by Halpern, considering both the average and maximum distances. We provide a complete characterization of the cent-dians in the case of a tree and present a new algorithm to determine this set in the case of a general network. Second, to deal with distributional justice considerations in the access to the facility, we introduce the concept of generalized center defined as the point which minimizes the difference between maximum and average distances. We show that this point coincides with the center in a tree. An algorithm to find the generalized center in a general network is proposed.*

Résumé. — *Cet article a deux objectifs. Tout d'abord, on reconsidère le problème centre-médiane étudié par Halpern, dans le cas où l'on retient les distances moyenne et maximale. On donne une caractérisation complète des centre-médianes dans le cas d'un arbre, tandis qu'on présente un nouvel algorithme permettant de déterminer l'ensemble de ces points dans le cas d'un réseau quelconque. En second lieu, afin d'appréhender les inégalités dans l'accessibilité à l'équipement que l'on cherche à implanter, nous introduisons le concept de centre généralisé défini comme le point minimisant la différence entre les distances maximale et moyenne. Dans le cas d'un arbre, nous montrons que ce point coïncide avec le centre. Pour le cas plus général d'un réseau quelconque, nous présentons un algorithme qui permet de trouver le point recherché.*

INTRODUCTION

In an effort to improve the performance of facility systems, planners have developed a host of operational models to deal with the location of a (public) facility on a network (*see* Hansen *et. al.* [5] for a recent survey). A large number of these models focus on the minimization of the total distance

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between clients situated at vertices of the network and the facility to be located; the solution is called the *median*. Ever since the pioneering contribution of Hakimi [4], this problem and its various extensions have been central to location theory.

However, the pertinence of this objective for placing a facility may appear to be questionable in the sense that total distance minimization favors, sometimes to a considerable degree, clients who are gathered in large population centers to the expense of clients who are spatially dispersed. One possible solution is to make the worst-off client as well-off as possible, that is, the facility is placed in order to minimize the maximum distance to a client (*see* Hakimi [4]); this solution is called the *center*.

In somes cases, locating a facility at the center may generate a substantial loss in efficiency through a large increase in total distance. This has led Halpern [1, 2, 3] to model the corresponding trade-off by minimizing a convex combination of total distance and maximum distance; this solution is called a *cent-dian*.

The primary purpose of this paper is to revisit a slightly modified version of Halpern's problem in which the total distance is replaced by the average distance. This is because we feel that average and maximum distances are directly comparable in terms of magnitude. Specifically, after recalling definitions and notations in Section 2, we derive in Section 3 a complete characterization of the λ-cent-dians in the case of a tree which extends Halpern's results (λ is defined as the weight of the maximum distance in the convex combination H_λ). We also determine an upper bound on the increase of the function H_λ when the median is chosen instead of a λ-cent-dian. In the case of a general network, the characterization of the λ-cent-dians turns out to be a very difficult task. Hence, in Section 4, we present a new algorithm to find the set of λ-cent-dians which is much simpler than that developed by Halpern [2].

The above solution concepts may be associated with a large range in the distribution of the distances separating the clients and the facility, thus contradicting any intuitive notion of distributional equity in the access to the facility. The secondary purpose of this paper is to consider a new solution concept, called the *generalized center*. This point minimizes the difference between the maximum distance and the average distance (when there exists a point equidistant to all clients, the difference is equal to zero). It corresponds to a λ-cent-dian for $\lambda \to \infty$. In a tree, we show that the center is a generalized center (Section 3), while an algorithm is presented to determine the generalized center in the case of a general network (Section 4).

When all the clients are in a bounded area, then the distances from a very far point to all clients are practically the same so that such a point might be a generalized center. To avoid such nonsensical situations, we restrict the generalized center to be an efficient point with respect to the distances to all clients (*i.e.* a point such that no other point is simultaneously closer to all clients).

II. THE MODEL

The following definitions allow to describe networks: a topological edge is the image of [0, 1] by a continuous mapping f from [0, 1] to $\mathbb{R}^3$ such that $f(\theta) \neq f(\theta')$ for any $\theta \neq \theta'$ in [0, 1]; a rectifiable edge is a topological edge of a well-defined length. A *network* is then defined as a subset N of $\mathbb{R}^3$ which satisfies the following conditions: (i) N is the union of a finite number of rectifiable edges; (ii) any two edges intersect at most at their extremities; (iii) N is connected.

The set of *vertices* of the network is made of the extremities of the edges defining N; it is denoted by $V = \{v_1, \ldots, v_n\}$. The set of edges defining the network is denoted by E and the length of an edge $[v_i, v_j] \in E$ is given and denoted by $l[v_i, v_j]$. Each point $s \in N$ belongs to some edge of E but s may or may not be a vertex. For any two points $s_1, s_2 \in [v_i, v_j]$ the subset of points of N between and including s_1 and s_2 is a subedge $[s_1, s_2]$ and its length is denoted by $l[s_1, s_2]$. A path $P(s_1, s_2)$ joining $s_1 \in N$ and $s_2 \in N$ is a minimal connected subset of N containing s_1 and s_2. The length of a path on the network is equal to the sum of the lengths of all its constituent edges and subedges. The *distance* $d(s_1, s_2)$ between $s_1 \in N$ and $s_2 \in N$ is equal to the length of a shortest path joining s_1 and s_2.

We now consider the demand: with each vertex $v_i \in V$ is associated a non-negative integer *weight* w_i representing the total number of times that the users, located at v_i, visit the facility. For a given subset $U \subseteq V$ of vertices, let $w(U) = \Sigma \{ w(v_i) : v_i \in U \}$. In particular, $w(V)$ represents the sum of the weights of all the vertices of N.

Finally, we consider our different solution concepts. The *average weighted distance* between a point $x \in N$ and the vertices $v_i \in V$ is given by

$$F(x) = \frac{1}{w(V)} \sum_{v_i \in V} w_i \, d(x, v_i).$$

A point minimizing $F(.)$ in N is called a *median*. Since the function $F(.)$ is known to be concave along each edge (*see* e. g., Hakimi [4]), a median can always be found in V.

The *maximum distance* between a point $x \in N$ and the vertices $v_i \in V$ with users is denoted by $G(x) = \max\{d(x, v_i) : v_i \in V \text{ and } w_i > 0\}$. A point minimizing $G(.)$ in N is called a *center*. The function $G(.)$ is piecewise linear with slope $+1$ or -1 on each edge so that a center can be found among the local minima of $G(.)$ on all edges.

In this paper, we consider the problem of finding a point $x \in N$ minimizing a linear combination of $F(x)$ and $G(x)$, given by

$$H_\lambda(x) = \lambda\, G(x) + (1-\lambda)\, F(x) \qquad \text{with} \quad \lambda \geqq 0.$$

Such a solution is called a λ-*cent-dian* and the set of all λ-cent-dians is noted λ-*CD*. In particular, if $\lambda = 0$ the λ-cent-dian is a center and if $\lambda = 1$ it is a median. For $0 < \lambda < 1$, the λ-cent-dian minimizes a convex combination of the average and maximum distances to the vertices. Hence it is an optimal solution to a location problem where both efficiency and equity criteria are important. The value of λ reflects the weight attributed to the maximum distance with respect to the average.

Assume now that the planner wishes to locate a facility in order to reduce as much as possible discrepancies in accessibility among users. More precisely, the selected point of the network, called a *generalized center*, has to minimize the difference between the largest and the average distances to the vertices. This may however lead to an "unreasonable" location. As an example, consider the network of Figure 1 with $w_1 = w_3 = 1$, $w_2 = n-2$, and $w_4 = 0$, and $k = 1$ so that v_4 is the only generalized center. Hence if the planner wants to reduce as much as possible the discrepancies between the distances the users have to cover, he/she would locate the facility at a point (v_4) which is very far from all users. Furthermore, it is easy to see that in such a case all users would prefer to have the facility at v_2 since this vertex is closer to all of them than v_4.

This suggests that if the planner is concerned with the discrepancies between the distances the users have to cover, he/she should restrict the set of feasible locations to the set *PO* of points which are *Pareto-optimal* (or *efficient*) with respect to the distances. A point $x \in N$ is Pareto-optimal with respect to the distances if there does not exist another point $y \in N$ for which

(i) $d(v_i, y) \leqq d(v_i, x)$, for all $v_i \in V$ such that $w_i > 0$
and

(ii) $d(v_i, y) < d(v_i, x)$ for at least one $v_i \in V$ such that $w_i > 0$.

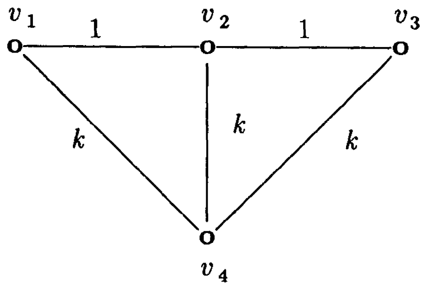

Figure 1. — Locating a facility at the generalized center.

Hence, formally, a point $x \in PO$ is a generalized center *iff* for any point $y \in PO$: $G(x) - F(x) \leqq G(y) - F(y)$.

Consider now the function $H(x)$ for $\lambda \geqq 1$. On one hand, $H_\lambda(x) = F(x) + \lambda(G(x) - F(x))$. On the other hand, when $\lambda \to \infty$ the limit of $H_\lambda(x)/\lambda$ equals $G(x) - F(x)$. Consequently, a λ-cent-dian for $\lambda > 1$ can be viewed as the solution to a location problem where both efficiency and egalitarism are important. The value chosen for λ reflects the weight given to the average distance with respect to the difference between the maximum and average distances. Since the same type of "unreasonable" decision may emerge for the λ-center-dian with $\lambda > 1$ as for the generalized center (see again the network of Figure 1) we also restrict the set of feasible locations for the λ-cent-dian with $\lambda > 1$ to *PO*.

III. THE λ-CENT-DIAN ON A TREE FOR $\lambda \geqq 0$

In this section, we consider the special case of trees, *i. e.*, networks without cycles. In a first part, we will present, several localization theorems. No proof will be given since most of these results (Theorems 1 to 3) are due to Halpern [1]. The second part of this section is devoted to the comparison of the λ-cent-dian with the median in terms of their values for the objective function $H_\lambda(.)$. We first recall some classical results concerning the distance and the functions $F(.)$ and $G(.)$ (*cf.* e. g. Hansen *et. al.* [5]).

For any vertex v_i of a tree T, the distance $d(v_i, x)$, when point x moves along a path, say $P(y, z)$, is a convex and piecewise linear function with slope -1 or $+1$.

The center c of a tree T is unique. Moreover, for any point $x \in T$:

$$G(x) = G(c) + d(c, x), \tag{1}$$

i.e., $G(x)$ strictly increases with slope $+1$ along any path starting from c.

The set of medians of a tree T is composed either of a single vertex or of a path $P(v_i, v_j)$, such that v_i and $v_j \in V$ and $|d(v_i, v_k) - d(v_j, v_k)| = d(v_i, v_j)$ for every $v_k \in V$ with $w_k > 0$. Moreover, $F(.)$ increases along any path starting from a median and if m is the median closest to a point $x \in T$, then

$$F(x) \geqq F(m) + \frac{d(m, x)}{w(V)}. \tag{2}$$

Finally, for a given value of λ, $0 \leqq \lambda \leqq 1$, the function $H_\lambda(.)$ is convex and piecewise linear along any path of a tree.

Let c be the center of a tree T and m be the median closest to c. Halpern [1] proved that $\bigcup \{\lambda - CD: 0 \leqq \lambda \leqq 1\} = P(m, c)$. The following theorem shows that these two sets also coincide with that of efficient points for $F(.)$ and $G(.)$. A point $x \in T$ is said to be *efficient* with respect to $F(.)$ and $G(.)$ iff there does not exist a point $y \in T$ such that $F(y) \leqq F(x)$ and $G(y) \leqq G(x)$ with at least one strict inequality. The set of all efficient points is denoted by EF.

THEOREM 1: $EF = P(m, c) = \bigcup \{\lambda - CD: 0 \leqq \lambda \leqq 1\}$.

Note that Theorem 1 is the network-couterpart of a well-known property in multiple objective optimization. Theorems 2 and 3 give values of λ for which a given point of $P(m, c)$ is a λ-cent-dian.

THEOREM 2: *On a tree, if* $0 \leqq \lambda \leqq 1/(w(V)+1)$, *then* m *is a* λ*-cent-dian. If* $(w(V)-2)/2(w(V)-1) \leqq \lambda \leqq 1$, *then* c *is a* λ*-cent-dian.*

Since $(w(V)-2)/2(w(V)-1) < 1/2$, Theorem 2 implies that the center is a λ-cent-dian for more than half of all values of λ corresponding to the bicriterion problems taking into account the maximum and average distances. Furthermore, the bounds $1/w(V)+1)$ and $(w(V)-2)/2(w(V)-1)$ are tight. As shown by the network of Figure 2 with $w_1 = k$ and $w_2 = k-1$ and $w_1 = k-1$ and $w_2 = 1$ respectively.

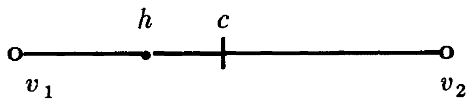

Figure 2. — Example for the bounds of Theorem 2

A vertex $v_i \in P(m, c)$ is said to be *active* if either $w_i > 0$ or there exists some vertex $v_j \in V$ such that $w_j > 0$ and both $d(v_j, m) = d(v_j, v_i) + d(v_i, m)$ and $d(v_j, c) = d(v_j, v_i) + d(v_i, c)$.

THEOREM 3: *Let v_i and v_j be two distinct active vertices of $P(m, c)$ such that $P(v_i, v_j)$ does not contain any other active vertex in its interior. Then, any interior point of $P(v_i, v_j)$ is a λ-cent-dian iff $\lambda = 1 - w(V)/2w(V_i)$, where $V_i = \{ v_k \in V : d(v_k, v_j) = d(v_k, v_i) + d(v_i, v_j) \}$.*

THEOREM 4: *On a tree T, the center is the unique* λ-cent-dian for all $\lambda \geqq 1$.

Proof: First, note that $c \in PO$. Then, let x be a point of T. It is easy to see that $F(x) \leqq F(c) + d(c, x)$. Hence,

$$\begin{aligned} H_\lambda(x) &= \lambda G(x) + (1-\lambda) F(x) \\ &= \lambda G(c) + \lambda d(c, x) + (1-\lambda) F(x), \text{ from Lemma 2} \\ &\geqq \lambda G(c) + (1-\lambda) F(c) + \lambda d(c, x) + (1-\lambda) d(c, x), \\ &\quad \text{from the above inequality together with } \lambda \geqq 1 \\ &\geqq H_\lambda(c). \quad \square \end{aligned}$$

The following theorem provides an upper bound on the increase in the value of the objective function when the median is chosen instead of the λ-cent-dian, for $0 \leqq \lambda \leqq 1$. It would be natural to investigate the similar question when the median is replaced by the center. However, we have not been able to find out a good upper bound.

THEOREM 5: *On a tree T, let h be a λ-cent-dian for $0 \leqq \lambda \leqq 1$, then*

$$\frac{H_\lambda(m)}{H_\lambda(h)} \leqq \begin{cases} 1 & \text{if } 0 \leqq \lambda \leqq \dfrac{1}{w(V)+1} \\ \dfrac{2\lambda(w(V)-1)+2}{\lambda(w(V)-3)+3} & \text{if } \dfrac{1}{w(V)+1} < \lambda \leqq 1. \end{cases}$$

Proof: We have:

$$\begin{aligned}
\frac{H_\lambda(m)}{H_\lambda(h)} &= \frac{\lambda G(m) + (1-\lambda) F(m)}{\lambda G(h) + (1-\lambda) F(h)} \\
&= \frac{\lambda(G(c) + d(m, c)) + (1-\lambda) F(m)}{\lambda(G(c) + d(c, h)) + (1-\lambda) F(h)}, \quad \text{by (1)} \\
&\leq \frac{\lambda(G(c) + d(m, c)) + (1-\lambda) F(m)}{\lambda(G(c) + d(c, h)) + (1-\lambda)(F(m) + (d(m, h)/w(V))}, \quad \text{by (2)} \\
&\leq \frac{2\lambda d(m, c) + (2(1-\lambda)/w(V)) d(m, c)}{\lambda(d(m, c) + d(c, h)) + ((1-\lambda)/w(V))(2d(m, c) + d(m, h))},
\end{aligned}$$

by subtracting up and down $\lambda G(c) - d(m, c)) + (1-\lambda)\left(F(m) - \dfrac{2d(m, c)}{w(V)}\right)$,

which is nonnegative since $G(c) \geqq d(m, c)$ and

$$F(m) \geqq \frac{1}{w(V)}(d(m, c) + G(c)) \geqq \frac{2d(m, c)}{w(V)}$$

$$\leqq \frac{2\lambda d(m, c) + (2(1-\lambda)/w(V)) d(m, c)}{\lambda d(m, c) + (2(1-\lambda)/w(V)) d(m, c) + \min\{\lambda, (1-\lambda)/w(V)\} d(m, c)}$$

$$= \begin{cases} 1 & \text{if } \; 0 \leqq \lambda \leqq 1/(w(v)+1) \\ \dfrac{2\lambda(w(V)-1)+2}{\lambda(w(V)-3)+3} & \text{if } \; 1/(w(V)+1) < \lambda < 1. \end{cases} \quad \square$$

Notice, that the bounds of Theorem 5 are best possible for $w(V) \leqq 3$. The network of Figure 2 with $w_1 = 2$ and $w_2 = 1$ illustrates such a situation when $w(V) = 3$. For $w(V) = 2$, since a median always coincides with the center and is, therefore, a λ-cent-dian for all $0 \leqq \lambda \leqq 1$, the bound is also tight. For $w(V) > 3$, the question whether or not the bounds are best possible remains open.

IV. THE λ-CENT-DIAN ON A GENERAL NETWORK FOR $\lambda \geq 0$

In this section, we present two algorithms to determine the sets $\bigcup_{\lambda=0}^{1} \lambda - CD$ and $\lambda - CD$ for any $\lambda > 1$, respectively. In the former case, an algorithm has already been proposed by Halpern [2]. The one we present here is much simpler, although it has same computational complexity. We first need some additional notation and definitions.

A point x on an edge $[v_i, v_j]$ is a *bottleneck point* if there exist some vertex v_k with $w_k > 0$ such that

$$d(v_k, x) = d(v_k, v_i) + l[v_i, x] = d(v_k, v_j) + l[v_j, x].$$

Let B_{ij} denote the set of bottleneck points on (v_i, v_j). Along a subedge limited by two successive vertices or bottleneck points (*i.e.* such that the subedge does not contain other points of B_{ij} in its interior), the distance from a vertex v_k is either linearly increasing or linearly decreasing (*cf.*, e.g. Hansen *et. al.* [5]).

Consider now the function $G(x) = \max \{ d(v_i, x) : v_i \in V \text{ and } w_i > 0 \}$ on $[v_i, v_j]$. Since it is the upper envelope of a finite family of piecewise linear and continuous functions, it is itself piecewise linear and continuous. Furthermore, its breakpoints are either bottleneck points or local minima. We denote by LM_{ij} the set containing the points of $[v_i, v_j]$ which are local minima of $G(.)$ and the two vertices v_i and v_j.

The following proposition, due to Halpern [2] identifies a finite set of points containing all $\lambda - CD$ for $0 \leq \lambda \leq 1$.

PROPOSITION 1: *For $x \in [v_i, v_j]$ and a given value for λ, $H_\lambda(x)$ is a piecewise linear function*

(i) *with a finite number of breakpoints, all belonging to $LM_{ij} \cup B_{ij}$, and*

(ii) *with a finite number of locally minimum values, all attained at points belonging to LM_{ij}.*

If for a given value of λ and for two consecutive points x and y of LM_{ij}, we have $H_\lambda(x) = H_x(y)$, then all points of $[x, y]$ are local minima of $H_\lambda(.)$. Our algorithm for finding $\bigcup_{\lambda=0}^{1} \lambda - CD$ is based on the following geometric interpretation of $\bigcup_{\lambda=0}^{1} \lambda - CD$ (*see* also Halpern [3]).

Let $g: N \to \mathbb{R}^2$ be a mapping from the network N into the plane $\mathbb{R}^2$ defined as: $g(x) = (G(x), F(x))$. Since N is connected and $G(.)$ and $F(.)$ are continuous and piecewise linear functions, the image $G(N)$ is a connected set composed of linear segments of $\mathbb{R}^2$. The extremities of these linear segments are the images of points of $\bigcup_{[v_i, v_j] \in E} (LM_{ij} \cup B_{ij})$.

Given the definition of $H_\lambda(.)$, all the λ-cent-dians $(0 \leqq \lambda \leqq 1)$ are the points $x \in N$ having an image $g(x)$ which belongs to the lower boundary of the convex hull of $g(N)$. Moreover, this convex hull coincides with the convex hull of $g(\bigcup_{[v_i, v_j] \in E} (LM_{ij} \cup B_{ij}))$.

We first describe the algorithm IMA yielding the image $g(LM_{ij} \cup B_{ij})$ for an edge $[v_i, v_j]$.

Algorithm IMA

The image $g(LM_{ij} \cup B_{ij})$ for edge $[v_i, v_j]$.

Step (a). The set LM_{ij} of local minima

Using the algorithm of Kariv and Hakimi [7], determine (i) the list $LM_{ij} = \langle m_0 = v_i, m_1, \ldots, m_q, m_{q+1} = v_j \rangle$, sorted in such a way that $l[v_i, m_0] = 0 < l[v_i, m_1] < \ldots < l[v_i, m_q] < l[v_i, m_{q+1}] = l[v_i, v_j]$ and (ii) the corresponding values of $G(.)$.

Step (b). The set B_{ij} of bottleneck points

Compute $F(v_i)$ and $F(v_j)$. Then, partition the set V of vertices into the following three subsets:

$$V_1 = \{ v \in V: d(v, v_j) = d(v, v_i) + l[v_i, v_j] \},$$

$$V_2 = \{ v \in V: d(v, v_i) = d(v, v_j) + l[v_i, v_j] \}, \qquad \text{and} \qquad V_3 = V \setminus (V_1 \cup V_2).$$

For all $v_k \in V_3$ compute $l[v_i, b_k] = [d(v_k, v_j) + l[v_i, v_j] - d(v_k, v_i)]/2$ and set $w(b_k) = w_k$.

Put the b_k's in a sorted list $B_{ij} = \langle b_0 = v_i, b_1, \ldots, b_{|V_3|}, b_{|V_3|+1} = v_j$ such that $l[v_i, b_0] = 0 \leqq l[v_i, b_1] \leqq \ldots \leqq l[v_i, b_{|V_3|}] \leqq l[v_i, b_{|V_3|+1}] = l[v_i, v_j]$. Set also $w(b_0) = w(V_1)$ and $w(b_{|V_3|+1}) = w(V_2)$.

Explore the list B_{ij} in sequence from $k=1$ to $|V_3|$ as follows.

If $b_k \neq b_{k+1}$, then set

$$F(b_k) = F(b_{k+1}) + \frac{w(V) - 2w(b_{|V_3|+1})}{w(V)} l[b_{k-1}, b_k] \tag{3}$$

and add $w(b_k)$ to $w(b_{|V_3|+1})$.

If $b_k = b_{k+1}$, then add $w(b_k)$ to $w(b_{k+1})$ and delete b_k from B_{ij}.

Step (c). $F(m_r)$ for $m_r \in LM_{ij}$ and $G(b_k)$ for $b_k \in B_{ij}$

Merge the two lists LM_{ij} and B_{ij} into a single sorted list $LM_{ij} \cup B_{ij}$. At the same time, for each b_k found, determine m_r and m_{r+1} such that $l[v_i, m_r] \leqq l[v_i, b_k] \leqq l[v_i, m_{r+1}]$ and compute:

$$G(b_k) = \begin{cases} G(m_r) + l[m_r, b_k], \\ \text{if} \quad l[v_i, b_k] \leqq l[v_i, m_r] + \frac{1}{2}(G(m_{r+1}) - G(m_r) + l[m_r, m_{r+1}]) \\ G(m_{r+1}) + l[b_k, m_{r+1}]), \qquad \text{otherwise.} \end{cases}$$

For each m_r found, determine b_k and b_{k+1}, such that

$$l[v_i, b_k] \leqq l[v_i, m_r] \leqq l[v_i, b_{k+1}]$$

and compute

$$F(m_r) = F(b_k) + \frac{F(b_{k+1}) - F(b_k)}{l[b_{k+1}, b_k]} l[b_k, m_r].$$

When all points of $LM_{ij} \cup B_{ij}$ have been considered, set $g(LM_{ij} \cup B_{ij}) = \{(G(x), F(x)) : x \in LM_{ij} \cup B_{ij}\}$. □

PROPOSITION 2: *Algorithm IMA determines the set $LM_{ij} \cup B_{ij}$ in at most $O(|V| \log |V|)$ operations.*

Proof: Step (*a*) determines the sorted list LM_{ij} and the corresponding values of $G(.)$. Step (*b*) determines the sorted list B_{ij} of bottleneck points as well as the corresponding values of $F(.)$. To this end, the set V of vertices is partitioned into three sets V_1, V_2 and V_3. The subset V_1 (resp. V_2) contains the vertices v_k for which $d(v_k, .)$ is increasing (resp. decreasing) along $[v_i, v_j]$. The subset V_3 contains the vertices v_k for which $[v_i, v_j]$ contains a bottleneck point b_k. Further, the b_k's are sorted by order of increasing values of $l[v_i, b_k]$. At each b_k of this list is associated a number $w(b_k)$ which represents the weight of the vertex v_k. Next, the value of $F(.)$ is computed for each b_k by exploring the sorted list in sequence. Each time that a b_k is found, $w(b_{|V_3|+1})$ [resp. $W(V)-b_{|V_3|+1}$] represents the sum of the weights of the vertices v_l for which $d(v_l, .)$ is decreasing (resp. increasing) along the subedge $[b_{k-1}, b_k]$. Hence $F(b_k)$ is computed by using (3).

Finally, at Step (c) the values of $G(.)$ [resp. $F(.)$] are computed for the elements of the list B_{ij} (resp. LM_{ij}). To this end, the following properties of functions $G(.)$ and $F(.)$ are used.

– Let $[m_r, m_{r+1}]$ be a subedge of $[v_i, v_j]$ limited by two consecutive elements of LM_{ij}. When x moves along this subedge, $G(x)$ is a piecewise linear function which first increases with slope $+1$ and then decreases with slope -1.

– Along $[v_i, v_j]$, $F(.)$ is piecewise linear with breakpoints belonging to B_{ij}.

Regarding computational complexity, step (a) requires $O(|V|\log|V|)$ operations (*see* Kariv and Hakimi [7]). Step (b) requires $O(|V|\log|V|)$ operations (the number of bottleneck points is bounded by $|V|$, the ranking of them is in $O(|V|\log|V|)$ and the computation of $F(.)$ is performed by exploring the sorted list once which is in $O(|V|)$). Finally, the merging of the sorted lists LM_{ij} and B_{ij} can be performed in $O(|V|)$ and the missing values of $G(.)$ and $F(.)$ are computed by exploring the new list $LM_{ij} \cup B_{ij}$ which also requires $O(|V|)$ operations. Thus, Step (d) is in $O(|V|)$. □

We now present the algorithm for finding the set $\bigcup_{\lambda=0}^{1} \lambda - CD$.

ALGORITHM 1: *The cent-dian set* $\bigcup_{\lambda=0}^{1} \lambda - CD$.

Step (a). For all edges $[v_i, v_j] \in E$, apply Algorithm IMA to determine the set $g(LM_{ij} \cup B_{ij})$. Set $g(LM \cup B) = \bigcup_{[v_i, v_j] \in E} (g(LM_{ij} \cup B_{ij}))$.

Step (b). Determine the convex hull of $g(LM \cup B)$. The lower boundary of this set is $g\left(\bigcup_{\lambda=0}^{1} \lambda - CD\right)$. □

PROPOSITION 3: *Algorithm 1 determines* $\bigcup_{\lambda=0}^{1} \lambda - CD$ *in* $O(|E||V|\log|E||V|)$ *operations.*

Proof: Step (a) requires $O(|E||V|\log|V|)$ operations as it applies Algorithm IMA to each edge $[v_i, v_j] \in E$. Step (b) requires $O(|E||V|\log|E||V|)$ operations since $g(LM \cup B)$ contains $O(|E||V|)$ points and the convex hull of a set of n points can be determined in $O(n \log n)$ operations (*see* e.g. Preparata and Shamos [9]). □

We now turn to the problem of finding the set $\lambda - CD$ for $\lambda > 1$. The following proposition allows us to limit the search to the set of local minima of the function $G(.)$.

PROPOSITION 4: *A point $x \in N$ is a local minimum of $H_\lambda(.)$ for $\lambda > 1$ iff x is a local minimum of $G(.)$.*

Proof: The function $H_\lambda(.)$ is piecewise linear and at a given point $x \in N$, its slope s is given by $s = \lambda s_1 + (1-\lambda) s_2$, where s_1 and s_2 are the slopes of $G(.)$ and $F(.)$ respectively. Furthermore, s_1 is equal to either $+1$ or -1 and $-1 \leqq s_2 \leqq +1$ (remember that $F(x)$ is the average weighted distance). Consequently, as $\lambda > 1$, the sign of s is equal to that of s_1, which implies that x is a local minimum of $H_\lambda(.)$ iff it is a local minimum of $G(.)$. □

On the other hand, remember that for $\lambda > 1$ we restrict the set of feasible solutions to the set PO of Pareto-optimal points with respect to the distances. This set can be determined in $O(|E|^2 |V|^2 \log |V|)$ operations (*see* Hansen *et. al.* [7]). Furthermore, it is constituted of several connected subnetworks which may contain some subedges. Let I be the set of all interior points limiting such subedges. It is easy to see that a point $x \in PO$ minimizes $H_\lambda(.)$ if $x \in (LM \cap PO) \cup I$. Hence the following algorithm can be easily adapted to find $\lambda - CD$ for $\lambda > 1$.

ALGORITHM 2: Argmin $\{H_\lambda(x) : x \in LM\}$, $\lambda > 1$.

Step (*a*). Apply Algorithm IMA to determine LM and the corresponding values of $G(.)$ and $F(.)$. [Note that as the points of $B = \bigcup_{[v_i, v_j] \in E} B_{ij}$ cannot be local minima of $H_\lambda(.)$, it is not necessary to compute $G(x)$ for $x \in B$. Nevertheless, we need this set and the corresponding values of $F(.)$ to compute $F(x)$ for all $x \in LM$].

Step (*b*). Explore LM to determine the point(s) minimizing $H_\lambda(.)$. □

PROPOSITION 5: *Algorithm* 2 *determines all $x \in LM$ minimizing $H_\lambda(.)$ ($\lambda > 1$) in $O(|E||V| \log |E||V|)$ operations.*

Proof: It follows directly from Proposition 4 that Algorithm $\lambda - CD$ for $\lambda > 1$. Regarding computational complexity, Step (*a*) requires $O(|V||E| \log |V|)$ operations. As LM has $O(|V||E|)$ points (*see* Kariv and Hakimi [7]) and $H_\lambda(.)$ is computed in constant time (using updating as explained above) for each point of LM, Step (b) requires $O(|V||E|)$ operations. In conclusion, the overall complexity is $O(|V||E|\log|V|)$. □

REFERENCES

1. J. HALPERN, The Location of a Center-Median Convex Combination on an Undirected Tree, *J. Regional Sci.*, 1976, 16, pp. 237-245.
2. J. HALPERN, Finding Minimal Center-Median Convex Combinations (Cent-Dian) of a Graph, *Management Sci.*, 1978, 24, pp. 535-544.
3. J. HALPERN, Duality in the Cent-Dian of a Graph, *Oper. Res.*, 1980, 28, pp. 722-735.
4. S. L. HAKIMI, Optimal Locations of Switching Centers and the Absolute Centers and Medians of a Graph, *Oper. Res.* , 1964, 12, pp. 450-459.
5. P. HANSEN, M. LABBÉ, D. PEETERS and J.-F. THISSE, Single Facility Location on Networks, *Ann. Discrete Math.*, 1987, 31, pp. 113-146.
6. P. HANSEN, J.-F. THISSE, and R. E. WENDELL, Efficient Points on a Network, *Networks*, 1986, 16, pp. 358-367.
7. O. KARIV and S. L. HAKIMI, An Algorithmic Approach to Network Location Problems I: The p-Centers, *S.I.A.M. J. Appl. Math.*, 1979, 37, pp. 513-538.
8. M. LABBÉ, Essays in Network Location Theory, Doctoral Dissertation, Université Libre de Bruxelles, 1985.
9. F. P. PREPARATA and M. I. SHAMOS, Computational Geometry: An Introduction, *Springer-Verlag*, New York, 1985.