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BOUNDS FOR A_α-EIGENVALUES

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Abstract. Let G be a graph with adjacency matrix $A(G)$ and degree diagonal matrix $D(G)$. In 2017, Nikiforov (V. Nikiforov, *Appl. Anal. Discret. Math.* **11** (2017) 81–107.) defined the matrix $A_\alpha(G)$, as a convex combination of $A(G)$ and $D(G)$, the following way, $A_\alpha(G) = \alpha A(G) + (1 - \alpha)D(G)$, where $\alpha \in [0, 1]$. In this paper we present some new upper and lower bounds for the largest, second largest and the smallest eigenvalue of A_α-matrix. Moreover, extremal graphs attaining some of these bounds are characterized.

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1. Introduction

Let $G = (V, E)$ be a simple graph such that $|V| = n$ and $|E| = m$. If v_i is adjacent to v_j we denote by $v_i \sim v_j$, otherwise $v_i \nsim v_j$. For $v \in V$, the set of its neighbours is denoted by N_v and $|N_v|$ is the cardinality of N_v. For each vertex $v \in V$ the degree of v, denoted by $d(v)$, is the number of neighbours to v. The minimum degree of G is $\delta(G) := \min\{d(v) : v \in V\}$ and the maximum degree of G is $\Delta(G) := \max\{d(v) : v \in V\}$. Eventually, we will use d_i, Δ and δ to represent $d(v_i)$, $\Delta(G)$ and $\delta(G)$ respectively. Also assume that the vertices are labeled such that $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$ and the degree sequence is defined by $d(G) = (d_1, \ldots, d_n)$. A graph is said to be bidegreed if and only if its degree sequence has only two different values.

The first Zagreb index is defined by $Z_1(G) = \displaystyle\sum_{i=1}^{n} d^2(v_i)$ and the study of its bounds and properties can be found at [4, 7–9, 27]. A nonempty subset $I \subset V$ is *independent* if and only if no two of its elements are adjacent. The independence number of G, $\gamma(G)$, is the largest cardinality among all independent sets of G.

A graph is called regular if all its vertices have the same degree. The complement of the graph G, denoted by $\overline{G}$, is the graph obtained from G with the same vertex set, $\overline{V} = V$, and $v_iv_j \in \overline{E}$ if and only if $v_iv_j \notin E$. We denote by K_n, $K_{a,b}$, $K_{1,n-1}$ and P_n the complete graph, the complete bipartite graph, the star and the path, respectively. The bipartite complement of a connected bipartite graph G with partitions V_1 and V_2 is a bipartite graph, denoted by $\widetilde{G}$, such that $\widetilde{G}$ has edges between V_1 and V_2 exactly where G does not, that

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is, $V(\widetilde{G}) = V(K_{|V_1|,|V_2|})$ and $E(\widetilde{G}) = E(K_{|V_1|,|V_2|}) - E(G)$. A graph G is called semi-regular bipartite, with parameters (n_1, n_2, r_1, r_2), if G is bipartite such that $V = V_1 \cup V_2$ where $n_1 = |V_1|$ and $n_2 = |V_2|$, and the vertices in the same partition have the same degree. In other words, n_1 vertices have degree r_1 and n_2 vertices have degree r_2, such that $n_1 r_1 = n_2 r_2$.

Let $x \in \mathbb{R}^n$, we denote by $|x|$ the Euclidean norm of x. Let M be a $n \times n$ matrix. If M is symmetric, the M-eigenvalues are real and we shall index them in non-increasing order, represented by $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$. The collection of M-eigenvalues together with their multiplicities is called the M-spectrum, denoted by $\sigma(M)$.

The adjacency matrix of G, $A = A(G) = [a_{ij}]$, is a square and symmetric matrix of order n, such that $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise. The degree matrix of G, denoted by $D(G) = [d_{ij}]$, is the diagonal matrix such that $d_{ii} = d(v_i)$. The Laplacian and signless Laplacian matrices are defined by $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. An interesting problem in Graph Spectral Theory is to obtain bounds for A-eigenvalues, L-eigenvalues and Q-eigenvalues involving invariants associated to graphs.

In 2017 Nikiforov, [25], defined for any real $\alpha \in [0, 1]$ the convex linear combination, $A_\alpha(G)$, of $A(G)$ and $D(G)$ in the following way:

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G), \quad \alpha \in [0, 1].$$

It is easy to see that $A(G) = A_0(G)$, $D(G) = A_1(G)$ and $Q(G) = 2A_{\frac{1}{2}}(G)$. So, obtaining bounds for A_α-eigenvalues is an interesting problem because it contemplates the study of bounds for the adjacency and signless Laplacian matrices.

Results involving bounds for A_α-eigenvalues have been obtained, as we can see in [1, 3, 20–23, 25, 28, 30, 31]. In this paper we obtain some bounds for the largest eigenvalue, the second largest eigenvalue and the smallest eigenvalue of the A_α matrix. This paper is organized as follows: in Section 2 we introduce some definitions and results required to prove the main results; after, in Section 3 we show the main results referring to bounds for A_α-eigenvalues.

2. Preliminaries

In this section we present some aspects of matrix theory that will be needed to prove the main results of this paper. The principal sub-matrix of a matrix is obtained by removing rows and columns with the same indices [12]. An important result about sub-matrices is presented in Theorem 2.1.

Theorem 2.1. *[18] Suppose $A \in M_n(\mathbb{R})$ is symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. If $B \in M_m(\mathbb{R})$ with $m \leq n$ is a principal submatrix of A with eigenvalues $\mu_1 \geq \cdots \geq \mu_m$, then $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$, for $i = 1, \ldots, m$.*

The following result is the theorem of Weyl and So, which is inequalities involving eigenvalues of sums of Hermitian matrices.

Theorem 2.2 (Weyl)**.** *[18] Let $A, B \in M_n(\mathbb{R})$ be Hermitian and let the spectrum of A, B, and $A+B$ be $\sigma(A) = \{\lambda_1(A), \ldots, \lambda_n(A)\}$, $\sigma(B) = \{\lambda_1(B), \ldots, \lambda_n(B)\}$ and $\sigma(A+B) = \{\lambda_1(A+B), \ldots, \lambda_n(A+B)\}$, respectively. Then,*

$$\lambda_{i+j-1}(A+B) \leq \lambda_i(A) + \lambda_j(B), \quad j = 1, \ldots, n-i+1 \tag{1}$$

for each $i = 1, \ldots, n$, with equality for some pair i, j if and only if there is a nonzero vector x such that $Ax = \lambda_i x$, $Bx = \lambda_j x$ and $(A+B)x = \lambda_{i+j-1} x$. Also,

$$\lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A+B), \quad j = i, \ldots, n \tag{2}$$

for each $i = 1, \ldots, n$, with equality for some pair i, j if and only if there is a nonzero vector x such that $Ax = \lambda_i x$, $Bx = \lambda_j x$ and $(A+B)x = \lambda_{i+j-n} x$. If A and B have no common eigenvector, then the inequalities in (1) and (2) are strict.

As a consequence of Theorem 2.2, follows Corollary 2.3.

Corollary 2.3. *[18] Let $A, B \in M_n(\mathbb{R})$ be Hermitian. Then,*

$$\lambda_i(A) + \lambda_n(B) \le \lambda_i(A+B) \le \lambda_i(A) + \lambda_1(B), \tag{3}$$

with $i = 1, \dots, n$. Equality in the upper bound holds if and only if there is a nonzero vector x that is the eigenvector of A, B and $A+B$ with corresponding eigenvalues λ_i, λ_1 and λ_i, respectively. Analogously, equality in the lower bound holds if and only if there is a nonzero vector x that is the eigenvector of A, B and $A+B$ with corresponding eigenvalues λ_i, λ_n and λ_i, respectively.

Lemma 2.4. *[11] Let G be a connected graph with n vertices and $A(G)$ be its adjacency matrix. Let $P(x)$ be any polynomial function and $S_v(P(A(G)))$ be the row sums of $P(A(G))$ corresponding to each vertex v. Then*

$$\min S_v(P(A)) \le P(\lambda_1(A(G))) \le \max S_v(P(A)).$$

Moreover, equality holds if and only if the row sums of $P(A(G))$ are all equal.

The proof of Lemma 2.5 is presented because it is used to prove Theorem 3.5 in the next section.

Lemma 2.5. *[17] Let G be a graph with n vertices, m edges and minimum degree δ. Then, $S_v(A^2(G) - (\delta - 1)A(G)) \le 2m - \delta(n-1)$.*

Proof. Note that $S_v(A^k(G))$ is exactly the number of walks of length k in G which begin at v. In particular, $S_v(A(G))$ is $d(v)$ and $S_v(A^2(G)) = \sum_{u \sim v} d(u)$. So,

$$\begin{aligned} S_v(A^2(G)) &= \sum_{u \sim v} d(u) \\ &= 2m - d(v) - \sum_{u \nsim v, u \neq v} d(u) \\ &\le 2m - d(v) - (n - d(v) - 1)\delta \\ &= 2m + (\delta - 1)d(v) - \delta(n-1). \end{aligned}$$

Hence,

$$S_v(A^2(G) - (\delta - 1)A(G)) \le 2m - \delta(n-1).$$

□

The next two theorems present a lower and an upper bound, respectively, for $Z_1(G)$ using Δ, δ, m and n.

Lemma 2.6. *[7] Let G be a simple graph with n vertices and m edges. Let δ and Δ be the minimum and the maximum degree of G, respectively. Then, for $n \ge 3$, $Z_1(G) \ge \Delta^2 + \delta^2 + \dfrac{(2m - \Delta - \delta)^2}{n-2}$. Furthermore, equality occurs if and only if $d_2 = \cdots = d_{n-1}$.*

Lemma 2.7. *[7] Let G be a connected graph with n vertices and m edges. Let δ be the minimum degree of G. Then, $Z_1(G) \le 2mn - n(n-1)\delta + 2m(\delta - 1)$. Moreover, the equality holds if and only if G is a star graph or a regular graph.*

The next results involve properties and bounds for the largest, the second largest and the smallest eigenvalues of $A(G)$.

Lemma 2.8. *[6] A graph G is bipartite if and only if its spectrum is symmetric about the origin.*

Proposition 2.9. *[6] Let G be a r-regular graph. Then*

(i) r is an eigenvalue of $A(G)$;
(ii) G is a connected graph if and only if the algebraic multiplicity of r is 1;
(iii) any λ eigenvalue of $A(G)$ satisfies $|\lambda| \leq r$.

Proposition 2.10. *[15] Let G be a graph with m edges, then*

$$\lambda_1(A(G)) \geq \frac{1}{m} \sum_{v_i \sim v_j} \sqrt{d_i d_j}.$$

Equality holds if and only if G is regular or semi-regular bipartite.

Lemma 2.11. *[16] Let G be a connected graph with m edges and n vertices. Then,*

$$\lambda_1(A(G)) \leq \sqrt{2m - n + 1},$$

with equality if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Lemma 2.12. *[2] Let G be a graph with n vertices and m edges. Let $\lambda_1(A(G))$ be the largest eigenvalue of the adjacency matrix $A(G)$. Then*

$$\lambda_1(A(G)) \leq \sqrt{\max_{1 \leq i \leq n} \sum_{\substack{j \\ v_i \sim v_j}} d_j}.$$

Lemma 2.13. *[24] Let G be a r-regular graph of order n and independence number $\gamma(G)$, then*

$$\lambda_2(A(G)) \geq -1 + \frac{2(n-1-r)^{\gamma(G)}}{\gamma(G) n^{\gamma(G)-1}}.$$

Lemma 2.14. *[29] Let G be a r-regular bipartite connected graph with n vertices. Then $\lambda_2(A(G)) \leq \dfrac{n}{2} - r$.*

Corollary 2.15. *[29] Let G be a r-regular bipartite graph with n vertices. Then $\lambda_1(A(G)) + \lambda_2(A(G)) \leq \dfrac{n}{2}$. Furthermore, $\lambda_1(A(G)) + \lambda_2(A(G)) = \dfrac{n}{2}$ if and only if its bipartite complement is disjoint.*

Lemma 2.16. *[13] Let G be a graph with minimum degree $\delta \neq 0$ and independence number $\gamma(G)$, then*

$$\lambda_n(A(G)) \leq \frac{\gamma(G)\delta^2}{\lambda_1(A(G))(\gamma(G) - n)}.$$

Lemma 2.17. *[6, 14] If G is a r-regular graph with independence number $\gamma(G)$, then*

$$\lambda_n(A(G)) \leq \frac{\gamma(G) r}{\gamma(G) - n}.$$

Lemma 2.18. *[5] Let G be a triangle-free graph on n vertices. Then,*

$$\lambda_n(A(G)) \leq \frac{\lambda_1^2(A(G))}{\lambda_1(A(G)) - n}.$$

Lemma 2.19. *[29] Let G be a r-regular bipartite connected graph with $2n$ vertices and $\widetilde{G}$ be its bipartite complement. Then,*

$$\frac{P_{A(G)}(\lambda)}{\lambda^2 - r^2} = \frac{P_{A(\widetilde{G})}(\lambda)}{\lambda^2 - (n-r)^2}.$$

Lemma 2.20. *[10] If G is a regular graph of order n, then $\lambda_1(A(G))+\lambda_2(A(G)) \leq n-2$. Moreover, $\lambda_1(A(G))+\lambda_2(A(G)) = n-2$ if and only if the complement of G has a component that is a bipartite graph.*

Proposition 2.21. *[6] Let $\sigma(L(G)) = \{\mu_1, \ldots, \mu_n\}$ be the $L(G)$-spectrum such that $\mu_1 \geq \cdots \geq \mu_n$. If G is a graph with n vertices, then $\mu_1 \leq n$, with equality occurring if and only if $\overline{G}$ is disconnected.*

The next results refer to the A_α-matrix.

Proposition 2.22. *[25] If $\alpha \in [0,1]$ and G is a graph of order n, then*

$$\lambda_1(A_\alpha(G)) = \max_{|x|=1} \langle A_\alpha(G)x, x\rangle \text{ and } \lambda_n(A_\alpha(G)) = \min_{|x|=1} \langle A_\alpha(G)x, x\rangle. \tag{4}$$

Furthermore, if x is a unit vector, then $\lambda_1(A_\alpha(G)) = \langle A_\alpha(G)x, x\rangle$ if and only if x is an eigenvector of $\lambda_1(A_\alpha(G))$, and $\lambda_n(A_\alpha(G)) = \langle A_\alpha(G)x, x\rangle$ if and only if x is an eigenvector of $\lambda_n(A_\alpha(G))$.

Lemma 2.23. *[25] If $\alpha \in [0,1]$ and $k = 1, \ldots, n$ and G is a r-regular graph of order n, then there exists a linear correspondence between the eigenvalues of $A_\alpha(G)$ and $A(G)$, the following way*

$$\lambda_k(A_\alpha(G)) = \alpha r + (1-\alpha)\lambda_k(A(G)). \tag{5}$$

In particular, if G is r-regular, then $\lambda_1(A_\alpha(G)) = r, \quad \forall \alpha \in [0,1]$.

Proposition 2.24. *[25] Let $\alpha \in [0,1)$, G be a graph and x be a nonnegative eigenvector of $\lambda_1(A_\alpha(G))$.*

(i) If G is connected, then x is positive and unique minus scalar;
(ii) If G is disconnected and P is the set of vertices with positive entries of x, then the subgraph induced by P is a union of H components of G with $\lambda_1(A_\alpha(H)) = \lambda_1(A_\alpha(G))$;
(iii) If G is connected and μ is an eigenvalue of $A_\alpha(G)$ with a non-negative eigenvector, then $\mu = \lambda_1(A_\alpha(G))$;
(iv) If G is connected, and H is a proper subgraph of G, then $\lambda_1(A_\alpha(H)) < \lambda_1(A_\alpha(G))$.

Proposition 2.25. *[25] The eigenvalues of $A_\alpha(K_n)$ are $\lambda_1(A_\alpha(K_n)) = n-1$ and $\lambda_k(A_\alpha(K_n)) = \alpha n - 1$ for $2 \leq k \leq n$.*

Proposition 2.26. *[25] Let $a \geq b \geq 1$. If $\alpha \in [0,1]$, then the eigenvalues of $A_\alpha(K_{a,b})$ are*

$$\begin{aligned}
\lambda_1(A_\alpha(K_{a,b})) &= \frac{1}{2}\left(\alpha(a+b) + \sqrt{\alpha^2(a+b)^2 + 4ab(1-2\alpha)}\right), \\
\lambda_{\min}(A_\alpha(K_{a,b})) &= \frac{1}{2}\left(\alpha(a+b) - \sqrt{\alpha^2(a+b)^2 + 4ab(1-2\alpha)}\right), \\
\lambda_k(A_\alpha(K_{a,b})) &= \alpha a \text{ for } 1 < k \leq b, \\
\lambda_k(A_\alpha(K_{a,b})) &= \alpha b \text{ for } b < k < a+b.
\end{aligned}$$

3. Main results

This section presents the main results of this paper which involve bounds for the largest, the second largest and the smallest eigenvalues of A_α-matrix.

3.1. Bounds for $\boldsymbol{\lambda_1(A_\alpha(G))}$

Theorem 3.1. *Let $\alpha \in [0,1]$ and G be a graph with $m \neq 0$ edges, $n \geq 3$ vertices, Δ and δ be the maximum and minimum degrees, respectively. Then,*

$$\lambda_1(A_\alpha(G)) \geq \left(\Delta^2 + \delta^2 + \frac{(2m-\Delta-\delta)^2}{n-2}\right)\frac{\alpha}{2m} + \frac{1-\alpha}{m}\sum_{i\sim j}\sqrt{d_i d_j}. \tag{6}$$

The equality occurs if and only if G is a regular graph or $G \cong \bigcup_{k=1}^{l-1} G_k \cup K_1$ such that G_k is r-regular.

Proof. From Proposition 2.22 we know that there exists an eigenvector $x \in \mathbb{R}^n$ associated with $\lambda_1(A_\alpha(G))$ that satisfies $\lambda_1(A_\alpha(G)) = \max_{x\in\mathbb{R}^n} \frac{x^T A_\alpha(G)x}{x^T x}$. So for all $y \neq kx,\ \ k \in \mathbb{R}$, we have

$$\begin{aligned}\lambda_1(A_\alpha(G)) &\geq \frac{y^T A_\alpha(G) y}{y^T y} = \frac{y^T(\alpha D(G) + (1-\alpha)A(G))y}{y^T y}\\ &= \frac{\alpha y^T D(G) y + (1-\alpha) y^T A(G) y}{y^T y}.\end{aligned}$$

Taking $y = (\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})$, follows that

$$\begin{aligned}\lambda_1(A_\alpha(G)) &\geq \frac{\alpha\sum_{i=1}^n d_i^2 + (1-\alpha)y^T A(G))y}{2m} = \frac{\alpha\sum_{i=1}^n d_i^2}{2m} + \frac{(1-\alpha)\sum_{v_i\sim v_j} 2\sqrt{d_i d_j}}{2m}\\ &= \frac{\alpha\sum_{i=1}^n d_i^2}{2m} + \frac{(1-\alpha)}{m}\sum_{v_i\sim v_j}\sqrt{d_i d_j}.\end{aligned}$$

From Lemma 2.6 follows that

$$\lambda_1(A_\alpha(G)) \geq \frac{\alpha}{2m}\left[\Delta^2 + \delta^2 + \frac{(2m-\Delta-\delta)^2}{n-2}\right] + \frac{(1-\alpha)}{m}\sum_{v_i\sim v_j}\sqrt{d_i d_j}. \tag{7}$$

Suppose initially that G is a r-regular graph. So $\Delta = \delta = r$ and $m = \frac{nr}{2}$. Then,

$$\begin{aligned}&\frac{\alpha}{2m}\left[\Delta^2 + \delta^2 + \frac{(2m-\Delta-\delta)^2}{n-2}\right] + \frac{(1-\alpha)}{m}\sum_{v_i\sim v_j}\sqrt{d_i d_j} =\\ &\frac{\alpha}{nr}\left[r^2 + r^2 + \frac{(nr-r-r)^2}{n-2}\right] + \frac{(1-\alpha)}{m}mr = \frac{\alpha}{nr}\left[2r^2 + \frac{r^2(n-2)^2}{n-2}\right] + (1-\alpha)r =\\ &\frac{\alpha}{nr}r^2(2+n-2) + (1-\alpha)r = \alpha r + (1-\alpha)r = r.\end{aligned}$$

Now, suppose that $G \cong \bigcup_{k=1}^{l-1} G_k \cup K_1$ such that G_k is r-regular. We have $\sum_{k=1}^{l-1} |V(G_k)| + 1 = n$, $\Delta = r$, $\delta = 0$ and $m = \frac{(n-1)r}{2}$. So,

$$\frac{\alpha}{2m}\left[\Delta^2 + \delta^2 + \frac{(2m-\Delta-\delta)^2}{n-2}\right] + \frac{(1-\alpha)}{m}\sum_{v_i \sim v_j}\sqrt{d_i d_j} =$$
$$\frac{\alpha}{(n-1)r}\left[r^2 + \frac{((n-1)r-r)^2}{n-2}\right] + \frac{(1-\alpha)}{m} mr =$$
$$\frac{\alpha}{(n-1)r}\left[r^2 + \frac{r^2(n-2)^2}{n-2}\right] + (1-\alpha)r = \frac{\alpha}{(n-1)r} r^2\,(n-1) + (1-\alpha)r =$$
$$\alpha r + (1-\alpha)r = r.$$

Moreover, from Lemma 2.23 we have $\lambda_1(A_\alpha(G)) = r$.

Now, suppose there is a graph G that satisfies the equality

$$\lambda_1(A_\alpha(G)) = \left(\Delta^2 + \delta^2 + \frac{(2m-\Delta-\delta)^2}{n-2}\right)\frac{\alpha}{2m} + \frac{1-\alpha}{m}\sum_{v_i \sim v_j}\sqrt{d_i d_j}.$$

This implies that $x = (\sqrt{d_1}, \ldots, \sqrt{d_n})$ is an eigenvector of $A_\alpha(G)$ associated to $\lambda_1(A_\alpha(G))$, which is $A_\alpha(G)x = \lambda_1(A_\alpha(G))x$.

If G is connected, we have

$$\alpha d_i + (1-\alpha)\frac{\sum_{v_j \sim v_i}\sqrt{d_i d_j}}{d_i} = \lambda_1(A_\alpha(G)),$$

for all $i = 1, \ldots n$. So, for arbitrary i and s, such that $i \neq s$ we have

$$\alpha d_i + (1-\alpha)\frac{\sum_{v_j \sim v_i}\sqrt{d_i d_j}}{d_i} = \alpha d_s + (1-\alpha)\frac{\sum_{v_j \sim v_s}\sqrt{d_s d_j}}{d_s}$$

which implies that $d_i = d_s$ and then G is regular.

Now, suppose that G is disconnected. Then $G \cong \bigcup_{k=1}^{l} G_k$ and consequently for each component such that $|E(G_k)| \neq 0$, we also have

$$\alpha d_i + (1-\alpha)\frac{\sum_{v_j \sim v_i}\sqrt{d_i d_j}}{d_i} = \lambda_1(A_\alpha(G_k)),$$

for all $i = 1, \ldots, |V(G_k)|$. So, for arbitrary i and s, such that $i \neq s$ we have

$$\alpha d_i + (1-\alpha)\frac{\sum_{v_j \sim v_i}\sqrt{d_i d_j}}{d_i} = \alpha d_s + (1-\alpha)\frac{\sum_{v_j \sim v_s}\sqrt{d_s d_j}}{d_s}$$

which implies that $d_i = d_s$ and then G_k, $\forall k$, is regular. From Lemma 2.6 $Z_1(G) = \Delta^2 + \delta^2 + \frac{(2m-\Delta-\delta)^2}{n-2}$ must occur if and only if $d_2 = \cdots = d_{n-1}$. Then, $G \cong \bigcup_{k=1}^{l} G_k$ or $G \cong \bigcup_{k=1}^{l-1} G_k \cup K_1$ where G_k is r-regular. $\square$

Theorem 3.2 presents other lower bound of $\lambda_1(A_\alpha(G))$. This bound is obtained similarly of the lower bound for $\lambda_1(A(G))$, obtained by Kumar, [19]. Denote by $(\mathbf{A}_\alpha)_i$, $\mathbf{A}_i$ and $\mathbf{D}_i$ to represent the i^{th} column vector of the matrices $A_\alpha(G)$, $A(G)$ and $D(G)$, respectively. Furthermore, we define $c_{ij} = |N_{v_i} \cap N_{v_j}|$.

Theorem 3.2. *Let G be a graph with $n \geq 2$ vertices and $\alpha \in [0,1]$. Then*

$$\lambda_1(A_\alpha(G)) \geq \max\left\{\max_{j<i}\sqrt{\frac{\alpha^2(d_i^2+d_j^2)+(1-\alpha)^2(d_i+d_j)+\sqrt{C}}{2}}, \max_{j<i}\sqrt{\frac{\alpha^2(d_i^2+d_j^2)+(1-\alpha)^2(d_i+d_j)+\sqrt{F}}{2}}\right\}, \tag{8}$$

where $C = \alpha^4(d_i^2-d_j^2)^2 + 2\alpha^2(1-\alpha)^2(d_i^3+d_j^3) - 8\alpha c_{ij}(\alpha-1)^3(d_i+d_j) + (\alpha-1)^2(5\alpha^2-2\alpha+1)(d_i^2+d_j^2) - 2\alpha^2(1-\alpha)^2 d_i d_j(d_i+d_j) + 4c_{ij}^2(\alpha-1)^4 + 2d_i d_j(1-\alpha)^2(1+\alpha)(3\alpha-1)$ and $F = (d_i^2-d_j^2)^2 + 2\alpha^2(1-\alpha)^2(d_i^3+d_i^2 d_j - d_i d_j^2 + d_j^3) + (1-\alpha)^4(4c_{ij}+(d_i+d_j)^2)$.

Proof. We know that $\lambda_1(M)$, for all symmetric matrix M, is greater than or equal to the largest eigenvalue of any principal sub-matrix of M. We also know that any principal sub-matrix of order two of $A_\alpha^2(G)$ is of the form $B = \begin{bmatrix}(\mathbf{A}_\alpha^T)_i(\mathbf{A}_\alpha)_i & (\mathbf{A}_\alpha^T)_i(\mathbf{A}_\alpha)_j \\ (\mathbf{A}_\alpha^T)_j(\mathbf{A}_\alpha)_i & (\mathbf{A}_\alpha^T)_j(\mathbf{A}_\alpha)_j\end{bmatrix}$. Then,

$$\begin{aligned}
&\begin{bmatrix}(\mathbf{A}_\alpha^T)_i(\mathbf{A}_\alpha)_i & (\mathbf{A}_\alpha^T)_i(\mathbf{A}_\alpha)_j \\ (\mathbf{A}_\alpha^T)_j(\mathbf{A}_\alpha)_i & (\mathbf{A}_\alpha^T)_j(\mathbf{A}_\alpha)_j\end{bmatrix} = \\
&\begin{bmatrix}(\alpha\mathbf{D}_i^T+(1-\alpha)\mathbf{A}_i^T)(\alpha\mathbf{D}_i+(1-\alpha)\mathbf{A}_i) & (\alpha\mathbf{D}_i^T+(1-\alpha)\mathbf{A}_i^T)(\alpha\mathbf{D}_j+(1-\alpha)\mathbf{A}_j) \\ (\alpha\mathbf{D}_j^T+(1-\alpha)\mathbf{A}_j^T)(\alpha\mathbf{D}_i+(1-\alpha)\mathbf{A}_i) & (\alpha\mathbf{D}_j^T+(1-\alpha)\mathbf{A}_j^T)(\alpha\mathbf{D}_j+(1-\alpha)\mathbf{A}_j)\end{bmatrix} = \\
&\begin{bmatrix}\alpha^2\mathbf{D}_i^T\mathbf{D}_i+\alpha(1-\alpha)(\mathbf{D}_i^T\mathbf{A}_i+\mathbf{A}_i^T\mathbf{D}_i)+(1-\alpha)^2\mathbf{A}_i^T\mathbf{A}_i & \alpha^2\mathbf{D}_i^T\mathbf{D}_j+\alpha(1-\alpha)(\mathbf{D}_i^T\mathbf{A}_j+\mathbf{A}_i^T\mathbf{D}_j)+(1-\alpha)^2\mathbf{A}_i^T\mathbf{A}_j \\ \alpha^2\mathbf{D}_j^T\mathbf{D}_i+\alpha(1-\alpha)(\mathbf{D}_j^T\mathbf{A}_i+\mathbf{A}_j^T\mathbf{D}_i)+(1-\alpha)^2\mathbf{A}_j^T\mathbf{A}_i & \alpha^2\mathbf{D}_j^T\mathbf{D}_j+\alpha(1-\alpha)(\mathbf{D}_j^T\mathbf{A}_j+\mathbf{A}_j^T\mathbf{D}_j)+(1-\alpha)^2\mathbf{A}_j^T\mathbf{A}_j\end{bmatrix}
\end{aligned} \tag{9}$$

We have two cases to consider.

Case 1. $v_i \sim v_j$

From (9) we have

$$\begin{bmatrix}(\mathbf{A}_\alpha^T)_i(\mathbf{A}_\alpha)_i & (\mathbf{A}_\alpha^T)_i(\mathbf{A}_\alpha)_j \\ (\mathbf{A}_\alpha^T)_j(\mathbf{A}_\alpha)_i & (\mathbf{A}_\alpha^T)_j(\mathbf{A}_\alpha)_j\end{bmatrix} = \begin{bmatrix}\alpha^2 d_i^2+(1-\alpha)^2 d_i & \alpha(1-\alpha)(d_i+d_j)+(1-\alpha)^2 c_{ij} \\ \alpha(1-\alpha)(d_i+d_j)+(1-\alpha)^2 c_{ij} & \alpha^2 d_j^2+(1-\alpha)^2 d_j\end{bmatrix}.$$

As $\lambda_1(A_\alpha^2(G)) \geq \lambda_1(B)$ we obtain

$$\lambda_1(A_\alpha(G)) \geq \frac{1}{\sqrt{2}}\sqrt{\alpha^2(d_i^2+d_j^2)+(1-\alpha)^2(d_i+d_j)+\sqrt{C}}$$

where $C = \alpha^4(d_i^2-d_j^2)^2 + 2\alpha^2(1-\alpha)^2(d_i^3+d_j^3) - 8\alpha c_{ij}(\alpha-1)^3(d_i+d_j) + (\alpha-1)^2(5\alpha^2-2\alpha+1)(d_i^2+d_j^2) - 2\alpha^2(1-\alpha)^2 d_i d_j(d_i+d_j) + 4c_{ij}^2(\alpha-1)^4 + 2d_i d_j(1-\alpha)^2(1+\alpha)(3\alpha-1)$.

Case 2. $v_i \nsim v_j$

From (9) we have that

$$B = \begin{bmatrix}(\mathbf{A}_\alpha^T)_i(\mathbf{A}_\alpha)_i & (\mathbf{A}_\alpha^T)_i(\mathbf{A}_\alpha)_j \\ (\mathbf{A}_\alpha^T)_j(\mathbf{A}_\alpha)_i & (\mathbf{A}_\alpha^T)_j(\mathbf{A}_\alpha)_j\end{bmatrix} = \begin{bmatrix}\alpha^2 d_i^2+(1-\alpha)^2 d_i & (1-\alpha)^2 c_{ij} \\ (1-\alpha)^2 c_{ij} & \alpha^2 d_j^2+(1-\alpha)^2 d_j\end{bmatrix}.$$

Then,

$$\lambda_1(A_\alpha(G)) \geq \frac{1}{\sqrt{2}}\sqrt{\alpha^2(d_i^2+d_j^2)+(1-\alpha)^2(d_i+d_j)+\sqrt{F}}$$

where $F = (d_i^2-d_j^2)^2 + 2\alpha^2(1-\alpha)^2(d_i^3+d_i^2 d_j - d_i d_j^2 + d_j^3) + (1-\alpha)^4(4c_{ij}+(d_i+d_j)^2)$ and the result follows. $\square$

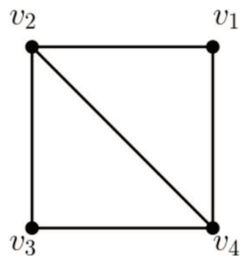

FIGURE 1. Graph G.

TABLE 1. Table with lower bounds of $\lambda_1(A_\alpha(G))$ with different α values.

	$\alpha=0.0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$
$\lambda_1(A_\alpha(G))$	2.56155	2.56815	2.57631	2.58661	2.6	2.61803	2.6434	2.68102	2.74031	2.83852
(6)	2.55959	2.55863	2.55767	2.55671	2.55576	2.5548	2.55384	2.55288	2.55192	2.55096
(8)	2.23607	2.16333	2.12603	2.12603	2.16333	2.23607	2.34094	2.47386	2.63059	2.80713

Example 3.3. Let G be a graph in Figure 1. Table 1 shows that the lower bounds presented in (6) and (8) are incomparable.

Theorem 3.4. *If G is a connected graph with n vertices, m edges, maximum degree Δ, minimum degree δ and $\alpha \in [0,1]$, then*

$$\lambda_1(A_\alpha(G)) \leq \sqrt{\alpha^2(2mn - n(n-1)\delta + 2m(\delta-1)) + 2m(1-\alpha)^2 + \delta(\alpha^2\delta + (1-\alpha)^2)(\Delta - n + 1)}. \tag{10}$$

Proof. Denote by $(A_\alpha)_i$ the i-th row of the matrix $A_\alpha(G)$. From Proposition 2.24, let $X = (x_1, \ldots, x_n)$ be the unit positive eigenvector associated to $\lambda_1(A_\alpha(G))$. Denote X^i the vector obtained from X by replacing x_j by 0 if v_j is not adjacent to v_i. Since $A_\alpha(G)X = \lambda_1(A_\alpha(G))X$, we have that $\lambda_1(A_\alpha(G))x_i = (A_\alpha)_i X = (A_\alpha)_i X^i$.

From the Cauchy-Schwarz inequality, we have

$$\lambda_1^2(A_\alpha(G))x_i^2 = |(A_\alpha)_i X^i|^2 \leq |(A_\alpha)_i|^2 |X^i|^2 = \left(d_i^2\alpha^2 + (1-\alpha)^2 d_i\right)\left(1 - \sum_{v_i \nsim v_j} x_j^2\right). \tag{11}$$

Taking the inequality (11) for all i, we have

$$\lambda_1^2(A_\alpha(G)) \leq \alpha^2 \sum_{i=1}^n d_i^2 + (1-\alpha)^2 \sum_{i=1}^n d_i - \sum_{i=1}^n \left(d_i^2\alpha^2 + (1-\alpha)^2 d_i\right)\left(\sum_{v_i \nsim v_j} x_j^2\right). \tag{12}$$

Since

$$\sum_{i=1}^n \left(d_i^2\alpha^2 + (1-\alpha)^2 d_i\right)\left(\sum_{v_i \nsim v_j} x_j^2\right) = \alpha^2 \sum_{i=1}^n d_i^2 \left(\sum_{v_i \nsim v_j} x_j^2\right) + (1-\alpha)^2 \sum_{i=1}^n d_i \left(\sum_{v_i \nsim v_j} x_j^2\right)$$

$$\geq \alpha^2\delta^2 \sum_{i=1}^n \left(\sum_{v_i \nsim v_j} x_j^2\right) + (1-\alpha)^2\delta \sum_{i=1}^n \left(\sum_{v_i \nsim v_j} x_j^2\right)$$

$$
\begin{aligned}
&= \alpha^2\delta^2 \sum_{i=1}^{n}(n-d_i-1)x_i^2 + (1-\alpha)^2\delta\sum_{i=1}^{n}(n-d_i-1)x_i^2 \\
&= \alpha^2\delta^2(n-1) + (1-\alpha)^2\delta(n-1) - \alpha^2\delta^2\sum_{i=1}^{n} d_i x_i^2 - (1-\alpha)^2\delta\sum_{i=1}^{n} d_i x_i^2. \qquad (13)
\end{aligned}
$$

Therefore, substituting (13) into (12), we have

$$
\begin{aligned}
\lambda_1^2(A_\alpha(G)) &\leq \alpha^2\sum_{i=1}^{n} d_i^2 + (1-\alpha)^2\sum_{i=1}^{n} d_i - \delta(n-1)(\alpha^2\delta + (1-\alpha)^2) + \alpha^2\delta^2\sum_{i=1}^{n} d_i x_i^2 + (1-\alpha)^2\delta\sum_{i=1}^{n} d_i x_i^2 \\
&\leq \alpha^2\sum_{i=1}^{n} d_i^2 + 2m(1-\alpha)^2 + \delta(\alpha^2\delta + (1-\alpha)^2)(\Delta - n + 1).
\end{aligned}
$$

From Lemma 2.7 we have

$$
\lambda_1^2(A_\alpha(G)) \leq \alpha^2(2mn - n(n-1)\delta + 2m(\delta-1)) + 2m(1-\alpha)^2 + \delta(\alpha^2\delta + (1-\alpha)^2)(\Delta - n + 1).
$$

Then

$$
\lambda_1(A_\alpha(G)) \leq \sqrt{\alpha^2(2mn - n(n-1)\delta + 2m(\delta-1)) + 2m(1-\alpha)^2 + \delta(\alpha^2\delta + (1-\alpha)^2)(\Delta - n + 1)} \qquad (14)
$$

and the result follows. □

Theorem 3.5. *Let G be a graph with n vertices, m edges, maximum degree Δ, minimum degree δ and $\alpha \in [0,1]$. Then*

$$
\lambda_1(A_\alpha(G)) \leq \frac{1}{2}\left(\delta - 1 + \sqrt{(\delta-1)^2 + 4(\alpha\Delta - \alpha(\delta-1)\delta + (1-\alpha)(2m - \delta(n-1))}\right). \qquad (15)
$$

Equality holds if and only if G is regular.

Proof. Let M be any matrix associated with a graph G and $S_v(M)$ be the sum of the row of M corresponding to the vertex v. It is easy to see that

$$
S_v(A_\alpha^2(G) - (\delta-1)A_\alpha(G)) = S_v(A_\alpha^2(G)) - (\delta-1)S_v(A_\alpha(G)).
$$

From [25], we have

$$
S_v(A_\alpha^2(G)) = \alpha S_v(D^2(G)) + (1-\alpha)S_v(A^2(G))
$$

and therefore

$$
\begin{aligned}
S_v(A_\alpha^2(G) - (\delta-1)A_\alpha(G)) &= \alpha S_v(D^2(G)) + (1-\alpha)S_v(A^2(G)) - \alpha(\delta-1)S_v(D(G)) - (1-\alpha)(\delta-1)S_v(A(G)) \\
&= \alpha S_v(D^2(G)) - \alpha(\delta-1)S_v(D(G)) + (1-\alpha)S_v(A^2(G) - (\delta-1)A(G)).
\end{aligned}
$$

From Lemma 2.5, we have

$$
S_v(A_\alpha^2(G) - (\delta-1)A_\alpha(G)) \leq \alpha S_v(D^2(G)) - \alpha(\delta-1)S_v(D(G)) + (1-\alpha)(2m - \delta(n-1)). \qquad (16)
$$

The inequality (16) holds for every vertex $v \in V(G)$ and for $\alpha \in [0,1]$. From Lemma 2.4 we have that

$$
\begin{aligned}
\lambda_1^2(A_\alpha(G)) - (\delta-1)\lambda_1(A_\alpha(G)) &\leq \alpha d^2(v) - \alpha(\delta-1)d(v) + (1-\alpha)(2m - \delta(n-1)) \\
&\leq \alpha\Delta^2 - \alpha(\delta-1)\delta + (1-\alpha)(2m - \delta(n-1)). \qquad (17)
\end{aligned}
$$

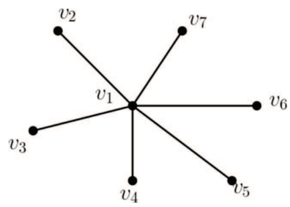

FIGURE 2. Graph G.

TABLE 2. Table with upper bounds of $\lambda_1(A_\alpha(G))$ with different α values.

	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
$\lambda_1(A_\alpha(G))$	2.44949	2.56867	2.72237	2.9215	3.17764	3.5	3.89165	4.34803	4.85913	5.41329
(10)	3.4641	3.18434	3.05941	3.10805	3.32265	3.67423	4.12795	4.65403	5.23068	5.84294
(15)	2.44949	3.0	3.4641	3.87298	4.24264	4.58258	4.89898	5.19615	5.47723	5.74456

Then, solving the quadratic inequality we obtain

$$\lambda_1(A_\alpha(G)) \leq \frac{1}{2}\left(\delta - 1 + \sqrt{(\delta-1)^2 + 4(\alpha\Delta^2 - \alpha(\delta-1)\delta + (1-\alpha)(2m - \delta(n-1))}\right).$$

Now, suppose that G is a r-regular graph. So $\delta = \Delta = r$ and $m = \frac{nr}{2}$. So, $\frac{1}{2}\left(\delta - 1 + \sqrt{(\delta-1)^2 + 4(\alpha\Delta^2 - \alpha(\delta-1)\delta + (1-\alpha)(2m - \delta(n-1))}\right) = r = \lambda_1(A_\alpha(G))$. Now, suppose the equality holds. Then, all inequalities in the above argument must be equalities. From Lemma 2.5

$$\sum_{u \nsim v, u \neq v} d(u) = (n - d(v) - 1)\delta,$$

for all $v \in V(G)$. Hence either $d(v) = n-1$ or $d(u) = \delta$, for all $u \in V(G)$ and $u \nsim v$, which implies that either G is a regular graph or G is a bidegreed graph in which each vertex is of degree either δ or $n-1$. As the second one can not occur because of inequality (17), we get the result. □

Example 3.6. Let $G \cong K_{1,6}$ as in Figure 2. The Table 2 presents a comparison of the upper bounds (10) and (15) and concludes that they are incomparable.

Remark 3.7. From Theorem 3.5 we have that the equality in (15) occurs if and only if G is regular. Taking this extremal graph and applying its information in (10) we obtain

$$\alpha^2(2mn - n(n-1)\delta + 2m(\delta-1)) + 2m(1-\alpha)^2 + \delta(\alpha^2\delta + (1-\alpha)^2)(\Delta - n + 1) =$$
$$\alpha^2(r^3 + 2r^2 + r) + (1-2\alpha)(r^2 + r) \geq r^2.$$

Then, the bounds obtained in (10) is bigger than the bound obtained in (15) for regular graphs.

3.2. Bounds for $\lambda_2(A_\alpha(G))$

Proposition 3.8. *Let G be a r-regular graph with n vertices and independence number $\gamma(G)$ and $\alpha \in [0,1]$. Then,*

$$\lambda_2(A_\alpha(G)) \geq \alpha r + (1-\alpha)\left(-1 + \frac{2(n-1-r)^{\gamma(G)}}{\gamma(G) n^{\gamma(G)-1}}\right). \quad (18)$$

Equality holds if $G \cong K_n$.

Proof. From Theorem 2.2 we know that

$$\lambda_2(A_\alpha(G)) = \lambda_2(\alpha D(G) + (1-\alpha)A(G)) \geq \alpha\lambda_n(D(G)) + (1-\alpha)\lambda_2(A(G)) = \alpha r + (1-\alpha)\lambda_2(A(G))$$

and from Lemma 2.13 the result follows.

Now, suppose that $G \cong K_n$. Then $\gamma(G) = 1$, $r = n-1$ and consequently

$$\alpha r + (1-\alpha)\left(-1 + \frac{2(n-1-r)^{\gamma(G)}}{\gamma(G)n^{\gamma(G)-1}}\right) = \alpha n - 1$$

and from Proposition 2.25 the equality holds. □

Proposition 3.9. *Let G be a connected bipartite graph with n vertices, $\widetilde{G}$ be the bipartite complement of G, $K_{p,q}$ $(1 \leq p \leq q)$ be the complete bipartite graph whose partitions are the same as those of G, and $\alpha \in [0,1]$. Then,*

$$q\alpha - \lambda_{3-j}(A_\alpha(\widetilde{G})) \leq \lambda_j(A_\alpha(G)) \leq q\alpha - \lambda_{2-j+n}(A_\alpha(\widetilde{G}))$$

for $j = 1, 2$.

Proof. It is easy to see that $A_\alpha(G) + A_\alpha(\widetilde{G}) = A_\alpha(K_{p,q})$. From Theorem 2.2, it follows that

$$\lambda_j(A_\alpha(G)) + \lambda_{i-j+n}(A_\alpha(\widetilde{G})) \leq \lambda_i(A_\alpha(K_{p,q})), \quad 1 \leq i \leq j \leq n \tag{19}$$

and

$$\lambda_i(A_\alpha(K_{p,q})) \leq \lambda_j(A_\alpha(G)) + \lambda_{i-j+1}(A_\alpha(\widetilde{G})), \quad 1 \leq j \leq i \leq n. \tag{20}$$

From inequalities (19) and (20) we have

$$\lambda_j(A_\alpha(G)) \leq \lambda_i(A_\alpha(K_{p,q})) - \lambda_{i-j+n}(A_\alpha(\widetilde{G})) \tag{21}$$

and

$$\lambda_i(A_\alpha(K_{p,q})) - \lambda_{i-j+1}(A_\alpha(\widetilde{G})) \leq \lambda_j(A_\alpha(G)). \tag{22}$$

From Proposition 2.26 we obtain $\lambda_2(A_\alpha(K_{p,q})) = \alpha q$ and taking $i = 2$ in the inequalities (21) and (22) we have

$$\lambda_j(A_\alpha(G)) \leq q\alpha - \lambda_{2-j+n}(A_\alpha(\widetilde{G}))$$

and

$$q\alpha - \lambda_{3-j}(A_\alpha(\widetilde{G})) \leq \lambda_j(A_\alpha(G)).$$

Then

$$q\alpha - \lambda_{3-j}(A_\alpha(\widetilde{G})) \leq \lambda_j(A_\alpha(G)) \leq q\alpha - \lambda_{2-j+n}(A_\alpha(\widetilde{G})), \quad j = 1, 2.$$

□

Corollary 3.10. *Let G be a connected bipartite and r-regular graph with n vertices. If $\widetilde{G}$ is the bipartite complement of G and $\alpha \in [0,1]$ then*

$$\lambda_2(A_\alpha(G)) \leq \frac{n}{2}(\alpha + 1) - r. \tag{23}$$

Proof. Let G be a connected bipartite and r-regular graph such that $V = V_1 \cup V_2$, where $|V_1| = p$ and $|V_2| = q$ $(p \leq q)$. From Proposition 3.9 we knows that

$$\lambda_j(A_\alpha(G)) \leq q\alpha - \lambda_{2-j+n}(A_\alpha(\widetilde{G})), \quad j = 1, 2.$$

Taking $j = 2$ we have

$$\lambda_2(A_\alpha(G)) \leq q\alpha - \lambda_n(A_\alpha(\widetilde{G}))$$

As G is r-regular bipartite, it follows that $q = \dfrac{n}{2}$. Furthermore, we know that the graph $\widetilde{G}$ is also bipartite and $\left(\dfrac{n}{2} - r\right)$-regular which implies $\lambda_1(A_\alpha(\widetilde{G})) = \dfrac{n}{2} - r$ and $\left(\dfrac{n}{2} - r\right) \geq -\lambda_n(A_\alpha(\widetilde{G}))$. Consequently, the result follows. □

The upper bound obtained by Corollary 3.10 can be improved, as we can see in Proposition 3.11.

Proposition 3.11. *Let G be a r-regular bipartite and connected graph with n vertices. Then*

$$\lambda_2(A_\alpha(G)) \leq \alpha\left(2r - \frac{n}{2}\right) + \frac{n}{2} - r. \tag{24}$$

Proof. Since G is r-regular and bipartite, from Lemma 2.8 and Proposition 2.9, it follows that $\lambda_1(A(G)) = r$ and $\lambda_n(A(G)) = -r$ are eigenvalues of $A(G)$. Moreover, we have that $\widetilde{G}$ is $\left(\dfrac{n}{2} - r\right)$-regular and bipartite, so $\lambda_1(A(\widetilde{G})) = \dfrac{n}{2} - r$ and $\lambda_n(A(\widetilde{G})) = -\dfrac{n}{2} + r$ are eigenvalues of $A(\widetilde{G})$. From Lemmas 2.19 and 2.23 we have $\lambda_k(A(G)) = \lambda_k(A(\widetilde{G})),$ for $2 \leq k \leq n-1$,

$$\lambda_k(A_\alpha(G)) = \alpha r + (1-\alpha)\lambda_k(A(G)), \text{ for } 1 \leq k \leq n \tag{25}$$

and

$$\lambda_k(A_\alpha(\widetilde{G})) = \alpha\left(\frac{n}{2} - r\right) + (1-\alpha)\lambda_k(A(\widetilde{G})), \text{ for } 1 \leq k \leq n. \tag{26}$$

So, $\lambda_1(A_\alpha(G)) = r$, $\lambda_1(A_\alpha(\widetilde{G})) = \dfrac{n}{2} - r$, $\lambda_n(A_\alpha(G)) = r(2\alpha - 1)$ and $\lambda_n(A_\alpha(\widetilde{G})) = \left(\dfrac{n}{2} - r\right)(2\alpha - 1)$.

Subtracting the equations (26) from (25) and using the relation between $\lambda_k(A_\alpha(G))$ and $\lambda_k(A_\alpha(\widetilde{G}))$ for $2 \leq k \leq n-1$ we get

$$\lambda_k(A_\alpha(G)) = \alpha\left(2r - \frac{n}{2}\right) + \lambda_k(A_\alpha(\widetilde{G})) \tag{27}$$

Taking $k = 2$ and considering $\lambda_1(A_\alpha(\widetilde{G})) \geq \lambda_2(A_\alpha(\widetilde{G}))$, we have that

$$\lambda_2(A_\alpha(G)) \leq \alpha\left(2r - \frac{n}{2}\right) + \frac{n}{2} - r = \frac{n}{2}(1-\alpha) + r(2\alpha - 1) \tag{28}$$

and the result follows. □

Remark 3.12. It is worth noting that if G is a regular connected and bipartite graph and the regularity is $\dfrac{n}{2}$, then the upper bounds shown in the equations (23) and (24) are equal.

Proposition 3.13. *Let G be a r-regular graph of order n. Then $\lambda_1(A_\alpha(G)) + \lambda_2(A_\alpha(G)) \leq 2r\alpha + (1-\alpha)(n-2)$. Equality holds if, and only if, G has a connected component that is a bipartite graph.*

Proof. Let G be a r-regular graph of order n. From Lemmas 2.20 and 2.23 we have

$$\lambda_1(A_\alpha(G)) + \lambda_2(A_\alpha(G)) = 2\alpha r + (1-\alpha)(\lambda_1(A(G)) + \lambda_2(A(G))) \leq 2\alpha r + (1-\alpha)(n-2) \tag{29}$$

with equality occuring if and only if G has a connected component that is a bipartite graph and the result follows. □

Proposition 3.14. *Let G be a r-regular bipartite graph of order n and $\alpha \in [0,1)$. Then $\lambda_1(A_\alpha(G)) + \lambda_2(A_\alpha(G)) \leq 2r\alpha + (1-\alpha)\dfrac{n}{2}$. Equality occurs if and only if $\widetilde{G}$ is disconnected.*

Proof. Let G be a r-regular bipartite graph. From Lemma 2.23 we have $\lambda_1(A_\alpha(G)) = r$. Moreover, from Proposition 3.11 we have

$$\lambda_1(A_\alpha(G)) + \lambda_2(A_\alpha(G)) \leq r + \alpha\left(2r - \frac{n}{2}\right) + \frac{n}{2} - r = \alpha\left(2r - \frac{n}{2}\right) + \frac{n}{2} = 2r\alpha + (1-\alpha)\frac{n}{2}.$$

Now, suppose that $\lambda_1(A_\alpha(G)) + \lambda_2(A_\alpha(G)) = 2r\alpha + (1-\alpha)\frac{n}{2}$. As G is r-regular, we have that

$$\lambda_1(A_\alpha(G)) = \alpha r + (1-\alpha)\lambda_1(A(G)) \text{ and } \lambda_2(A_\alpha(G)) = \alpha r + (1-\alpha)\lambda_2(A(G))$$

and consequently

$$\lambda_1(A_\alpha(G)) + \lambda_2(A_\alpha(G)) = 2\alpha r + (1-\alpha)(\lambda_1(A(G)) + \lambda_2(A(G))).$$

So,

$$\lambda_1(A(G)) + \lambda_2(A(G)) = \frac{n}{2}$$

and from Corollary 2.15, it follows that $\widetilde{G}$ is disconnected.

Now, suppose that the $\widetilde{G}$ is disconnected. From Corollary 2.15, $\lambda_1(A(G)) + \lambda_2(A(G)) = \frac{n}{2}$. As G is r-regular we have

$$\lambda_1(A_\alpha(G)) + \lambda_2(A_\alpha(G)) = 2\alpha r + (1-\alpha)(\lambda_1(A(G)) + \lambda_2(A(G))) = 2\alpha r + (1-\alpha)\frac{n}{2}.$$

and the result follows. □

3.3. Bounds for $\lambda_n(A_\alpha(G))$

Proposition 3.15. *Let G be a graph with n vertices, minimum degree $\delta \neq 0$, maximum degree Δ and independence number $\gamma(G)$. Then*

$$\lambda_n(A_\alpha(G)) \leq \alpha\Delta + (1-\alpha)\frac{\gamma(G)\delta^2}{\lambda_1(A(G))(\gamma(G)-n)} \tag{30}$$

In particular, if G is an r-regular graph we have

$$\lambda_n(A_\alpha(G)) \leq \alpha r + (1-\alpha)\frac{\gamma(G)r}{\gamma(G)-n}, \tag{31}$$

whose equality holds if $G \cong K_n$ or, when n is even, $G \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$, or $G \cong K_{\frac{n}{2}} \cup K_{\frac{n}{2}}$.

Proof. From Corollary 2.3 and Lemma 2.16 we obtain the bound in (30) and from Corollary 2.3 and Lemma 2.17 we have the bound in (31). Initially suppose that $G \cong K_n$. Then $\gamma(G) = 1$ and from Proposition 2.25, $\lambda_n(A_\alpha(G)) = \alpha n - 1$. So, $\alpha r + (1-\alpha)\frac{\gamma(G)r}{\gamma(G)-n} = \alpha n - 1$.

Now suppose that n is even and $G \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$. We know that $\gamma(G) = \frac{n}{2}$, $r = 1$, $\lambda_n(A_\alpha(G)) = 2\alpha - 1$ and consequently $\alpha r + (1-\alpha)\frac{\gamma(G)r}{\gamma(G)-n} = 2\alpha - 1$. Finally, suppose that n is even and $G \cong K_{\frac{n}{2}} \cup K_{\frac{n}{2}}$. In this case, $\gamma(G) = 2$, $r = \frac{n}{2} - 1$ and $\lambda_n(A_\alpha(G)) = \frac{n}{2}\alpha - 1$. So $\alpha r + (1-\alpha)\frac{\gamma(G)r}{\gamma(G)-n} = \alpha\frac{n}{2} - 1$, and the result follows. □

Proposition 3.16. *Let G be a triangle-free graph with n vertices and independence number $\gamma(G)$. Then*

$$\lambda_n(A_\alpha(G)) \le \alpha\Delta + (1-\alpha)\frac{\lambda_1^2(A(G))}{\lambda_1(A(G)) - n}$$

Proof. From Theorem 2.2 we have that $\lambda_n(A_\alpha(G)) \le \alpha\Delta + (1-\alpha)\lambda_n(A(G))$ and applying Lemma 2.18 the result follows. ☐

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