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CONTINUITY OF THE SOLUTION SET MAPPINGS TO PARAMETRIC UNIFIED WEAK VECTOR EQUILIBRIUM PROBLEMS *VIA* FREE-DISPOSAL SETS

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Abstract. In this paper, the lower and upper semicontinuity of the solution mappings for parametric unified weak vector equilibrium problem (PUWVEP) are established, *via* free-disposal set and non-linear scalarization function. Moreover, example is given to illustrate the obtained results. The results improve the corresponding ones in the literature [Z.Y. Peng and S.S. Chang, *Optim. Lett.* (2014) **8** 159–169.], [Z.Y. Peng, X.B. Li, X.J. Long and X.D. Fan, *Optim. Lett.* (2018) **12** 1339–1356.], [Z.Y. Peng, J.W. Peng, X.J. Long and J.C. Yao, *J. Global Optim.* **70** (2018) 55–69.].

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1. Introduction

It is well know that vector equilibrium problem provides a unified model for several classes problems, such as the vector variational inequality problem, the vector complementarity problem, the vector optimization problem and the vector saddle point problem, and so on. Moreover, the stability analysis of the solution for vector equilibrium problems play an important role in the vector optimization theory. As a significant aspect of stability analysis, semicontinuity of the solution mappings for vector equilibrium problems is considerable interest. In the development the semicontinuity of solution mappings for vector equilibrium problems, the method of scalarization (for more details, see [2, 3, 5–8, 15]) are proved to be useful tools. Under new assumptions by using linear method, Peng and cheng [12] investigated weak vector solutions and global vector solution to a parametric generalized system, by employing scalarization method. Wang and Li [16] studied a projection iterative algorithm for strong vector equilibrium problem, by using nonlinear scalarization function ξ. In 2013, Wangkeeree *et al.* [17] established the continuity of efficient solution mappings to a parametric generalized strong vector equilibrium problem, depending on the Hölder relation. Under a new assumption, Zhang *et al.* [18] established the lower semicontinuity of the efficient solution mappings for parametric vector equilibrium problems without amonotonicity assumption. Recently, Luo *et al.* [11] obtained the continuity of the solution

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set map to parametric weak vector equilibrium problems, due to the nonlinear scalarization function ξ. Han and Huang [10] discussed some characterizations of the approximate solutions to generalized vector equilibrium problems. Other studies on vector optimization problems are described in [13, 14] and so on.

Motivated by the results in [8, 9, 11, 16, 18], in this paper, we study the lower and upper semicontinuity of the solution mappings for parametric unified weak vector equilibrium problem (PUWVEP) *via* nonlinear scalarization function $\varphi_{D,e}$. Moreover, the correctness of main result is verified by an example. Our results improve the corresponding ones in [11, 16–18].

2. Preliminaries

Throughout this paper, let X and Z be metric spaces, Y be a linear metric spaces, Y^* be topological dual spaces of Y, we also assume that C be a pointed closed convex cone in Y with nonempty topological interior $\mathrm{int}C$ and C^* be the dual cone of C, *i.e.*,

$$C^* = \{\xi \in Y^* | \langle \xi, c\rangle \geq 0, \forall c \in C\}.$$

Let A be a nonempty subset of Y, the cone hull of A is defined as

$$\mathrm{cone}(A) = \{ta : t \geq 0, a \in A\}.$$

Denote the closure A of by $\mathrm{cl}(A)$ and the interior of A by $\mathrm{int}(A)$. A nonempty convex subset B of the convex cone C is called a base of C if $C = \mathrm{cone}(B)$ and $0 \notin \mathrm{cl}(B)$. Given a point $e \in \mathrm{int}C$.

$$B_e^* = \{\xi \in C^* \backslash \{0\} \,|\, \langle \xi, e\rangle = 1\},$$

then B_e^* is a weak*-compact base of C^*.

Let $D \subseteq Y$ be a nonempty proper ($i.e.$, $D \neq Y$) set with nonempty interior $\mathrm{int}D$ and $k^0 \in Y \backslash \{0_Y\}$.

Definition 1. ([6–8]) The nonlinear scalarization function $\varphi_{D,k^0}(\cdot) : Y \to R \cup \{\pm\infty\}$ is defined by

$$\varphi_{D,k^0}(y) = \inf\{t \in R \,|\, y \in tk^0 - D\}.$$

In the special case of $D = C$ and $k^0 \in \mathrm{int}C, \varphi_{C,k^0} : Y \to R$ is a finite-valued, Lipschitz continuous, sublinear and convex function, and it is $C-$monotone ($i.e.$, $y^2 - y^1 \in C \Rightarrow \varphi_{C,k^0}(y^1) \leq \varphi_{C,k^0}(y^2)$) and strictly $C-$monotone ($i.e.$, $y^2 - y^1 \in \mathrm{int}C \Rightarrow \varphi_{C,k^0}(y^1) < \varphi_{C,k^0}(y^2)$) (for more detail see [2, 5, 8]).

We shall use the following concepts proposed in [8] and [15], respectively:

Assumption (P): D is proper closed, and satisfies the free-disposal assumption $D + C = D$;

Assumption (P_w): D is proper closed, and satisfies the weak free-disposal assumption $D + C \subseteq D$.

The following properties of φ come from Lemma 2.1 of [3], Theorem 2.3.1 and Proposition 2.3.4 of [8], and Theorem 3.1 of [15].

Proposition 1. *Let $k^0 \in \mathrm{int}C$. If assumption (P_w) holds, then $\varphi := \varphi_{D,k^0}$ is finite-valued and continuous. Moreover,*

(i) $\{y \in Y | \varphi(y) \leq r\} = rk^0 - D, \forall r \in R$;
(ii) $\{y \in Y | \varphi(y) < r\} = rk^0 - \mathrm{int}D, \forall r \in R$;
(iii) $\{y \in Y | \varphi(y) = r\} = rk^0 - \mathrm{bd}D, \forall r \in R$, *where* $\mathrm{bd}D$ *denotes the boundary of* D;
(iv) $\varphi(y + rk^0) = \varphi(y) + r, \forall y \in Y, \forall r \in R$ *(translation property)*;
(v) φ *is convex if and only if* D *is convex*;
(vi) $\forall y \in Y, \forall r \in R, \varphi(ry) = r\varphi(y)$ *if and only if* D *is a cone*;
(vii) φ *is subadditive if and only if* $D + D \subseteq D$.

Definition 2. ([4],[9]) A nonempty proper set $D \subseteq Y$ is said to be a free-disposal set with respect to C if $D + C = D$.

Remark 1. It is clear that if $D \subseteq Y$ is a proper closed free-disposal set with respect to C and $k^0 \in \mathrm{int}C$, then all properties of the function $\varphi := \varphi_{D,k^0}$ in Proposition 1 still hold.

Suppose that $A \subseteq X$, let $f : A \times A \to Y$ be a vector-valued mapping. In the rest of this paper, we always suppose that $D \subseteq Y$ is a closed free-disposal set with respect to C. Now, we consider the follows weak vector equilibrium problem (WVEP) *via* free-disposal set D of finding $\overline{x} \in A$ such that

$$f(\overline{x}, y) \notin -\mathrm{int}D, \forall y \in A.$$

When the set A and the mapping f are perturbed by μ which varies over a set Λ of Z, we have the follows: parametric unified weak vector equilibrium problem (PUWVEP) *via* free-disposal set D of finding $\overline{x} \in A(\mu)$ such that

$$f(\overline{x}, y, \mu) \notin -\mathrm{int}D,\ \forall y \in A(\mu).$$

Where $A(\cdot) : \Lambda \to 2^X \backslash \{\varnothing\}$ is a set-valued mapping, $f : B \times B \times \Lambda \subseteq X \times X \times Z \to Y$ be a vector-valued mapping, moreover $A(\Lambda) = \cup_{\mu \in \Lambda} A(\mu) \subseteq B$.

For each $\mu \in \Lambda$, let $S(\mu)$ denote the solution set of (PUWVEP), *i.e.*,

$$S(\mu) = \{x \in A(\mu) : f(\overline{x}, y, \mu) \notin -\mathrm{int}D,\ \forall y \in A(\mu)\}.$$

Obviously, for any fixed $k^0 \in \mathrm{int}C$, from Proposition 1(ii), we have

$$S(\mu) = \{x \in A(\mu) : \varphi(f(\overline{x}, y, \mu)) \geq 0,\ \forall y \in A(\mu)\}.$$

The following problems are special cases of (PUWVEP).

(i) If the free-disposal set $D = C$, then the problem (PUWVEP) is parametric weak vector equilibrium problem (for short, (PWVEP)) of finding $x \in A(\mu)$ such that

$$f(x, y, \mu) \notin -\mathrm{int}C,\ \forall y \in A(\mu).$$

It was investigated by Luo [11] and Peng [12].

(ii) If the free-disposal set $D = C$, $f(x, y, u) = f(x, y)$ for any $(x, y, \mu) \in X \times X \times Z$, the (PUWVEP) reduces to the weak vector equilibrium problem (WVEP) which was considered in [15].

Throughout the rest of this paper, we assume that $S(\mu) \neq \phi$ for any $\mu \in \Lambda$, and for $\mu \in \Lambda, x \in A(\mu)$, let $f(x, A(\mu), \mu) = \{f(x, y, \mu) : y \in A(\mu)\}$. In this paper, by using the Gerstewitz (Tammer) nonlinear scalarization function φ_{D,k^0}, we will discuss the lower and upper semicontinuity of the solution mapping from Λ to X. Now we recall some basic definitions and their properties.

Definition 3. ([1]) Let X and Y be topological spaces, $T : X \to 2^Y$ be a set-valued mapping.

(i) T is said to be upper semicontinuous (u.s.c., for short) at $x_0 \in X$ iff for any open set V containing $T(x_0)$, there exists an open set U containing x_0 such that $T(x) \subseteq V$ for all $x \in U$.

(ii) T is said to be lower semicontinuous (l.s.c., for short) at $x_0 \in X$ iff for any open set V with $T(x_0) \cap V \neq \phi$, there exists an open set U containing x_0 such that $T(x) \cap V \neq \phi$ for all $x \in U$.

T is said to be continuous at $x_0 \in X$, if it is both l.s.c. and u.s.c. at $x_0 \in X$. T is said to be l.s.c. (resp. u.s.c.) on X, iff it is l.s.c. (resp. u.s.c.) at each $x \in X$.

Proposition 2. *([1]) Let X and Y be topological spaces, $T : X \to 2^Y$ be a set-valued mapping.*

(i) T is l.s.c. at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x_0$ and any $y_0 \in T(x_0)$, there exists $y_\alpha \in T(x_\alpha)$ such that $y_\alpha \to y_0$.

(ii) If T has compact values (i.e., $T(x)$ is a compact set for each $x \in X$), then T is u.s.c. at x_0 if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x_0$ and for any $y_\alpha \in T(x_\alpha)$, there exist $y_0 \in T(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$, such that $y_\beta \to y_0$.

3. The continuity of solution set mappings for (PUWVEP)

Lemma 1. *Let $k^0 \in \text{int}C$ be any given point. Let $f : X \to Y$ be a vector-valued mapping.*

(i) If f is l.s.c., then $\varphi_{D,k^0} \circ f$ is l.s.c.;
(ii) If f is u.s.c., then $\varphi_{D,k^0} \circ f$ is u.s.c.;
(iii) If f is continuous, then $\varphi_{D,k^0} \circ f$ is continuous.

Proof. (i) Take any $r \in R$ and let r be fixed. Let

$$L = \{x \in X : \varphi_{D,k^0}(f(x)) \leq r\}.$$

We only need to show that L is closed in X.

In fact, let $\{x_\alpha\} \subseteq L$ be any net with $x_\alpha \to x_0 \in X$. We shall prove that $x_0 \in L$. Suppose to the contrary that $x_0 \notin L$, then $\varphi_{D,k^0}(f(x_0)) > r$. It follows from Proposition 1(i) that $f(x_0) \notin rk^0 - D$. And then, by the closedness of D, there exists a neighborhood V of 0 such that

$$[f(x_0) + V] \cap [rk^0 - D] = \phi. \tag{1}$$

On the other hand, since f is l.s.c. at x_0, then for the net $x_\alpha \to x_0$, we have $f(x_\alpha) \to f(x_0)$. This fact, together with (1), leads to that there exists α_0, such that

$$f(x_{\alpha_0}) \notin rk^0 - D, \, \forall \alpha \geq \alpha_0. \tag{2}$$

It follows from Proposition 1(i) that

$$\varphi_{D,k^0}(f(x_\alpha)) > r, \, \forall \alpha \geq \alpha_0.$$

This contradicts that $\{x_\alpha\} \subseteq L$. Thus, $x_0 \in L$.

(ii) Take any $r \in R$ and let r be fixed. Let

$$L = \{x \in X : \varphi_{D,k^0}(f(x)) \geq r\}.$$

We only need to show that L is closed in X.

In fact, let $\{x_\alpha\} \subseteq L$ be any net with $x_\alpha \to x_0 \in X$. we shall prove that $x_0 \in L$. Suppose to the contrary that $x_0 \notin L$, then $\varphi_{D,k^0}(f(x_0)) < r$. It follows from Proposition 1(ii) that $f(x_0) \in rk^0 - \text{int}D$, *i.e.*,

$$0 \in rk^0 - f(x_0) - \text{int}D.$$

which implies that $rk^0 - f(x_0) - \text{int}D$ is a neighborhood of 0 in Y. Since f is u.s.c., there exists a neighborhood U of x_0 in X such that

$$f(x) \in rk^0 - \text{int}D, \, \forall x \in U \cap X.$$

It follows from Proposition 1(ii) that

$$\varphi_{D,k^0}(f(x)) < r, \, \forall x \in U \cap X. \tag{3}$$

On the other hand, since $x_\alpha \to x_0$, there exists some α_0 such that $x_\alpha \in U \cap X$, $\forall \alpha \geq \alpha_0$. This fact together with (3) yields

$$\varphi_{D,k^0}(f(x_\alpha)) < r, \, \forall \alpha \geq \alpha_0.$$

This contradicts that $\{x_\alpha\} \subseteq L$. Thus, $x_0 \in L$.

(iii) Since f is continuous, it is both l.s.c. and u.s.c. Then, by (i) and (ii), we know that the real-valued function $\varphi_{D,k^0} \circ f$ is l.s.c. and u.s.c. at the same time, and so is continuous. This completes the proof. □

Theorem 1. *(l.s.c of solution set mappings) Suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with nonempty compact values on* Λ*;*
(ii) $f(\cdot,\cdot,\cdot)$ *is l.s.c. on* $B\times B\times\Lambda$*;*
(iii) For each $\mu\in\Lambda, x\in A(\mu)\backslash S(\mu)$*, there exists* $y\in S(\mu)$ *such that.*

$$d(x,y)\leq d(\varphi_{D,k^0}(f(x,y,\mu)),R_+).$$

Then, $S(\cdot)$ *is l.s.c. on* Λ*.*

Proof. Suppose to the contrary that there exists $\mu_0\in\Lambda$ such that $S(\cdot)$ is not l.s.c. at μ_0. Then, there exist a sequence $\{\mu_n\}$ with $\mu_n\to\mu_0$ and $x_0\in S(\mu_0)$ such that for any $x_n\in S(\mu_n), x_n\nrightarrow x_0$. $x_0\in S(\mu_0)$ implies that $x_0\in A(\mu_0)$, from Proposition 1(ii), we have

$$\varphi_{D,k^0}(f(x_0,y,\mu_0))\geq 0,\ \forall y\in A(\mu_0). \tag{4}$$

□

Since $A(\cdot)$ is l.s.c. at μ_0, there exists a sequence $\{\overline{x}_n\}\subseteq A(\mu)$, such that $\overline{x}_n\to x_0$. Obviously, $\overline{x}_n\in A(\mu_n)\backslash S(\mu_n)$. We derive from (iii) that there exists $y_n\in S(\mu_n)$, such that

$$d(\overline{x}_n,y_n)\leq d(\varphi_{D,k^0}(f(\overline{x}_n,y_n,\mu_n)),R_+). \tag{5}$$

Since $y_n\in A(\mu_n)$, it follows from the upper semicontinuity and compactness of $A(\cdot)$ at μ_0 that, there exist $y_0\in A(\mu_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k}\to y_0$. In particular, by (5), it holds that

$$d(\overline{x}_{n_k},y_{n_k})\leq d(\varphi_{D,k^0}(f(\overline{x}_{n_k},y_{n_k},\mu_{n_k})),R_+). \tag{6}$$

The fact that $f(\cdot,\cdot,\cdot)$ is l.s.c. on $B\times B\times\Lambda$, together with Lemma 1(i), yields that $\varphi_{D,k^0}\circ f$ is l.s.c. on $B\times B\times\Lambda$. Then for $(\overline{x}_{n_k},y_{n_k},\mu_{n_k})\to(x_0,y_0,\mu_0)$, we have

$$\varphi_{D,k^0}(f(x_0,y_0,\mu_0))\leq\lim\inf_{k\to\infty}\varphi_{D,k^0}(f(\overline{x}_{n_k},y_{n_k},\mu_{n_k})).$$

Take the limit as $k\to+\infty$ on both sides of (6), by continuity of the distance function $d(\cdot,\cdot)$, we have

$$d(x_0,y_0)\leq d(\varphi_{D,k^0}(f(x_0,y_0,\mu_0)),R_+). \tag{7}$$

Assume that $x_0\neq y_0$, then (7) implies that

$$0<d(x_0,y_0)\leq d(\varphi_{D,k^0}(f(x_0,y_0,\mu_0)),R_+).$$

Which contradicts with (4). Hence, we have $x_0=y_0$. This is impossible by the contradiction assumption. Thus, we have $S(\cdot)$ is l.s.c. on Λ. The proof is completed.

Theorem 2. *(u.s.c of solution set mappings) Suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with nonempty compact values on* Λ*;*
(ii) $f(\cdot,\cdot,\cdot)$ *is u.s.c. on* $B\times B\times\Lambda$*;*
(iii) For each $\mu\in\Lambda, x\in A(\mu)\backslash S(\mu)$*, there exists* $y\in S(\mu)$ *such that.*

$$d(x,y)\leq d(\varphi_{D,k^0}(f(x,y,\mu)),R_+).$$

Then, $S(\cdot)$ *is u.s.c. on* Λ*.*

Proof. Suppose, by contradiction, that $S(\cdot)$ is not u.s.c. at $\mu_0 \in \Lambda$. Then there exists a open set W_0 containing $S(\mu_0)$, for any open set $U(\mu_0)$ containing μ_0, there exists $\mu' \in U(\mu_0)$ such that $S(\mu') \nsubseteq W_0$. Hence, there exists a sequence $\{\mu_n\}$ with $\mu_n \to \mu_0$ such that

$$S(\mu_n) \nsubseteq W_0, \ \forall n \in N.$$

Consequently, there exists $x_n \in S(\mu_n)$, such that

$$x_n \notin W_0, \ \forall n \in N.$$

□

Since $x_n \in S(\mu_n) \subseteq A(\mu_n)$, and $A(\cdot)$ is u.s.c. at μ_0 with compact values, then there exist $x_0 \in A(\mu_0)$ and subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$.

In the sequel, we proof that $x_0 \in S(\mu_0)$. By the way of contradiction, we assume that $x_0 \notin S(\mu_0)$. It follows from (iii) that there exist $y_0 \in S(\mu_0)$ such that

$$d(x_0, y_0) \leq d(\varphi_{D,k^0}(f(x_0, y_0, \mu_0)), R_+).$$

Since $A(\cdot)$ is l.s.c. at μ_0 and $y_0 \in A(\mu_0)$, there exists a sequence $y_n \in A(\mu_n)$, such that $y_n \to y_0$. Note that $x_n \in S(\mu_n)$ and $y_n \in A(\mu_n)$, then

$$\varphi_{D,k^0}(f(x_n, y_n, \mu_n)) \geq 0.$$

Especially, we have

$$\varphi_{D,k^0}(f(x_{n_k}, y_{n_k}, \mu_{n_k})) \geq 0. \tag{8}$$

By $f(\cdot,\cdot,\cdot)$ is u.s.c. at (x_0, y_0, μ_0) and $(x_{n_k}, y_{n_k}, \mu_{n_k}) \to (x_0, y_0, \mu_0)$, we obtain that $\varphi_{D,k^0} \circ f$ is u.s.c. at (x_0, y_0, μ_0). Which implies that

$$\limsup_{k\to\infty} \varphi_{D,k^0}(f(x_{n_k}, y_{n_k}, \mu_{n_k})) \leq \varphi_{D,k^0}(f(x_0, y_0, \mu_0)).$$

Thus, as $k \to +\infty$, we have

$$d(x_0, y_0) \leq d(\varphi_{D,k^0}(f(x_0, y_0, \mu_0)), R_+) \leq d(\varphi_{D,k^0}(f(x_{n_k}, y_{n_k}, \mu_{n_k})), R_+). \tag{9}$$

we derive from (8) and (9) that $y_0 = x_0$, which is a contradiction. Therefore, $x_0 \in S(\mu_0) \subseteq W_0$.

On the other hand, combining $x_n \notin W_0$ and the openness of W_0 yields $x_0 \notin W_0$. Which is a contradiction. Hence, $S(\cdot)$ is u.s.c. on Λ. This completes the proof.

Remark 2. under a general order set(*i.e.*, the free-disposal set), by using nonlinear scalarization technic(the nonlinear scalarization Function $\varphi_{D,k^0}(\cdot)$), we obtain the sufficient condition of for the u.s.c and l.s.c solution set mappings S($\cdot$) to (PUWVEP) respectively.

By virtue of Theorems 1 and 2, one can establish the following corollary directly.

Corollary 1. *Suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with nonempty compact values on* Λ*;*
(ii) $f(\cdot,\cdot,\cdot)$ *is continuous on* $B \times B \times \Lambda$*;*
(iii) For each $\mu \in \Lambda, x \in A(\mu)\backslash S(\mu)$*, there exists* $y \in S(\mu)$ *such that.*

$$d(x, y) \leq d(\varphi_{D,k^0}(f(x, y, \mu)),\ R_+).$$

Then, $S(\cdot)$ *is continuous on* Λ*.*

The following example illustrates the assumption (iii) in Corollary 1 is indispensable.

Example 1. Let $X = Z = R, Y = R^2, C = R^2_+, D = [1, +\infty) \times [1, +\infty), \Lambda = [0, 1], A(\mu) = [0, 1], k^0 = (1, 1) \in \text{int}C = R^2_+$, and $f(x, y, \mu) = (-4, \mu(y - x))$.
It easy to verify that D is an free-disposal set with respect to C. By direct compute, we have

$$S(\mu) = \begin{cases} [0,1], & \mu = 0; \\ \{0\}, & \mu \in (0,1]. \end{cases}$$

Obviously, $S(\cdot)$ is not l.s.c at $\mu = 0$. The reason is that the assumption (iii) in Corollary 1 dose not holds. In fact, for any $\mu \in (0, 1)$, there exists $x = \frac{2}{3} \in A(\mu)\backslash S(\mu) = (0, 1]$, such that $y = 0 \in S(\mu)$, we obtain

$$d(x, y) = \frac{2}{3}, \text{and}\, d(\varphi_{D,k^0}(f(x, y, \mu)), R_+) = \frac{2}{3}\mu.$$

which implies that $d(x, y) = \frac{2}{3} > d(\varphi_{D,k^0}(f(x, y, \mu)), R_+) = \frac{2}{3}\mu$. Clearly, assumption (iii) not holds, therefore, assumption (iii) in Corollary 1 is indispensable.

4. Conclusion

By using the technique of nonlinear scalarization, under a new kind of hypothesis, we obtain the sufficient conditions of u.s.c and l.s.c for the solution set mappings s(.) to (PUWVEP), our results are different from the exist ones in [11–13, 18]. In addition, whether we can obtain the result of the density and connectivity of the solution set mappings for (PUWVEP) is another interesting subject, which is important problem and needs to be solved.

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