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WELL *ev*-COVERED TREES

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Abstract. An edge in a graph $G = (V, E)$ is said to ev-dominate the vertices incident to it as well as the vertices adjacent to these incident vertices. A subset $F \subseteq E$ is an edge-vertex dominating set (or simply, ev-dominating set) if every vertex is ev-dominated by at least one edge of F. The ev-domination number $\gamma_{ev}(G)$ is the minimum cardinality of a ev-dominating set of G. An ev-dominating set is independent if its edges are independent. The independent ev-domination number $i_{ev}(G)$ is the minimum cardinality of an independent ev-dominating set and the upper independent ev-domination number $\beta_{ev}(G)$ is the maximum cardinality of a minimal independent ev-dominating set of G. In this paper, we show that for every nontrivial tree T, $\gamma_{ev}(T) = i_{ev}(T) \le \gamma(T) \le \beta_{ev}(T)$, where $\gamma(T)$ is the domination number of T. Moreover, we provide a characterization of all trees T with $i_{ev}(T) = \beta_{ev}(T)$, which we call well ev-covered trees, as well as a characterization of all trees T with $\gamma_{ev}(T) = i_{ev}(T) = \gamma(T)$.

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1. Introduction

Let $G = (V, E)$ be a simple graph. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighborhood* is $N(S) = \cup_{v \in S} N(v)$, the *closed neighborhood* is $N[S] = N(S) \cup S$, and $G[S]$ is the *subgraph* induced by the vertices of S. The *private neighborhood* of a vertex $u \in S$ with respect to S is defined as $\mathrm{pn}[u, S] = \{v \in V \mid N[v] \cap S = \{u\}\}$. The *degree* of a vertex v denoted by $\deg_G(v)$ is the cardinality of its open neighborhood. A vertex of degree one is called a *pendant vertex* or a *leaf*, and its neighbor is called a *support vertex*. An edge incident with a leaf is called a *pendant edge*.

We denote the *star* consisting of one central vertex and k leaves as $K_{1,k}$. A *subdivided star* SS_k is obtained from a star $K_{1,k}$ by subdividing each edge by exactly one vertex. A tree T is a *double star* if it contains exactly two vertices that are not leaves. Let $\mathrm{diam}(G)$ denote the *diameter* of a graph G. Throughout this paper, we only consider nontrivial connected graphs, called *ntc graphs*.

A set $S \subseteq V$ of a graph G is a *dominating* set if every vertex in $V - S$ has at least a neighbor in S. The set S is *independent* if no two vertices in S are adjacent. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. The *independence number* $\beta_0(G)$ and the *independent domination number* $i(G)$ are the maximum and the minimum cardinality of a set that is both independent and dominating in G, respectively.

Keywords. Domination, independent edge-vertex domination, well ev-covered trees.

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Clearly, for every graph G, $\gamma(G) \le i(G) \le \beta_0(G)$. A graph G is called *well-covered* if every maximal independent set is maximum, that is, $i(G) = \beta_0(G)$. Well-covered graphs were introduced by Plummer [6]. We refer the reader to Plummer [7] for a comprehensive survey on well-covered graphs and their properties.

In this paper, we are interested in the concept of edge-vertex domination introduced by Peters [5] in his 1986 Ph.D. thesis, and studied further in the 2007 Ph.D. thesis by Lewis [4]. Edge-vertex domination has been introduced in [4, 5] in connection with vertex-edge domination, but the study of these two concepts has only begun to receive more attention in recent years. The literature on the latter concept is more abundant than on the former, and a survey paper on them would be very interesting.

A set $F \subseteq E$ is an *edge-vertex dominating set* (or simply an *ev-dominating set*), where an edge dominates the vertices incident to it as well as the vertices adjacent to these incident vertices. An ev-dominating set F is a *minimal* if, for every edge $e \in F$, $F - \{e\}$ is not an ev-dominating set in G. The minimum cardinality of an ev-dominating set of G is called the *edge-vertex domination number* and is denoted by $\gamma_{ev}(G)$. Trivially, $\gamma_{ev}(G) \le \gamma(G)$ holds for every nontrivial graph G.

An edge $e = uv \in F \subseteq E$ has a *private vertex* $w \in V$ with respect to a set F, if (i) w is incident to e or w is adjacent to either u or v, and (ii) for all edges $e' = xy \in F - \{e\}$, e' is neither incident to w and neither x nor y is adjacent to w. In other words, e ev-dominates the vertex w and no other edge in F ev-dominates w. We denote by $\mathrm{pn}(e, F)$ the set of private vertices of an edge $e \in F$ with respect to F.

In his Ph.D. thesis, Lewis [4] introduced the concept of independent ev-domination which substantially extends the work of Peters [5] on ev-domination. A set $F \subseteq E$ is an *independent edge-vertex dominating set* (or simply, *independent ev-dominating set*) if F is both an independent set of edges and ev-dominating set. The *independent edge-vertex domination number*, $i_{ev}(G)$, of G is the minimum cardinality of an independent ev-dominating set and the *upper independent edge-vertex domination number* $\beta_{ev}(G)$ is the maximum cardinality of a minimal independent ev-dominating set of G. If S is a minimal independent ev-dominating set of G of size $\beta_{ev}(G)$, then we call S a $\beta_{ev}(G)$-set. It is worth mentioning that since any independent ev-dominating set is also a minimal ev-dominating set, we have $\gamma_{ev}(G) \le i_{ev}(G) \le \beta_{ev}(G)$ for every ntc graph G. Moreover, Lewis [4] showed that equality between $\gamma_{ev}(G)$ and $i_{ev}(G)$ holds for all ntc graphs G.

Theorem 1 ([4])**.** *For any ntc graph G, $\gamma_{ev}(G) = i_{ev}(G)$.*

In his dessertation, Lewis [4] established some results on independent ev-domination mainly related to complexity and algorithmic aspect. But, as far as we know, no other work has been done on the upper independent edge-vertex domination number. This largely motivates our study of this parameter, where in addition, a list of open problems will be given at the end.

We define a graph G as being *well ev-covered* if $i_{ev}(G) = \beta_{ev}(G)$. In this paper, we show that for any nontrivial tree T,

$$\gamma_{ev}(T) = i_{ev}(T) \le \gamma(T) \le \beta_{ev}(T). \tag{1}$$

Moreover, we provide a characterization of well ev-covered trees as well as a characterization of all trees T with $\gamma_{ev}(T) = i_{ev}(T) = \gamma(T)$.

We close this section by the following observations that will be useful for the next.

Observation 2. Let F be an ev-dominating set of an ntc graph G. Then F is a minimal ev-dominating set if and only if every edge $e \in F$ has at least one private vertex with respect to F.

Proof. Suppose that F is a minimal ev-dominating set of G. Then for every edge $e \in F$, $F - \{e\}$ does not ev-dominate G. Hence, there is a vertex v which is not ev-dominated by $F - \{e\}$, and thus v is a private vertex of e with respect to F.

Conversely, assume that F is an ev-dominating set of G such that every edge of F has a private vertex with respect to F. If F is not minimal, then there is an edge $e \in F$ such that $F - \{e\}$ remains an ev-dominating set of G. It follows that each vertex of G is ev-dominated by $F - \{e\}$, contradicting the fact e has a private neighbor with respect to F. Therefore, F is a minimal ev-dominating set. □

Observation 3. If G is a connected graph of order at least three, then there exists a $\gamma(G)$-set that contains all support vertices.

Observation 4. For any connected graph G with diameter at least three, there exists a $\gamma_{ev}(G)$-set that contains no pendant edge.

2. Proof of (1)

We need only to prove the right inequality of (1).

Proposition 5. *For every nontrivial tree T, $\gamma(T) \leq \beta_{ev}(T)$.*

Proof. We use induction on the order n of T. Clearly, the result holds if $n \in \{2, 3\}$ which establishes the base cases. Let $n \geq 4$ and assume that every nontrivial tree T' of order $n' < n$ satisfies $\gamma(T') \leq \beta_{ev}(T')$. Let T be a tree of order n. Since stars and double stars have equal domination and upper independent edge-vertex domination numbers, we can assume that T has diameter at least four.

If any support vertex, say x, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to x. Then $\beta_{ev}(T') = \beta_{ev}(T)$ and $\gamma(T') = \gamma(T)$ yielding $\beta_{ev}(T) \geq \gamma(T)$. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T) \geq 4$, and let u be a support vertex at maximum distance from r. Let v be the parent of u and w be the parent of v in the rooted tree. Let u' be the unique leaf adjacent to u. Note that w has degree at least two, since $\mathrm{diam}(T) \geq 4$. Also, we recall that T_b denotes the subtree induced by a vertex b and its descendants in the rooted tree T. We distinguish between two cases.

Case 1. v is a support vertex. Let T' be the tree obtained from T by removing u and u'. Then every $\beta_{ev}(T')$-set F' can be extended to an independent ev-dominating set of T by adding the edge uu'. Observe that since $u' \in \mathrm{pn}(uu', F' \cup \{uu'\})$, set $F' \cup \{uu'\}$ is a minimal and thus $\beta_{ev}(T) \geq \beta_{ev}(T') + 1$. Moreover, if S' is any $\gamma(T')$-set, then $S' \cup \{u\}$ is a dominating set of T, implying that $\gamma(T) \leq \gamma(T') + 1$. Now by combining the previous inequalities and using the fact $\gamma(T') \leq \beta_{ev}(T')$, we obtain $\gamma(T) - 1 \leq \gamma(T') \leq \beta_{ev}(T') \leq \beta_{ev}(T) - 1$, and therefore $\gamma(T) \leq \beta_{ev}(T)$.

Case 2. v is not a support vertex. Let $k = \deg_T(v) - 1 \geq 1$. Clearly, every child of v besides u (if any) is a support vertex of degree two. Thus T_v is either a path $P_3 : u'uv$ or a subdivided star SS_k. Let T' be the tree obtained from T by removing T_v. Since $\mathrm{diam}(T) \geq 4$, the subtree T' is nontrivial. Also, since every $\beta_{ev}(T')$-set can be extended to a minimal independent ev-dominating set of T by adding $u'u$ if $T_v = P_3$ or all pendant edges in T_v for otherwise, we obtain $\beta_{ev}(T) \geq \beta_{ev}(T') + k$. On the other hand, any $\gamma(T')$-set can be extended to a dominating set of T by adding all support vertices of T_v, implying that $\gamma(T) \leq \gamma(T') + k$. Hence $\gamma(T) - k \leq \gamma(T') \leq \beta_{ev}(T') \leq \beta_{ev}(T) - k$, and therefore $\gamma(T) \leq \beta_{ev}(T)$. □

We note that Proposition 5 is not valid for all graphs, as can be seen by the cycle C_4, where $\gamma(C_4) = 2 > \beta_{ev}(C_4) = 1$. Also, nontrivial stars are the simplest example of trees T with $\gamma(T) = \beta_{ev}(T)$. Moreover, the next example of a tree shows that the inequality in Proposition 5 may be strict. Let H be a tree obtained from a path $P_2 : x$–y and a path P_{10} whose vertices are labelled in order $u_1, u_2, \ldots, u_{10}$ by adding the edge xu_4. One can easily see that $\gamma(H) = 4$ (for instance, $S = \{x, u_2, u_6, u_9\}$ is a $\gamma(H)$-set) while the set $F = \{xy, u_1u_2, u_4u_5, u_7u_8, u_9u_{10}\}$ is a minimal independent ev-dominating set of H and thus $\beta_{ev}(H) \geq 5$. In addition, using the previous tree H, we can construct a tree T_k for any positive integer k such that the difference $\beta_{ev}(T_k) - \gamma(T_k)$ can be arbitrarily large. Indeed, let T_k be the tree obtained from a path P_k and k copies of the tree H by identifying vertex u_4 of the ith copy of H with the ith vertex of the path P_k so that no two copies of H have a common vertex. Then $\gamma(T_k) \leq 4k$ and $\beta_{ev}(T_k) \geq 5k$ yielding $\beta_{ev}(T_k) - \gamma(T_k) \geq k$.

3. Well ev-covered trees

According to (1), for every graph G, $\gamma_{ev}(G) \leq \gamma(G)$. In the sequel, we give a necessary condition for ntc graphs G with $\gamma_{ev}(G) = \gamma(G)$.

Theorem 6. *If G is an ntc graph such that $\gamma_{ev}(G) = \gamma(G)$, then every $\gamma(G)$-set is independent.*

Proof. Let G be an ntc graph such that $\gamma_{ev}(G) = \gamma(G)$. Suppose to the contrary that there is a $\gamma(G)$-set S which is not independent. Let x and y be two adjacent vertices of S. Note that since G is connected, the minimality of S implies that every vertex in S has at least one neighbor in $V - S$. Let E' be a set of edges between S and $V - S$ chosen so that each vertex $z \in S - \{x, y\}$ is an end-vertex of exactly one edge of E'. Clearly, $|E'| = |S| - 2$ and $E' \cup \{xy\}$ is an ev-dominating of G. Hence $\gamma_{ev}(G) \leq |E' \cup \{xy\}| = |S| - 1$, a contradiction. Therefore, every $\gamma(G)$-set is independent. □

Using Theorem 6, we obtain the following corollary which provides a necessary condition for well ev-covered trees T.

Corollary 7. *If T is a nontrivial well ev-covered tree, then every $\gamma(T)$-set is independent.*

We note that the converse of Corollary 7 is not true as can be seen by the path P_8, where any $\gamma(P_8)$-set is independent. But $i_{ev}(P_8) = 2$ and $\beta_{ev}(P_8) = 3$.

Proposition 8. *If T is a tree of order $n \geq 3$, then there exists a $\beta_{ev}(T)$-set containing for each support vertex of T one pendant edge incident to it.*

Proof. Clearly, the result is valid for stars and double stars. Hence we can assume that T has diameter at least four. Let F be a $\beta_{ev}(T)$-set, and assume that for some support vertex x, set F contains no pendant edge incident with x. Recall that by Observation 2, $\mathrm{pn}(e, F) \neq \emptyset$ for every edge e of F. Let x' be a leaf neighbor of x. To ev-dominate xx', F must contain an edge xy for some non-leaf neighbor y of x. If $\mathrm{pn}(xy, F) \subseteq N(x)$, then $\{xx'\} \cup F - \{xy\}$ is a $\beta_{ev}(T)$-set that contains xx' as desired. Hence we can assume that $\mathrm{pn}(xy, F) - N(x) \neq \emptyset$. Let $\mathrm{pn}(xy, F) - N(x) = \{z_1, z_2, \ldots, z_k\}$. Clearly, each $z_i \in N(y)$. If z_i is a leaf for some i, then $\{xx', yz_i\} \cup F - \{xy\}$ is a minimal independent ev-dominating set of T larger than F, a contradiction. Thus every z_i is different from a leaf. Now, for every $i \in \{1, 2, \ldots, k\}$, let $A_i = N(z_i) - \{y\}$. For every vertex v of A_i, let e_v denote the edge of F that ev-dominates v, and let $E_{A_i} = \{e_v \in F : v \in A_i\}$. Note that since T is a tree, the A_i's are disjoint sets and no edge of E_{A_i} dominates two vertices of A_i. If for some i, no vertex of A_i is a unique private vertex of an edge of E_{A_i}, then $\{xx', yz_i\} \cup F - \{xy\}$ is a minimal independent ev-dominating set of T larger than F, a contradiction. Therefore for each $i \in \{1, 2, \ldots, k\}$, there is a vertex, say u_i, in A_i which is the unique private neighbor of e_{u_i}, that is $\mathrm{pn}(e_{u_i}, F) = \{u_i\}$. For every i, let u_i' be the endvertex of e_{u_i} which is adjacent to u_i. In that case, $F^* = \{u_1'u_1, \ldots, u_k'u_k\} \cup F - \{e_{u_1}, \ldots, e_{u_k}\}$ is a minimal independent ev-dominating set of T with cardinality $\beta_{ev}(T)$, that is, F^* is a $\beta_{ev}(T)$-set. It is worth mentioning that for set F^*, no z_i is a private vertex of an edge of F^*. But then $\{xx'\} \cup F^* - \{xy\}$ is a $\beta_{ev}(T)$-set that contains the pendant edge xx' as desired, and the proof is complete. □

Our aim in the next is to characterize the well ev-covered trees T. Let $\mathcal{T}$ be the family of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k (k \geq 1)$ of trees such that $T_1 = P_2, T = T_k$, and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the following operations. Let one vertex of P_2 be considered as a support vertex.

- **Operation $\mathcal{O}_1$:** add a vertex by joining it to a support vertex of T_i.
- **Operation $\mathcal{O}_2$:** add a pendant path $P_2 = ab$ attached by an edge bc at a vertex c of T_i already containing at least one pendant path cc_1c_2 of length 2.
- **Operation $\mathcal{O}_3$:** add a pendant path $P_3 = abc$ attached by an edge cd at a vertex d of T_i such that d is either a support vertex or contained in a pendant path of length 2 or 3.

– **Operation $\mathcal{O}_4$:** add a pendant path $P_4 = uvwx$ attached by an edge xy at a support vertex y of T_i.

Lemma 9. *If $T \in \mathcal{T}$, then T is well ev-covered.*

Proof. We use induction on the number of operations k performed to construct T. Clearly, the property is true for $T_1 = P_2$. Suppose the property is true for all trees of $\mathcal{T}$ constructed with $k-1 \geq 0$ operations. Let $T = T_k$ with $k \geq 2$, and $T' = T_{k-1}$. By induction on T', T' is well ev-covered, that is $\gamma_{ev}(T') = \beta_{ev}(T')$. We examine the following situations depending on whether T is obtained from T' by Operation $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$, or $\mathcal{O}_4$. By Proposition 8, let F be a $\beta_{ev}(T)$-set that contains all pendant edges of T. Also, by Observation 4, let D be a $\gamma_{ev}(T)$-set that contains no pendant edge.

Clearly, if T was obtained from T' by Operation $\mathcal{O}_1$, then $\beta_{ev}(T') = \beta_{ev}(T)$, $\gamma_{ev}(T') = \gamma_{ev}(T)$, and thus $\beta_{ev}(T) = \gamma_{ev}(T)$.

Assume that T was obtained from T' by Operation $\mathcal{O}_2$. Then $ab, c_1c_2 \in F$ and thus $F - \{ab\}$ is a minimal independent ev-dominating set of T'. Hence $\beta_{ev}(T') \geq \beta_{ev}(T) - 1$. Moreover, since $bc, cc_1 \in D$, it follows that $D - \{bc\}$ is an ev-dominating set of T', and thus $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 1$. Therefore

$$\gamma_{ev}(T) \leq \beta_{ev}(T) \leq \beta_{ev}(T') + 1 = \gamma_{ev}(T') + 1 \leq \gamma_{ev}(T),$$

and so $\gamma_{ev}(T) = \beta_{ev}(T)$.

Assume now that T was obtained from T' by Operation $\mathcal{O}_3$. Let H be a pendant path of T' of length $1, 2$, or 3 that contains vertex d. Note that if H has length 1, then d is a support vertex. Clearly $ab \in F$. Also, if H has length 1, then F contains the pendant edge incident with d, and if H has length 2, say dxy, then F contains xy. Finally, if H has length 3, say $dxyz$, then $yz \in F$. If d is ev-dominated by at least two edges of F, then $F - \{ab\}$ is a minimal independent ev-dominating set of T'. Now if d is ev-dominated only by ab, then clearly H has length 3, and so $F' = \{xy\} \cup F - \{yz\}$ is also a $\beta_{ev}(T)$-set, implying that $F' - \{ab\}$ is a minimal independent ev-dominating set of T'. In any case, we have $\beta_{ev}(T') \geq \beta_{ev}(T) - 1$. On the other hand, since $bc \in D$ and D contains an edge of $E(T')$ incident with d or its neighbor, we deduce that $D - \{bc\}$ is an ev-dominating set of T', and thus $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 1$. Therefore

$$\gamma_{ev}(T) \leq \beta_{ev}(T) \leq \beta_{ev}(T') + 1 = \gamma_{ev}(T') + 1 \leq \gamma_{ev}(T),$$

and so $\gamma_{ev}(T) = \beta_{ev}(T)$.

Finally, assume that T was obtained from T' by Operation $\mathcal{O}_4$. Let y' be a leaf neighbor of y. Since $uv, yy' \in F$, we obtain that $F - \{uv\}$ is a minimal independent ev-dominating set of T', and thus $\beta_{ev}(T') \geq \beta_{ev}(T) - 1$. On the other hand, $vw \in D$ and D contains an edge of $E(T')$ incident with y. Hence $D - \{vw\}$ is an ev-dominating set of T', and thus $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 1$. Therefore,

$$\gamma_{ev}(T) \leq \beta_{ev}(T) \leq \beta_{ev}(T') + 1 = \gamma_{ev}(T') + 1 \leq \gamma_{ev}(T),$$

and hence $\gamma_{ev}(T) = \beta_{ev}(T)$. ∎

Theorem 10. *A nontrivial tree T is well ev-covered if and only if $T \in \mathcal{T}$.*

Proof. If $T \in \mathcal{T}$, then by Lemma 9, T is well ev-covered. To prove the necessity, we use an induction on the order n of T. Clearly, if $n = 2$, then T is a path P_2 that belongs to $\mathcal{T}$. Let $n \geq 3$ and assume that every well ev-covered tree T' of order n' with $2 \leq n' < n$ is in $\mathcal{T}$.

Let T be a well ev-covered tree of order n. If $\operatorname{diam}(T) = 2$, then T is a star that belongs to $\mathcal{T}$ since it is obtained from P_2 by using Operation O_1. Since double stars are not well ev-covered (by Cor. 7), we can assume that T has diameter at least four.

If any support vertex, say x, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to x. It can be seen easily that $\beta_{ev}(T') = \beta_{ev}(T)$, $\gamma_{ev}(T') = \gamma_{ev}(T)$ and so T' is well ev-covered. By induction on T', we have $T' \in \mathcal{T}$. It follows that $T \in \mathcal{T}$ because it is obtained from T' by using Operation $\mathcal{O}_1$. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

If T has two adjacent support vertices, say a and b, then by Observation 3, T has a $\gamma(T)$-set that contains a and b, which contradicts Corollary 7. Therefore, no two support vertices of T are adjacent.

Root T at a vertex r of maximum eccentricity $\operatorname{diam}(T) \geq 4$, and let u be a support vertex at maximum distance from r, v be the parent of u, w be the parent of v, and x be the parent of w in the rooted tree. Since every support vertex is adjacent to exactly one leaf, let u' be the leaf neighbor of u. Note that v is different from a support vertex and $\deg_G(w) \geq 2$, since $\operatorname{diam}(T) \geq 4$. We distinguish between two cases.

Case 1. $\deg_T(v) \geq 3$. Hence every child of v is a support vertex of degree two. Let T' be the tree obtained from T by removing vertices u and u'. Note that $\operatorname{diam}(T') \geq 4$ Since every $\beta_{ev}(T')$-set can be extended to a minimal independent ev-dominating set of T by adding edge uu', $\beta_{ev}(T) \geq \beta_{ev}(T') + 1$. On the other hand, any $\gamma_{ev}(T')$-set containing no pendant edge can be extended to an ev-dominating set of T by adding edge uv, and thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Combining the previous two inequalities and using the fact that T is well ev-covered, we obtain that $\gamma_{ev}(T) = \beta_{ev}(T) \geq \beta_{ev}(T') + 1 \geq \gamma_{ev}(T') + 1 \geq \gamma_{ev}(T)$. Consequently, $\gamma_{ev}(T) = \gamma_{ev}(T')+1$, $\beta_{ev}(T) = \beta_{ev}(T')+1$ and thus T' is well ev-covered. Applying the inductive hypothesis to T', we have $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$ because it is obtained from T' by using Operation $\mathcal{O}_2$.

Case 2. $\deg_T(v) = 2$. We consider the following two subcases.

Subcase 2.1. $\deg_T(w) \geq 3$. Then $r \neq x$; otherwise w would be a support vertex with at least two leaves which contradicts the assumption. Also, seeing the previous situations, every vertex in T_w different from w has degree 1 or 2. More precisely, every rooted tree a child t of w is a path of order $1, 2$ or 3. Let $T' = T - T_v$. Clearly, $\beta_{ev}(T) \geq \beta_{ev}(T') + 1$ since every $\beta_{ev}(T')$-set can be extended to a minimal independent ev-dominating set of T by adding the edge uu'. Likewise, every $\gamma_{ev}(T')$-set can be extended to an ev-dominating set of T by adding edge uv, and hence $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. It follows that $\gamma_{ev}(T) = \beta_{ev}(T) \geq \beta_{ev}(T') + 1 \geq \gamma_{ev}(T') + 1 \geq \gamma_{ev}(T)$, and so $\gamma_{ev}(T') = \beta_{ev}(T')$, that is T' is well ev-covered. By induction on T', we have $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$, since T is obtained from T' by using Operation $\mathcal{O}_3$.

Subcase 2.2. $\deg_T(w) = 2$. If $r = x$, then $T = P_5$ that belongs to $\mathcal{T}$ since it can be obtained from $T_1 = P_2$ by using Operation $\mathcal{O}_1$ and Operation $\mathcal{O}_2$, where a vertex of T_1 is a support vertex as we assumed. Hence we can assume that $x \neq r$, and so let y be the parent of x in the rooted tree.

Suppose first that x is a support vertex, and let $T' = T - T_w$. If F' is a $\beta_{ev}(T')$-set containing the unique pendant edge incident with x (by Prop. 8), then $F' \cup \{u'u\}$ is a minimal independent ev-dominating set of T, and thus $\beta_{ev}(T) \geq \beta_{ev}(T') + 1$. Moreover, any $\gamma_{ev}(T')$-set can be extended to an ev-dominating set of T by adding edge uv, and so $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Therefore, $\gamma_{ev}(T) = \beta_{ev}(T) \geq \beta_{ev}(T') + 1 \geq \gamma_{ev}(T') + 1 \geq \gamma_{ev}(T)$, and so we have $\gamma_{ev}(T') = \beta_{ev}(T')$. Thus T' is well ev-covered, and by induction on T', we have $T \in \mathcal{T}$. Since T is obtained from T' by using Operation $\mathcal{O}_4$, we deduce that $T \in \mathcal{T}$.

From now on, we will assume that x is not a support vertex, and thus $r \neq y$. Let z be the parent of y in the rooted tree.

Subcase 2.2.1. $\deg_T(x) \geq 3$. Let F be a minimal independent ev-dominating set in T containing edges $u'u$ and vw. Note that such a set exists, and $\beta_{ev}(T) \geq |F|$. Also, since F is minimal, F contains no edge incident with x. Now, if $x \notin \operatorname{pn}(vw, F)$, then $F' = \{vu\} \cup F - \{u'u, vw\}$ is an ev-dominating set in T, implying that $\gamma_{ev}(T) \leq |F'| = |F| - 1 < \beta_{ev}(T)$, a contradiction. Hence we assume that $x \in \operatorname{pn}(vw, F)$. It follows that x belongs to no pendant path of length 2, and thus x belongs to some pendant path $R_1 : xx_1x_2x_3$ or $R_2 : xx_1x_2x_3x_4$ or x has a child t of degree $k \geq 3$ for which T_t is isomorphic to a subdivided tree SS_{k-1} centered at t. If x belongs to R_1, then $x_2x_3 \in F$ because of $x \in \operatorname{pn}(vw, F)$. But then $F' = \{vu, x_1x_2\} \cup F - \{u'u, vw, x_2x_3\}$ is an ev-dominating set in T, implying that $\gamma_{ev}(T) \leq |F'| = |F| - 1 < \beta_{ev}(T)$, contradicting the fact that T is well ev-covered. Suppose that x belongs to R_2. Observe that $x_3x_4 \notin F$ for otherwise x_1 cannot be ev-dominated because of $x \in \operatorname{pn}(vw, F)$. Hence $x_2x_3 \in F$. But then $F'' = \{x_3x_4, x_1x_2\} \cup F - \{x_2x_3\}$ is a minimal independent ev-dominating set in T of size $|F| + 1$, implying that $\gamma_{ev}(T) \leq |F| < |F''| \leq \beta_{ev}(T)$, a contradiction. Finally, let $T_t = \mathrm{SS}_{k-1}$ centered at t. Let h be a support vertex of T_t and h' its leaf neighbor. Clearly, since $x \in \mathrm{pn}(vw, F)$, set F contains all pendant edges of T_t, in particular $h'h \in F$. But then $F' = \{vu, ht\} \cup F - \{u'u, vw, h'h\}$ is an ev-dominating set in T, implying that $\gamma_{ev}(T) \le |F'| = |F| - 1 < \beta_{ev}(T)$, a contradiction.

Subcase 2.2.2. $\deg_T(x) = 2$. We first note that y is different from a support vertex; otherwise, by Observation 3, there exists a $\gamma(T)$-set containing both u and y. But to dominate w, such a $\gamma(T)$-set may also contain v or x, which contradicts Corollary 7. Thus, $r \neq z$. Let s be a neighbor of y chosen so that s is either a support vertex adjacent to y (if any) or $s = z$. Now, let F be a minimal independent ev-dominating set in T containing edges $u'u, vw$ and ys. Clearly, such a set exists, and $\beta_{ev}(T) \ge |F|$. It follows that $F' = \{vu\} \cup F - \{u'u, vw\}$ is an ev-dominating set in T, implying that $\gamma_{ev}(T) \le |F'| = |F| - 1 < \beta_{ev}(T)$, contradicting the fact that T is well ev-covered. This completes the proof. □

4. Trees T with $\gamma_{ev}(T) = i_{ev}(T) = \gamma(T)$

The purpose of this section is to characterize the trees T with $\gamma_{ev}(T) = i_{ev}(T) = \gamma(T)$. But first, we need to point out that Hedetniemi *et al.* [2] have given an interesting result, but not common, characterizing the ev-domination number in terms of the paired-domination number. We recall that the paired-domination number of a graph with no isolated vertices is the minimum cardinality of a dominating set whose induced subgraph has a perfect matching. Hedetniemi *et al.* [2] showed that for every graph G with no isolated vertices $2\gamma_{ev}(G) = \gamma_{pr}(G)$. Therefore characterizing the trees T such that $\gamma_{ev}(T) = i_{ev}(T) = \gamma(T)$ is equivalent to characterizing the trees T such that $\gamma_{pr}(T) = 2\gamma(T)$. In 2006, Haynes and Henning [1] have given a constructive characterization of trees T such that $\gamma_{pr}(T) = 2\gamma(T)$. However, a year later, Henning and Vestergaard [3] gave another constructive characterization of such trees using labelings, arguing that is simpler than the one presented in [1]. The constructive characterization we present here is also simpler using only 4 operations. Moreover, the proof of our result remains rather short.

To do this we define the family $\mathcal{F}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k (k \ge 1)$ of trees such that $T_1 = P_2$, $T = T_k$, and if $k \ge 2$, T_{i+1} is obtained recursively from T_i by one of the operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ defined in Section 3 or operation $\mathcal{O}_5$ defined below. Let one vertex of P_2 be considered as a support vertex.

- **Operation $\mathcal{O}_5$:** add a pendant path $P_4 = abcd$ attached by an edge dx at a vertex x of T_i belonging to a $\gamma(T_i)$-set.

In the rest of the paper, we shall prove:

Theorem 11. *Let T be a nontrivial tree. Then $\gamma_{ev}(T) = \gamma(T)$ if and only if $T \in \mathcal{F}$.*

It is worth mentioning that if T is a well ev-covered tree, then by (1) we have $\gamma_{ev}(T) = \gamma(T)$, and thus $T \in \mathcal{F}$. However, not every tree $T \in \mathcal{F}$ is well ev-covered. This can be seen by the path P_9, where $P_9 \in \mathcal{F}$ (obtained from $T_1 = P_2$ by using Operations $\mathcal{O}_1, \mathcal{O}_2$, and $\mathcal{O}_5$) but $4 = \beta_{ev}(P_9) > i_{ev}(P_9) = 3$.

Lemma 12. *If $T \in \mathcal{F}$, then $\gamma_{ev}(T) = \gamma(T)$.*

Proof. We use induction on the number of operations k performed to construct T. Clearly, the property is true for $T_1 = P_2$. Suppose the property is true for all trees of $\mathcal{F}$ constructed with $k - 1 \ge 0$ operations. Let $T = T_k$ with $k \ge 2$, and $T' = T_{k-1}$. By induction on T', $\gamma_{ev}(T') = \gamma(T')$. Obviously, if T is a star, then the result holds. Moreover, since double stars do not belong to $\mathcal{F}$, we can assume that $\mathrm{diam}(T) \ge 4$. We examine the following situations. By Observation 4, let F be a $\gamma_{ev}(T)$-set that contains no pendant edge.

If T was obtained from T' by Operation $\mathcal{O}_1$, then clearly $\gamma(T') = \gamma(T)$, $\gamma_{ev}(T') = \gamma_{ev}(T)$, and so $\gamma(T) = \gamma_{ev}(T)$.

Assume that T is obtained from T' by Operation $\mathcal{O}_2$. Then $\gamma(T) \leq \gamma(T')+1$ since $\gamma(T')$-set can be extended to a dominating set of T by adding b. On the other hand, since $bc, cc_1 \in F$, we have $F-\{bc\}$ is an ev-dominating set of T', and thus $\gamma_{ev}(T') \leq \gamma_{ev}(T)-1$. Therefore

$$\gamma_{ev}(T) \leq \gamma(T) \leq \gamma(T')+1 = \gamma_{ev}(T')+1 \leq \gamma_{ev}(T),$$

and thus $\gamma_{ev}(T) = \gamma(T)$.

Assume now that T is obtained from T' by Operation $\mathcal{O}_3$. Clearly, $\gamma(T) \leq \gamma(T')+1$ and $bc \in F$. Also, since d is contained in a pendant path of length at most 3, there must exist an edge of $E(T') \cap F$ incident with either d or its neighbor. We deduce that $F-\{bc\}$ is an ev-dominating set of T'. Hence $\gamma_{ev}(T') \leq \gamma_{ev}(T)-1$. Therefore

$$\gamma_{ev}(T) \leq \gamma(T) \leq \gamma(T')+1 = \gamma_{ev}(T')+1 \leq \gamma_{ev}(T),$$

and thus $\gamma_{ev}(T) = \gamma(T)$.

Finally, assume that T is obtained from T' using Operation $\mathcal{O}_5$. Since x belongs to some $\gamma(T')$-set, such a set can be extended to dominating set of T by adding vertex b, and so $\gamma(T) \leq \gamma(T')+1$. Also, $bc \in F$ and without loss of generality, we can assume that $cd, dx \notin F$. It follows that $F-\{bc\}$ ev-dominates T', and thus $\gamma_{ev}(T') \leq \gamma_{ev}(T)-1$. Therefore

$$\gamma_{ev}(T) \leq \gamma(T) \leq \gamma(T')+1 = \gamma_{ev}(T')+1 \leq \gamma_{ev}(T),$$

and so $\gamma_{ev}(T) = \gamma(T)$. □

Lemma 13. *If T is a nontrivial tree with $\gamma_{ev}(T) = \gamma(T)$, then $T \in \mathcal{F}$.*

Proof. We use an induction on the order n of T. Clearly, if $n \in \{2,3\}$, then T is a star and $T \in \mathcal{F}$. Let $n \geq 4$, and assume that every nontrivial tree T' of order n' with $2 \leq n' < n$ satisfying $\gamma_{ev}(T') = \gamma(T')$ is in $\mathcal{T}$.

Let T be a tree of order n such that $\gamma_{ev}(T) = \gamma(T)$. If $\operatorname{diam}(T) = 2$, then T is a star belonging to $\mathcal{F}$. Since Theorem 6 implies that T is not a double star, we can assume that T has diameter at least four.

If any support vertex, say x, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to x. Then $\gamma(T') = \gamma(T)$, $\gamma_{ev}(T') = \gamma_{ev}(T)$ and so $\gamma(T') = \gamma_{ev}(T')$. By induction on T', we have $T' \in \mathcal{T}$. Thus $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_1$. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

Root T at a vertex r of maximum eccentricity, and let u be a support vertex at maximum distance from r. Let v be the parent of u, w be the parent of v, and x be the parent of w in the rooted tree. Note that w has degree at least two, since $\operatorname{diam}(T) \geq 4$. Let S be a $\gamma(T)$-set that contains all support vertices of T. Then $u \in S$. We distinguish between two cases.

Case 1. $\deg_T(v) \geq 3$. Then every child of v is either a leaf or a support vertex. If v is a support vertex, then S contains u and v, which contradicts Theorem 6. Hence, every child of v is a support vertex of degree two and all such children are in S. Let $T' = T - T_u$. Since v is dominated by $S-\{u\}$, we have $\gamma(T') \leq \gamma(T)-1$. Also, $\gamma_{ev}(T) \leq \gamma_{ev}(T')+1$ since any $\gamma_{ev}(T')$-set can be extended to an ev-dominating set of T by adding the edge uv. Now combining the previous inequalities and using the fact $\gamma_{ev}(T) = \gamma(T)$, we obtain that $\gamma_{ev}(T) = \gamma(T) \geq \gamma(T')+1 \geq \gamma_{ev}(T')+1 \geq \gamma_{ev}(T)$. Consequently, $\gamma_{ev}(T') = \gamma(T')$, and by the inductive hypothesis on T', $T' \in \mathcal{F}$. Therefore $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_2$.

Case 2. $\deg_T(v) = 2$. We consider the following two subcases.

Case 2.1. $\deg_T(w) \geq 3$. Then w is a support vertex or every leaf in T_w is at distance two or three from w, that is w belongs to a pendant path of length 1, 2 or 3. Let $T' = T - T_v$. Then $S-\{u\}$ is a dominating set of T', and hence $\gamma(T') \leq \gamma(T)-1$. As discussed above, it is easy to see that $\gamma_{ev}(T) \leq \gamma_{ev}(T')+1$. Hence $\gamma_{ev}(T) = \gamma(T) \geq \gamma(T')+1 \geq \gamma_{ev}(T')+1 \geq \gamma_{ev}(T)$, and thus $\gamma_{ev}(T') = \gamma(T')$. By induction on T', we have $T' \in \mathcal{F}$. Since T is obtained from T' by using Operation $\mathcal{O}_3$, we obtain that $T \in \mathcal{F}$.

Case 2.2. $\deg_T(w) = 2$. If $r = x$, then $T = P_5$ that belongs to $\mathcal{F}$ because it can be obtained from $T = P_2$ by using Operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Hence we assume that $x \neq r$. Let $T' = T - T_w$. We assume that $w \notin S$ for otherwise we can replace it by x in S. Thus $x \in S$, and $S - \{u\}$ dominates T', implying that $\gamma(T') \leq \gamma(T) - 1$. We also have $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Hence $\gamma_{ev}(T) = \gamma(T) \geq \gamma(T') + 1 \geq \gamma_{ev}(T') + 1 \geq \gamma_{ev}(T)$. Thus $\gamma_{ev}(T') = \gamma(T')$ and x belongs to a $\gamma(T')$-set, namely $S - \{u\}$. By induction on T', we have $T' \in \mathcal{F}$. Now since T is obtained from T' by using Operation $\mathcal{O}_5$, we deduce that $T \in \mathcal{F}$. $\square$

According to Lemmas 12 and 13, we have proven Theorem 11. We conclude this paper with some open problems.

Problem 1. Can you recognize a well *ev*-covered graph in polynomial time?

Problem 2. Characterize the well *ev*-covered graphs.

Problem 3. Characterize the trees T such that $\gamma(T) = \beta_{ev}(T)$.

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