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HIGHER-ORDER OPTIMALITY CONDITIONS WITH SEPARATED DERIVATIVES AND SENSITIVITY ANALYSIS FOR SET-VALUED OPTIMIZATION

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Abstract. In this paper, we establish optimality conditions and sensitivity analysis of set-valued optimization problems in terms of higher-order radial derivatives. First, we obtain the optimality conditions with separated derivatives for a set-valued optimization problem, here separated derivatives means the derivatives of objective and constraint functions are different. Then, some duality theorems for a mixed type of primal-dual set-valued optimization problem are gained. Finally, several results concerning higher-order sensitivity analysis are presented. The main results of this paper are illustrated by some concrete examples.

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1. Introduction

Recently, set-valued problems have been received much attention by many researchers [1–6, 8–11, 13–15, 17–20, 22, 23, 25, 26]. It is well known that derivative is a significant tool in the study of optimization problems, so various concepts of generalized derivatives have been introduced, such as tangent derivative [10], contingent derivative [10], adjacent derivative [10], Studniarski derivative [1] and radial derivative [18]. In addition, in order to obtain more information about the optimal solutions, higher-order derivatives have become one of the core contents of set-valued optimization. Particularly, the so-called higher-order radial derivatives [5] have been extensively studied because of its weaker existence conditions and independent of lower-order derivatives. For example, Anh *et al.* [5] utilized these derivatives to gain optimality conditions for various efficient solutions. Yu [25] established optimality conditions under both vector and set order criterions. Anh [2] discussed a mixed type of primal-dual set-valued optimization problem. It is noteworthy that the derivatives of the objective and constraint functions of the above optimality conditions and duality theorems are not separated, and they are presented in the form of a cross product. In fact, in the classic KKT conditions, the derivatives of the objective and constraint functions are separated, even different. Therefore, it is a meaningful work to establish higher-order optimality of the derivative separation form for set-valued optimization problems. Very recently, Peng *et al.* [13] introduced a second-order tangent epiderivative, and used it to investigate the optimality theorems

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with separated derivatives. In [14], Peng *et al.* proposed a kind of higher-order weak lower epiderivative, and developed higher-order optimality conditions with separated derivatives. One aim of the paper is to discuss the optimality and duality with separated derivatives to a set-valued optimization problem in terms of the higher-order radial derivatives.

On the other hand, sensitivity analysis provides information about derivatives of the perturbation map in case of an optimization problem being perturbed. Recently, by means of various higher-order generalized derivatives, many results for higher-order sensitivity analysis have been obtained. Wang and Li [21] established higher-order sensitivity analysis of a nonconvex vector optimization by utilizing the higher-order adjacent derivatives. Anh [3] applied higher-order Studniarski derivatives to gain sensitivity results of a constrained set-valued optimization problem. Tung and Bao [20] introduced a kind of higher-order set-valued Hadamard directional derivative, and used it to study higher-order sensitivity analysis of the solution mapping for a parametric vector equilibrium problem. However, as far as we know, there are a few results on sensitivity analysis in terms of the higher-order radial derivatives. Moreover, the above mentioned derivatives only incorporate local information. Thus, in order to reflect global information, another aim of our paper is to establish the quantitative information on the behavior of the weakly perturbation and Benson proper perturbation maps in set-valued optimization by using the higher-order radial derivatives.

The structure of this paper is arranged as follows. Section 2 gives several notations, definitions and lemmas, which will be used in later sections. Then, we mention their applications to optimality conditions, duality and sensitivity analysis. In detail, Section 3 establishes the necessary optimality conditions for the separation type of higher-order radial derivatives for weakly efficient solution to a set-valued optimization problem, and the corresponding sufficient optimality conditions for Benson proper efficient solution are given. Section 4 obtains some duality theorems for a mixed type of primal-dual set-valued optimization problem. Finally, we obtain the relationships between the higher-order radial derivatives of a given map and that of the weakly perturbation, Benson proper perturbation maps in Section 5.

2. Preliminaries

Throughout this paper, without specified otherwise, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^n$ be the sets of the natural numbers, real numbers and n-dimensional Euclidean space, respectively. Let X, Y, Z be real normed spaces, X^* denote the topological dual space of X, $A \subset X$ be a nonempty subset, the interior, closure and cone hull of the set A be denoted by $\mathrm{int}A$, $\mathrm{cl}A$ and $\mathrm{cone}A$, respectively. The zero element of any space is denoted by 0. $Q \subset Y$ and $P \subset Z$ are closed convex pointed cones with $\mathrm{int}Q \neq \emptyset$ and $\mathrm{int}P \neq \emptyset$. The dual cone of Q is

$$Q^* := \left\{\omega^* \in Y^* : \omega^{*T} q \geq 0, \forall q \in Q\right\}.$$

Let $J : X \to 2^Y$ be a set-valued map. The domain, image, graph and epigraph of J are defined as, respectively,

$$\mathrm{dom}J := \{v \in X : J(v) \neq \emptyset\},$$
$$\mathrm{gr}J := \{(v, m) \in X \times Y : m \in J(v), v \in X\}$$

and

$$\mathrm{epi}J := \{(v, m) \in X \times Y : m \in J(v) + Q\}.$$

Let $\hat{m} \in Y$. Denote

$$J_+(\cdot) := J(\cdot) + Q,\ \left(J - \hat{m}\right)(v) := J(v) - \hat{m},\ J(A) := \bigcup_{v \in A} J(v).$$

Definition 2.1. [12] Let $A \subset X$ be a convex subset, $J : A \to 2^Y$. J is said to be Q-convex on A if for all $v, m \in A$ and $\alpha \in [0, 1]$,

$$\alpha J(v) + (1 - \alpha)J(m) \subset J(\alpha v + (1 - \alpha)m) + Q.$$

Definition 2.2. [24] Let $A \subset X$, $J : A \to 2^Y$. J is said to be nearly Q-subconvexlike on A, if clcone$(J(A)+Q)$ is convex.

Remark 2.3. If J is Q-convex on a convex subset A, then J is nearly Q-subconvexlike on A. But the converse is not necessarily true, here is an example to illustrate the case.

Example 2.4. Let $X = Y = \mathbb{R}, A = [0,3], Q = \mathbb{R}_+, J : A \to 2^Y$ be defined by $J(v) = \{m \in Y : m \geq v^{\frac{2}{3}}\}$ for all $v \in A$. It is obvious that clcone$(J(A)+Q)$ is convex, but J is not Q-convex on A.

Lemma 2.5. [24] Let $A \subset X$, $J : A \to 2^Y$. If J is nearly Q-subconvexlike on A, then there is one and only one of the following conclusions:
(i) $\exists v \in A$ such that $J(v) \cap (-\text{int}Q) \neq \emptyset$;
(ii) $\exists m^* \in Q^* \backslash \{0\}$ such that $m^{*T} m \geq 0, \forall m \in J(v), \forall v \in A$.

Definition 2.6. Let $\Theta \subset Y$ be a nonempty subset and $\hat{m} \in \Theta$.
(i) $\hat{m}$ is called an efficient point of Θ ($\hat{m} \in \text{Min}_Q\Theta$) if

$$(\Theta - \hat{m}) \cap (-Q\backslash\{0\}) = \emptyset.$$

(ii) $\hat{m}$ is called a weakly efficient point of Θ ($\hat{m} \in \text{WMin}_Q\Theta$) if

$$(\Theta - \hat{m}) \cap (-\text{int}Q) = \emptyset.$$

(iii) $\hat{m}$ is called a Benson proper efficient point of Θ ($\hat{m} \in \text{BPMin}_Q\Theta$) if

$$\text{clcone}(\Theta + Q - \hat{m}) \cap (-Q\backslash\{0\}) = \emptyset.$$

Remark 2.7. By the definitions, it is obvious that

$$\text{BPMin}_Q\Theta \subset \text{Min}_Q\Theta \subset \text{WMin}_Q\Theta.$$

However, the reverse inclusions may not hold by Examples 2.2 and 2.3 in [19].

Definition 2.8. [18] Let $A \subset X$, $x_0 \in \text{cl}A$. The radial cone $R(A, x_0)$ of A at x_0 is defined by

$$R(A, x_0) = \{v \in X : \exists t_n > 0, v_n \to v, x_0 + t_n v_n \in A\}.$$

We now recall the higher-order radial derivatives of set-valued maps, and always assume $l \in \mathbb{N}$.

Definition 2.9. [4,5] Let $J : X \to 2^Y$ be a set-valued map, $(\hat{v}, \hat{m}) \in \text{gr}J$, $v \in X$ and $l \in \mathbb{N}$.
(i) The higher-order outer radial derivative of J at $(\hat{v}, \hat{m})$ is defined as

$$\overline{D}^l_R J(\hat{v}, \hat{m})(v) = \left\{m \in Y : \exists t_n > 0, (v_n, m_n) \to (v, m), \hat{m} + t^l_n m_n \in J\left(\hat{v} + t_n v_n\right)\right\}.$$

(ii) The higher-order inner radial derivative of J at $(\hat{v}, \hat{m})$ is defined as

$$\underline{D}^l_R J(\hat{v}, \hat{m})(v) = \{m \in Y : \forall t_n > 0, v_n \to v, \exists m_n \to m, \hat{m} + t^l_n m_n \in J(\hat{v} + t_n v_n)\}.$$

(iii) The higher-order weak inner radial derivative of J at $(\hat{v}, \hat{m})$ is defined as

$$\underline{D}^l_{Rw} J(\hat{v}, \hat{m})(v) = \{m \in Y : \forall t_n > 0, \exists (v_n, m_n) \to (v, m), \hat{m} + t^l_n m_n \in J(\hat{v} + t_n v_n)\}.$$

According to Definition 2.9, it is obvious that

$$\underline{D}^l_R J(\hat{v}, \hat{m})(v) \subset \underline{D}^l_{Rw} J(\hat{v}, \hat{m})(v) \subset \overline{D}^l_R J(\hat{v}, \hat{m})(v). \tag{1}$$

However, the inverse conclusion of (1) may not hold as shown in the following examples.

Example 2.10. Consider $J : \mathbb{R} \to 2^{\mathbb{R}}$ given by

$$J(v) = \begin{cases} \{m \in Y : m \geq 0\}, & \text{if } v \geq 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Take $(\hat{v}, \hat{m}) = (0, 0)$. By calculating, we get

$$\underline{D}^l_{Rw} J(\hat{v}, \hat{m})(0) = \begin{cases} \{m \in Y : m \geq 0\}, & \text{if } v \geq 0, \\ \emptyset, & \text{otherwise} \end{cases}$$

and $\underline{D}^l_R J(\hat{v}, \hat{m})(0) = \emptyset$. Thus, $\underline{D}^l_{Rw} J(\hat{v}, \hat{m})(v) \not\subset \underline{D}^l_R J(\hat{v}, \hat{m})(v)$.

Example 2.11. Consider $J : \mathbb{R} \to 2^{\mathbb{R}^2}$ given by

$$J(v) = \begin{cases} \left\{(m_1, m_2) \in \mathbb{R}^2 : m_1 \geq v^4, m_2 \geq 0\right\}, & \text{if } v \geq 0, \\ \left\{(m_1, m_2) \in \mathbb{R}^2 : m_1 \geq 0, m_2 \leq v^4\right\}, & \text{otherwise.} \end{cases}$$

Take $l = 4, \hat{v} = 0, \hat{m} = (0, 0)$. By a simple calculation, we have

$$\begin{aligned} \overline{D}^4_R J(\hat{v}, \hat{m})(0) &= \left\{(m_1, m_2) \in \mathbb{R}^2 : m_1 \geq 0, m_2 \in \mathbb{R}\right\}, \\ \underline{D}^4_R J(\hat{v}, \hat{m})(0) &= \left\{(m_1, m_2) \in \mathbb{R}^2 : m_1 \geq 0, m_2 = 0\right\}. \end{aligned}$$

Obviously, $\overline{D}^4_R J(\hat{v}, \hat{m})(v) \not\subset \underline{D}^4_R J(\hat{v}, \hat{m})(v)$.

Example 2.12. Consider $J : \mathbb{R} \to 2^{\mathbb{R}}$ given by

$$J(v) = \begin{cases} \{m \in \mathbb{R} : m \leq 2v\}, & \text{if } v \leq 0, \\ -v, & \text{if } v = \frac{1}{n}, (n \in \mathbb{N}), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Take $(\hat{v}, \hat{m}) = (0, 0)$. By directly calculating, one has

$$\overline{D}^1_R J\left(\hat{v}, \hat{m}\right)(v) = \begin{cases} \{m \in \mathbb{R} : m \leq 2v\}, & \text{if } v \leq 0, \\ \{m \in \mathbb{R} : m = -v\}, & \text{otherwise.} \end{cases}$$

and

$$\underline{D}^1_{Rw} J\left(\hat{v}, \hat{m}\right)(v) = \begin{cases} \{m \in \mathbb{R} : m \leq 2v\}, & \text{if } v \leq 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Obviously, when $v > 0$, we have

$$\overline{D}^1_R J\left(\hat{v}, \hat{m}\right)(v) \not\subset \underline{D}^1_{Rw} J\left(\hat{v}, \hat{m}\right)(v).$$

Lemma 2.13. [5] Let $J : X \to 2^Y$ and $(\hat{v}, \hat{m}) \in \mathrm{gr}J$. Then for all $v \in X$, one has

$$J(v) - \hat{m} \subset \overline{D}^l_R J(\hat{v}, \hat{m})(v - \hat{v}).$$

Remark 2.14. Using the property in Lemma 2.13, it is quite convenient to establish sufficient optimality conditions through the higher-order outer radial derivative. An important point to mention is that convexity assumptions are not required for the higher-order outer radial derivative to be effective in the process of establishing optimality conditions. However, other kinds of higher-order derivatives may not possess this property, such as the higher-order generalized contingent epiderivative, higher-order weakly contingent epiderivative, etc (see [6, 11]) enjoy this property under cone-convexity condition.

3. Higher-order optimality conditions

Let $J : X \to 2^Y$ and $K : X \to 2^Z$ be set-valued maps. In this section, by using the higher-order radial derivatives, we concentrate on establishing derivatives separation type higher-order optimality conditions to a set-valued optimization problem (SOP):

$$\text{(SOP)} \quad \min J(v), \quad \text{s.t.} \quad \forall v \in \Omega,$$

where $\Omega = \{v \in A : K(v) \cap (-P) \neq \emptyset\}$.

Definition 3.1. Let $(\hat{v}, \hat{m}) \in \mathrm{gr}J$.
(i) $(\hat{v}, \hat{m})$ is called an efficient solution of problem (SOP) iff

$$(J(v) - \hat{m}) \cap (-Q\backslash\{0\}) = \emptyset, \quad \forall v \in \Omega.$$

(ii) $(\hat{v}, \hat{m})$ is called a weakly efficient solution of problem (SOP) iff

$$(J(v) - \hat{m}) \cap (-\mathrm{int}Q) = \emptyset, \quad \forall v \in \Omega.$$

(iii) $(\hat{v}, \hat{m})$ is called a Benson proper efficient solution of problem (SOP) iff

$$\mathrm{clcone}(J(v) + Q - \hat{m}) \cap (-Q\backslash\{0\}) = \emptyset, \quad \forall v \in \Omega.$$

Next, we establish several necessary optimality conditions for weakly efficient solutions to problem (SOP). We always assume that $\nabla = \mathrm{dom}\underline{D}^l_R J(\hat{v}, \hat{m}) \cap \mathrm{dom}\overline{D}^l_R K(\hat{v}, \hat{z}) \subset A$.

Theorem 3.2. Let $\hat{v} \in A, \hat{m} \in J(\hat{v}), \hat{z} \in K(\hat{v}) \cap (-P)$. If $(\hat{v}, \hat{m})$ is a weakly efficient solution to problem (SOP), then

$$\left(\underline{D}^l_R J(\hat{v}, \hat{m})(v), \overline{D}^l_R K(\hat{v}, \hat{z})(v)\right) \cap (-\mathrm{int}Q) \times (-\mathrm{int}P) = \emptyset, \tag{2}$$

where $v \in \nabla$.

Proof. Due to $(\hat{v}, \hat{m})$ is a weakly efficient solution of problem (SOP),

$$(J(v) - \hat{m}) \cap (-\mathrm{int}Q) = \emptyset, \quad \forall v \in \Omega. \tag{3}$$

We assume that (2) does not hold. Then there are $\bar{v} \in \nabla$, $\bar{m} \in \underline{D}^l_R J(\hat{v}, \hat{m})(\bar{v})$ and $\bar{z} \in \overline{D}^l_R K(\hat{v}, \hat{z})(\bar{v})$ such that

$$\bar{m} \in -\mathrm{int}Q \text{ and } \bar{z} \in -\mathrm{int}P.$$

From $\bar{z} \in \overline{D}^l_R K(\hat{v}, \hat{z})(\bar{v})$, it yields that there are $\bar{t}_n > 0, \bar{v}_n \to \bar{v}$ and $\bar{z}_n \to \bar{z}$ such that

$$\hat{z} + \bar{t}^l_n \bar{z}_n \in K(\hat{v} + \bar{t}_n \bar{v}_n).$$

Then there exists $\hat{z}_n \in K(\hat{v} + \bar{t}_n \bar{v}_n)$ such that

$$\bar{z}_n = \frac{1}{\bar{t}^l_n}(\hat{z}_n - \hat{z}) \to \bar{z} \in -\mathrm{int}P.$$

Thus, there exists N_1 such that

$$\bar{z}_n \in -\mathrm{int}P, \quad \forall n > N_1.$$

Since P is a convex cone and $\hat{z} \in -P$,

$$\hat{z}_n \in -\mathrm{int}P - P \subset -\mathrm{int}P.$$

Therefore,

$$\hat{z}_n \in K\left(\hat{v} + \bar{t}_n \bar{v}_n\right) \cap (-P) \neq \emptyset,$$

which implies that $\hat{v} + \bar{t}_n \bar{v}_n \in \Omega$.

Since $\bar{m} \in \underline{D}^l_R J(\hat{v}, \hat{m})(\bar{v})$, for the above $\bar{t}_n$ and $\bar{v}_n$, there exists $\bar{m}_n \to \bar{m}$ such that

$$\hat{m} + \bar{t}^l_n \bar{m}_n \in J\left(\hat{v} + \bar{t}_n \bar{v}_n\right).$$

Then there exist $\hat{m}_n \in J\left(\hat{v} + \bar{t}_n \bar{v}_n\right)$ and N_2 such that

$$\bar{m}_n = \frac{1}{\bar{t}^l_n}\left(\hat{m}_n - \hat{m}\right) \to \bar{m} \in -\mathrm{int}Q$$

and

$$\bar{m}_n \in -\mathrm{int}Q, \quad \forall n > N_2,$$

which leads to

$$\hat{m}_n - \hat{m} \in -\mathrm{int}Q.$$

Therefore,

$$\hat{m}_n - \hat{m} \in \left(J\left(\hat{v} + \bar{t}_n \bar{v}_n\right) - \hat{m}\right) \cap (-\mathrm{int}Q),$$

which contradicts (3). □

Theorem 3.3. Let $\hat{v} \in A, \hat{m} \in J(\hat{v}), \hat{z} \in K(\hat{v}) \cap (-P)$. If $(\hat{v}, \hat{m})$ is a weakly efficient solution to problem (SOP), then

$$\underline{D}^l_R J(\hat{v}, \hat{m})(v) \cap (-\mathrm{int}Q) = \emptyset, \tag{4}$$

where $v \in \left\{u \in \nabla : \overline{D}^l_R K(\hat{v}, \hat{z})(u) \cap (-\mathrm{int}P) \neq \emptyset\right\}$.

Proof. Suppose to the contrary, assume that (4) is not true. Then there is $\bar{v} \in \nabla$ with $\overline{D}^l_R K(\hat{v}, \hat{z})(\bar{v}) \cap (-\mathrm{int}P) \neq \emptyset$ such that

$$\underline{D}^l_R J(\hat{v}, \hat{m})(\bar{v}) \cap (-\mathrm{int}Q) \neq \emptyset. \tag{5}$$

Let $\bar{z} \in \overline{D}^l_R K(\hat{v}, \hat{z})(\bar{v}) \cap (-\mathrm{int}P)$. According to the proof of Theorem 3.2, there exist $\bar{t}_n > 0$, $\bar{v}_n \to \bar{v}$, $\bar{z}_n \to \bar{z}$ and $N_1 \in \mathbb{N}$ such that

$$\hat{z} + \bar{t}^l_n \bar{z}_n \in K\left(\hat{v} + \bar{t}_n \bar{v}_n\right) \cap (-\mathrm{int}P), \quad \forall n \geq N_1,$$

which leads to $\hat{v} + \bar{t}_n \bar{v}_n \in \Omega$ for $n \geq N_1$. By (5), there is

$$\bar{m} \in \underline{D}^l_R J(\hat{v}, \hat{m})(\bar{v}) \cap (-\mathrm{int}Q).$$

By the proof of Theorem 3.2, for the above $\bar{t}_n$ and $\bar{v}_n$, there are $\bar{m}_n \to \bar{m}$ and $N_2 \in \mathbb{N}$ such that

$$\hat{m} + \bar{t}^l_n \bar{m}_n - \hat{m} \in \left(J\left(\hat{v} + \bar{t}_n \bar{v}_n\right) - \hat{m}\right) \cap (-\mathrm{int}Q), \quad \forall n \geq N_2,$$

which leads to

$$\hat{m} + \bar{t}^l_n \bar{m}_n - \hat{m} \in \left(J\left(\hat{v} + \bar{t}_n \bar{v}_n\right) - \hat{m}\right) \cap (-\mathrm{int}Q), \quad \forall n \geq \max\{N_1, N_2\},$$

which contradicts the weakly efficiency of $(\hat{v}, \hat{m})$ to problem (SOP). □

Now, we give a similar result with the difference in orders of inner and outer radial derivatives of J and K, respectively.

Theorem 3.4. Let $\hat{v} \in A, \hat{m} \in J(\hat{v}), \hat{z} \in K(\hat{v}) \cap (-P)$. If $(\hat{v}, \hat{m})$ is a weakly efficient solution to problem (SOP), then

$$\underline{D}^l_R J(\hat{v}, \hat{m})(v) \cap (-\text{int}Q) = \emptyset, \tag{6}$$

where $v \in R(A, \hat{v}) \cap \left\{u : \overline{D}^k_R K(\hat{v}, \hat{z})(u) \cap (-\text{int}P) \neq \emptyset\right\}$.

Proof. Suppose that (6) does not hold. Then there are $k \geq 1$, $\bar{v} \in R(A, \hat{v})$ with $\overline{D}^k_R K(\hat{v}, \hat{z})(\bar{v}) \cap (-\text{int}P) \neq \emptyset$ such that

$$\underline{D}^l_R J(\hat{v}, \hat{m})(\bar{v}) \cap (-\text{int}Q) \neq \emptyset. \tag{7}$$

Let $\bar{z} \in \overline{D}^k_R K(\hat{v}, \hat{z})(\bar{v}) \cap (-\text{int}P)$. From the proof of Theorem 3.2 and $\bar{v} \in R(A, \hat{v})$, it follows that there exist $\bar{t}_n > 0$, $\bar{v}_n \to \bar{v}$, $\bar{z}_n \to \bar{z}$ and $N_1 \in \mathbb{N}$ such that

$$\hat{z} + \bar{t}^k_n \bar{z}_n \in K(\hat{v} + \bar{t}_n \bar{v}_n) \cap (-\text{int}P), \quad \forall n \geq N_1,$$

which leads to $\hat{v} + \bar{t}_n \bar{v}_n \in \Omega$ for $n \geq N_1$. By (7), there is

$$\bar{m} \in \underline{D}^l_R J(\hat{v}, \hat{m})(\bar{v}) \cap (-\text{int}Q).$$

Then, for the preceding sequences $\bar{t}_n$ and $\bar{v}_n$, there are $\bar{m}_n \to \bar{m}$ and $N_2 \in \mathbb{N}$ such that

$$\hat{m} + t^l_n \bar{m}_n - \hat{m} \in (J(\hat{v} + \bar{t}_n \bar{v}_n) - \hat{m}) \cap (-\text{int}Q), \quad \forall n \geq N_2,$$

which implies that

$$\hat{m} + t^l_n \bar{m}_n - \hat{m} \in (J(\hat{v} + \bar{t}_n \bar{v}_n) - \hat{m}) \cap (-\text{int}Q), \quad \forall n \geq \max\{N_1, N_2\},$$

which contradicts the weakly efficiency of $(\hat{v}, \hat{m})$ to problem (SOP). □

Theorem 3.5. Let $\hat{v} \in A, \hat{m} \in J(\hat{v}), \hat{z} \in K(\hat{v}) \cap (-P)$ and $((J - \hat{m}) \times K)$ be nearly $(Q \times P)$-subconvexlike on A. If $(\hat{v}, \hat{m})$ is a weakly efficient solution of problem (SOP), then there exists $(m^*, z^*) \in (Q^* \times P^*) \setminus \{(0, 0)\}$ such that

$$z^{*T} \hat{z} = 0$$

and

$$m^{*T} m + z^{*T} z \geq 0,$$

where $v \in \nabla, m \in \underline{D}^l_R J(\hat{v}, \hat{m})(v)$ and $z \in \overline{D}^l_R K(\hat{v}, \hat{z})(v)$. If, in addition, there exists $v' \in \nabla$ such that $\overline{D}^l_R K(\hat{v}, \hat{z})(v') \cap (-\text{int}P) \neq \emptyset$, then $m^* \neq 0$.

Proof. Due to $(\hat{v}, \hat{m})$ is a weakly efficient solution of problem (SOP),

$$((J - \hat{m}), K)(v) \cap (-\text{int}Q) \times (-\text{int}P) = \emptyset, \quad v \in \Omega.$$

By assumption, $((J - \hat{m}) \times K)$ is $(Q \times P)$-subconvexlike on A, according to Lemma 2.5 , there is $(m^*, z^*) \in (Q^* \times P^*) \setminus \{(0, 0)\}$ such that

$$m^{*T}(m - \hat{m}) + z^{*T} z \geq 0, \quad \forall (m, z) \in (J, K)(v), \quad v \in \Omega. \tag{8}$$

Take $m = \hat{m}$ and $z = \hat{z}$ in (8), one gets $z^{*T} \hat{z} \geq 0$. If follows from $\hat{z} \in -P$ that $z^{*T} \hat{z} \leq 0$. We can conclude that $z^{*T} \hat{z} = 0$, which implies that

$$m^{*T}(m - \hat{m}) + z^{*T}(z - \hat{z}) \geq 0, \quad \forall (m, z) \in (J, K)(v), \quad v \in \Omega. \tag{9}$$

Let $v \in \nabla, z \in \overline{D}^l_R K(\hat{v}, \hat{z})(v)$. There exist $t_n > 0$ and $(v_n, z_n) \to (v, z)$ such that

$$z_n \in \frac{K(\hat{v} + t_n v_n) - \hat{z}}{t_n^l}, \quad \hat{v} + t_n v_n \in \Omega. \tag{10}$$

Let $m \in \underline{D}^l_R J(\hat{v}, \hat{m})(v)$. For the preceding sequences t_n, v_n, there exists $m_n \to m$ such that

$$m_n \in \frac{J(v + t_n v_n) - \hat{m}}{t_n^l}, \quad \hat{v} + t_n v_n \in \Omega. \tag{11}$$

According to (10) and (11), there are $\bar{z}_n \in K(\hat{v} + t_n v_n)$ and $\bar{m}_n \in J(\hat{v} + t_n v_n)$ such that

$$m_n = \frac{\bar{m}_n - \hat{m}}{t_n^l}, \quad z_n = \frac{\bar{z}_n - \hat{z}}{t_n^l}.$$

From (9), it follows that

$$m^{*T}\left(\frac{\bar{m}_n - \hat{m}}{t_n^l}\right) + z^{*T}\left(\frac{\bar{z}_n - \hat{z}}{t_n^l}\right) \geq 0,$$

that is,

$$m^{*T} m_n + z^{*T} z_n \geq 0. \tag{12}$$

Take $n \to \infty$ in (12), for any $v \in \nabla, m \in \underline{D}^l_R J(\hat{v}, \hat{m})(v)$ and $z \in \overline{D}^l_R K(\hat{v}, \hat{z})(v)$, one has

$$m^{*T} m + z^{*T} z \geq 0. \tag{13}$$

We now prove $m^* \neq 0$. By contradiction, assume that $m^* = 0$. Then due to (13), for any $v \in \nabla, z \in \overline{D}^l_R K(\hat{v}, \hat{z})(v)$, we have

$$z^{*T} z \geq 0. \tag{14}$$

By assumption, there is $z' \in \overline{D}^l_R K(\hat{v}, \hat{z})(v')$ with $z' \in -\text{int}P$. Since $z^* \in P^* \backslash \{0\}, z^{*T} z' < 0$, this contradicts (14). Thus, $m^* \neq 0$. □

Remark 3.6. It is under the condition of nearly cone-subconvexlikeness that we obtain the necessary conditions for weakly efficient solution to problem (SOP) in Theorem 3.5. However, literatures [14, 16] gained the similar conclusion under the assumption of cone-convexity. Thus, by Remark 2.3, our condition is weaker than Theorem 4 of [14] and Theorem 3.1 of [16].

Example 3.7. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, Q = \mathbb{R}^2_+, P = \mathbb{R}_+, A = \mathbb{R}_+$ and

$$J(v) = \{(m_1, m_2) \in Y : m_1 \geq v^{\frac{6}{7}}, m_2 \geq v^{\frac{8}{9}}\}, \quad \forall v \in A,$$
$$K(v) = \mathbb{R}, \quad \forall v \in A.$$

Thus, $\Omega = \{v \in A : K(v) \cap -P \neq \emptyset\} = A$.

Take $(\hat{v}, \hat{m}) = (0, (0, 0))$ and $\hat{z} = 0$. We calculate that $(\hat{v}, \hat{m})$ is a weakly efficient solution to problem (SOP), $\text{clcone}\big((J - \hat{m}, K)_+ (A)\big)$ is convex,

$$\underline{D}^l_R J(\hat{v}, \hat{m})(v) = \{(m_1, m_2) \in Y : m_1 \geq 0, m_2 \geq 0\}, \quad \forall v \in \mathbb{R}_+$$

and

$$\overline{D}^l_R K(\hat{v}, \hat{z})(v) = \mathbb{R}, \quad \forall v \in \mathbb{R}_+.$$

Take $m^* = (2, 3) \in Q^*, z^* = 0 \in P^*$. Then, for all $v \in \nabla, m \in \underline{D}^l_R J(\hat{v}, \hat{m})(v)$ and $z \in \overline{D}^l_R K(\hat{v}, \hat{z})(v)$, we have

$$z^{*T} \hat{z} = 0 \text{ and } m^{*T} m + z^{*T} z \geq 0.$$

It is notice that for all $v \in A, J$ is not Q-convex on A. Thus, the necessary condition of Theorem 4 in [14] and Theorem 3.1 of [16] are not applicable here. So, we illustrate Theorem 3.5 and Remark 3.6 here.

Corollary 3.8. Let $\hat{v} \in A, \hat{m} \in J(\hat{v}), \hat{z} \in K(\hat{v}) \cap (-P)$ and $((J - \hat{m}) \times K)$ be nearly $(Q \times P)$-subconvexlike on A. If $(\hat{v}, \hat{m})$ is a weakly efficient solution of problem (SOP), then there exists $(m^*, z^*) \in (Q^* \times P^*) \setminus \{(0,0)\}$ such that

$$z^{*T} \hat{z} = 0$$

and

$$m^{*T} m + z^{*T} z \geq 0,$$

where $v \in \nabla, m \in \underline{D}^l_R J(\hat{v}, \hat{m})(v)$ and $z \in \underline{D}^l_R K(\hat{v}, \hat{z})(v)$.

In order to discuss the derivatives separation type sufficient optimality conditions for Benson proper efficient solutions to problem (SOP), we introduce the next Assumption.

Assumption $\mathfrak{B}$: For all $v \in X$,

$$J(v) - \hat{m} \subset \underline{D}^l_R J(\hat{v}, \hat{m})(v - \hat{v}). \tag{15}$$

Theorem 3.9. Let $\hat{v} \in A, \hat{m} \in J(\hat{v}), \hat{z} \in K(\hat{v}) \cap (-P)$ and Assumption $\mathfrak{B}$ hold for J. If there exist $m^* \in Q^* \setminus \{0\}$ and $z^* \in P^*$ such that

$$m^{*T} m + z^{*T} z \geq 0, \quad z^{*T} \hat{z} = 0, \tag{16}$$

where $m \in \underline{D}^l_R J(\hat{v}, \hat{m})(v)$, $z \in \overline{D}^l_R K(\hat{v}, \hat{z})(v)$ and $v \in \nabla$, then $(\hat{v}, \hat{m})$ is a Benson proper efficient solution to problem (SOP).

Proof. Suppose that $(\hat{v}, \hat{m})$ is not a Benson proper efficient solution of problem (SOP). Then there exist $\bar{v} \in \Omega$, $\bar{m} \in J(\bar{v}), q \in Q$ and $\lambda > 0$ such that

$$0 \neq \lambda(\bar{m} + q - \hat{m}) \in \operatorname{clcone}(J(\Omega) + Q - \hat{m}) \cap (-Q)$$

Let $m^* \in Q^* \setminus \{0\}$ and $z^* \in P^*$. We claim that $\hat{m} \neq \bar{m}$. In fact, if $\hat{m} = \bar{m}$, then $0 \neq \varepsilon(\bar{m} + q - \hat{m}) = \varepsilon q \in Q \cap (-Q)$, this is a contradiction to point cone Q. So, $0 \neq \bar{m} - \hat{m} \in -Q$, which leads to

$$m^{*T}(\bar{m} - \hat{m}) < 0.$$

From $z^{*T} \hat{z} = 0$, it follows that for all $\bar{z} \in K(\bar{v}) \cap (-P)$, we have

$$m^{*T}(\bar{m} - \hat{m}) + z^{*T}(\bar{z} - \hat{z}) < 0. \tag{17}$$

According to condition (15) and Lemma 2.13, one gets

$$\bar{m} - \hat{m} \in J(\bar{v}) - \hat{m} \subset \underline{D}^l_R J(\hat{v}, \hat{m})(\bar{v} - \hat{v})$$

and

$$\bar{z} - \hat{z} \in K(\bar{v}) - \hat{z} \subset \overline{D}^l_R K(\hat{v}, \hat{z})(\bar{v} - \hat{v}).$$

By (16), it yields that

$$m^{*T}(\bar{m} - \hat{m}) + z^{*T}(\bar{z} - \hat{z}) \geq 0,$$

which contradicts (17). $\square$

Remark 3.10. The following example shows that Assumption $\mathfrak{B}$ of Proposition 3.9 is necessary.

Example 3.11. Let $A = X = Z = \mathbb{R}, Y = \mathbb{R}^2, Q = \mathbb{R}^2_+, P = \mathbb{R}_+$. We consider the function J defined in Example 2.11 and $K(v) = \mathbb{R}, \forall v \in A$.

Take $l = 4, (\hat{v}, \hat{m}) = (0, (0, 0))$. Then, we have

$$\underline{D}^4_R J(\hat{v}, \hat{m})(0) = \big\{(m_1, m_2) \in \mathbb{R}^2 : m_1 \geq 0, m_2 = 0\big\}$$

and

$$\overline{D}^4_R K(\hat{v}, \hat{z})(v) = \mathbb{R}, \quad \forall v \in \mathbb{R}.$$

It is obvious that

$$J(0) - \hat{m} \not\subset \underline{D}^4_R J(\hat{v}, \hat{m})(0 - \hat{v}),$$

which implies that Assumption $\mathfrak{B}$ does not hold.

Take $m^* = (1, 0) \in Q^*\backslash\{0\}$ and $z^* = 0 \in P^*$. So, condition (16) holds for all $m \in \underline{D}^4_R J(\hat{v}, \hat{m})(v)$, $z \in \overline{D}^4_R K(\hat{v}, \hat{z})(v)$. However,

$$\text{clcone}(J(-1) + Q - \hat{m}) \cap (-Q\backslash\{0\}) \neq \emptyset,$$

i.e., $(\hat{v}, \hat{m})$ is not a Benson proper efficient solution of problem (SOP). Thus, we illustrate Remark 3.10 here.

Corollary 3.12. Let $\hat{v} \in A, \hat{m} \in J(\hat{v})$ and $\hat{z} \in K(\hat{v}) \cap (-P)$. If there exist $m^* \in Q^*\backslash\{0\}$ and $z^* \in P^*$ such that

$$m^{*T} m + z^{*T} z \geq 0, \quad z^{*T} \hat{z} = 0,$$

where $m \in \overline{D}^l_R J(\hat{v}, \hat{m})(v)$, $z \in \overline{D}^l_R K(\hat{v}, \hat{z})(v)$ and $v \in \nabla$, then $(\hat{v}, \hat{m})$ is a Benson proper efficient solution to problem (SOP).

4. Mixed type duality

In this section, we propose a mixed type duality problem (DSOP) of problem (SOP), where the derivatives of the objective and the constraint functions are separated, and establish some duality results. Let $(\hat{v}, \hat{m}, \hat{z}) \in \text{gr}(J, K), e \in \text{int}Q, \delta \in \{0, 1\}$.

Consider the mixed type dual problem (DSOP):

$$\begin{aligned}
\max\ \ & \Upsilon(\hat{v}, \hat{m}, \hat{z}, m^*, z^*) = \hat{m} + \frac{e}{m^{*T} e}(1 - \delta) z^{*T} \hat{z}, \\
\text{s.t.}\ \ & m^{*T} m + z^{*T} z \geq 0, \\
& \forall m \in \underline{D}^l_R J(\hat{v}, \hat{m})(\nabla),\ z \in \overline{D}^l_R K(\hat{v}, \hat{z})(\nabla), \\
& \delta(z^{*T} \bar{z}) \geq 0,\ m^* \in Q^*\backslash\{0\},\ z^* \in P^*,
\end{aligned} \tag{18}$$

where $\nabla = \text{dom}\underline{D}^l_R J(\hat{v}, \hat{m}) \cap \text{dom}\overline{D}^l_R K(\hat{v}, \hat{z})$.

The feasible set of problem (DSOP) is denoted by

$$K_{(\text{DSOP})} := \{(\hat{v}, \hat{m}, \hat{z}, m^*, z^*) : (\hat{v}, \hat{m}, \hat{z}, m^*, z^*) \text{ satisfies condition (18) of problem (DSOP)}\}.$$

A feasible point $(\hat{v}, \hat{m}, \hat{z}, m^*, z^*)$ is said to be a weakly efficient solution of problem (DSOP), if for any $(v, m, z, \hat{m}^*, \hat{z}^*) \in K_{(\text{DSOP})}$, one has

$$(\Upsilon(v, m, z, \hat{m}^*, \hat{z}^*) - \Upsilon(\hat{v}, \hat{m}, \hat{z}, m^*, z^*)) \cap \text{int}Q = \emptyset.$$

Observe that, by taking $\delta = 0$ and $\delta = 1$ from problem (DSOP), we can get Wolfe type dual and Mond-Weir type dual problems, respectively.

Theorem 4.1. (Weak duality theorem) Let $v_0, \hat{v} \in A$, $m_0 \in J(v_0)$, $\hat{m} \in J(\hat{v})$, $\hat{z} \in K(\hat{v}) \cap -P$, and let Assumption $\mathfrak{B}$ hold for J. If (v_0, m_0) is a feasible point for problem (SOP) and $(\hat{v}, \hat{m}, \hat{z}, m^*, z^*)$ is a feasible point for problem (DSOP), then

$$\hat{m} + \frac{e}{m^{*T}e}(1-\delta)z^{*T}\hat{z} - m_0 \notin \mathrm{int}Q.$$

Proof. Suppose to the contrary, *i.e.*,

$$\hat{m} + \frac{e}{m^{*T}e}(1-\delta)z^{*T}\hat{z} - m_0 \in \mathrm{int}Q. \tag{19}$$

Since (v_0, m_0) is a feasible point for problem (SOP), there exists $z_0 \in K(v_0) \cap -P$. By assumption $\mathfrak{B}$ and Lemma 2.13, there are $m' \in \underline{D}^l_R J(\hat{v}, \hat{m})(\Omega)$ and $z' \in \overline{D}^l_R K(\hat{v}, \hat{z})(\Omega)$ such that

$$m_0 - \hat{m} - m' = 0, z_0 - \hat{z} - z' = 0. \tag{20}$$

Combining (19) with (20), one gets

$$\frac{e}{m^{*T}e}(1-\delta)z^{*T}\hat{z} - m' \in \mathrm{int}Q.$$

Hence,

$$m^{*T}(\frac{e}{m^{*T}e}(1-\delta)z^{*T}\hat{z} - m') > 0 = z^{*T}(z_0 - \hat{z} - z'),$$

which implies that $-\delta(z^{*T}\hat{z}) + z^{*T}z_0 > m^{*T}m' + z^{*T}z'$. Moreover, $\delta(z^{*T}\hat{z}) \geq 0$ and $z_0 \in -P$, we have $m^{*T}m' + z^{*T}z' < 0$, which contradicts (18). ☐

Theorem 4.2. (Strong duality theorem) Let $\hat{v} \in A$, $\hat{m} \in J(\hat{v})$, $\hat{z} \in K(\hat{v}) \cap -P$ and Assumption $\mathfrak{B}$ hold for J, and let $((J - \hat{m}) \times K)$ be nearly $(Q \times P)$-subconvexlike on A. If $(\hat{v}, \hat{m})$ is a weakly efficient solution of problem (SOP), then there exist $m^* \in Q\backslash\{0\}$ and $z^* \in P^*$ such that $(\hat{v}, \hat{m}, \hat{z}, m^*, z^*)$ is a weakly efficient solution of problem (DSOP).

Proof. It follows from Theorem 3.5 that there are $m^* \in Q\backslash\{0\}$ and $z^* \in P^*$ such that $(\hat{v}, \hat{m}, \hat{z}, m^*, z^*)$ is a feasible solution of problem (DSOP).

If $(\hat{v}, \hat{m}, \hat{z}, m^*, z^*)$ is not a weakly efficient solution of problem (DSOP), then there exists $(v_1, m_1, z_1, m_1^*, z_1^*) \in K_{(\mathrm{DSOP})}$ such that

$$m_1 + \frac{e}{m_1^{*T}e}(1-\delta)z_1^{*T}z_1 - \hat{m} \in \mathrm{int}Q,$$

which contradicts Theorem 4.1. ☐

Theorem 4.3. (Converse duality theorem) Let $\hat{v} \in A$, $\hat{m} \in J(\hat{v})$ and $\hat{z} \in K(\hat{v}) \cap -P$, and let Assumption $\mathfrak{B}$ hold for J. If $(\hat{v}, \hat{m}, \hat{z}, m^*, z^*)$ is a feasible point of problem (DSOP), then $(\hat{v}, \hat{m})$ is a Benson proper efficient solution to problem (SOP).

Proof. It follows directly from Theorem 3.9. ☐

5. Higher-order outer radial derivatives of perturbation maps

Let X be a normed space of perturbation parameters, Y be an objective normed space and $J : X \to 2^Y$. We define two set-valued maps $\mathcal{W} : X \to 2^Y$ and $\mathcal{B} : X \to 2^Y$ by

$$\mathcal{W}(v) := \mathrm{WMin}_Q\, J(v) = \{m \in J(v) : (J(v) - m) \cap (-\mathrm{int}Q) = \emptyset\}$$

and

$$\mathcal{B}(v) := \mathrm{BPMin}_Q\, J(v) = \{m \in J(v) : \mathrm{clcone}(J(v) + Q - m) \cap (-Q\backslash\{0\}) = \emptyset\}.$$

The maps $\mathcal{W}$ and $\mathcal{B}$ are called the weak perturbation map and Benson proper perturbation map, respectively. In this section, we mainly study the relations between higher-order outer radial derivatives of J and that of $\mathcal{W}$ and $\mathcal{B}$.

Inspired by [7], we introduce the following higher-order compactness of J, which will be used in the later of this section.

Definition 5.1. For $v \in X, J$ is called higher-order v-directionally radial compact at $(\hat{v}, \hat{m}) \in \operatorname{gr} J$ if, for any $t_n > 0$ and $v_n \to v$, m_n with $\hat{m} + t_n^l m_n \in J(\hat{v} + t_n v_n)$ has a convergent subsequence.

We now establish the existence of the above compactness.

Proposition 5.2. Let $(\hat{v}, \hat{m}) \in \operatorname{gr} J$ and Y be finite dimensional. If

$$\overline{D}_R^l J(\hat{v}, \hat{m})(0) = \{0\}, \tag{21}$$

then J is higher-order v-directionally radial compact at $(\hat{v}, \hat{m})$ for all $v \in X$.

Proof. Let $v_n \to v, t_n > 0$ and $\hat{m} + t_n^l m_n \in J(\hat{v} + t_n v_n)$. We only need to prove the boundedness of m_n. Assume $\lim_{n\to\infty} \|m_n\| = +\infty$. Set

$$v_n = \frac{m_n}{\|m_n\|}, \ u_n = \frac{v_n}{\sqrt[l]{\|m_n\|}} \text{ and } s_n = t_n \sqrt[l]{\|m_n\|}.$$

Then, $s_n^l v_n = t_n^l m_n, s_n u_n = t_n v_n$, $s_n > 0$ and $u_n \to 0$. There exists $\bar{m}$ with $\|\bar{m}\| = 1$ such that $v_n \to \bar{m}$ (for a subsequence if necessary). Therefore, $\bar{m} \in D_R^l J(\hat{v}, \hat{m})(0)$, which contradicts (21). □

Definition 5.3. [12] The Q-domination property is said to be hold for a subset Θ of Y if $\Theta \subset \operatorname{Min}\Theta + Q$.

Proposition 5.4. Let $(\hat{v}, \hat{m}) \in \operatorname{gr} J$ and Q have a compact base B. Then for any $v \in H := \operatorname{dom} \overline{D}_R^l J(\hat{v}, \hat{m})(v)$,

$$\operatorname{Min}_Q \overline{D}_R^l J_+(\hat{v}, \hat{m})(v) \subset \overline{D}_R^l J(\hat{v}, \hat{m})(v). \tag{22}$$

Proof. Let $v \in H, m \in \operatorname{Min}_Q \overline{D}_R^l J_+(\hat{v}, \hat{m})(v)$. Then there are $t_n > 0$ and $(v_n, m_n) \to (v, m)$ such that

$$\hat{m} + t_n^l m_n \in J(\hat{v} + t_n x_n) + Q,$$

and there is a sequence $q_n \in Q$ such that

$$\hat{m} + t_n^l (m_n - \frac{q_n}{t_n^l}) \in J(\hat{v} + t_n x_n).$$

Since Q has a compact base B and $q_n \in Q$, there are $\lambda_n > 0$ and $b_n \in B$ such that $q_n = \lambda_n b_n$. Then

$$\hat{m} + t_n^l \left(m_n - \frac{\lambda_n}{t_n^l} b_n \right) \in J(\hat{v} + t_n v_n). \tag{23}$$

By the compactness of B, without loss of generality, there exists some $b \in B$ such that $b_n \to b$.

We claim that $\frac{\lambda_n}{t_n^l} \to 0$. Suppose to the contrary, there is $\varepsilon > 0$ such that $\frac{\lambda_n}{t_n^l} \geq \varepsilon$, for all n. Thus, from (23), we have

$$\begin{aligned} \hat{m} + t_n^l (m_n - \varepsilon b_n) =& \hat{m} + t_n^l \left(m_n - \frac{\lambda_n}{t_n^l} b_n \right) + (\lambda_n b_n - t_n^l \varepsilon b_n) \\ =& \hat{m} + t_n^l \left(m_n - \frac{\lambda_n}{t_n^l} b_n \right) + t_n^l b_n \left(\frac{\lambda_n}{t_n^l} - \varepsilon \right) \in J(\hat{v} + t_n v_n) + Q. \end{aligned}$$

Since $m_n \to m$ and $b_n \to b$,

$$m_n - \varepsilon b_n \to m - \varepsilon b.$$

It follows from $v_n \to v$ and $t_n > 0$ that

$$m - \varepsilon b \in \overline{D}_R^l J_+(\hat{v}, \hat{m})(v),$$

which contradicts $m \in \mathrm{Min}_Q \overline{D}_R^l J_+(\hat{v}, \hat{m})(v)$. Therefore, the claim is verified and $m_n - \frac{\lambda_n}{t_n^l} b_n \to m$. From (23), it follows that

$$m \in \overline{D}_R^l J(\hat{v}, \hat{m})(v).$$

Thus, (22) holds. ∎

Now, we discuss relations between the higher-order outer radial derivative of a set-valued map and those of its profile map.

Proposition 5.5. Let $(\hat{v}, \hat{m}) \in \mathrm{gr}J$, $v \in X$. Suppose that one of the following conditions holds:

(i) Q has a compact base B, $\overline{D}_R^l J_+(\hat{v}, \hat{m})(v)$ fulfills the Q-domination property for all $v \in H$;

(ii) for $v \in X$, if J is higher-order v-directionally radial compact at $(\hat{v}, \hat{m})$;

(iii) Y is finite dimensional and $\overline{D}_R^l J(\hat{v}, \hat{m})(0) \cap (-Q) = 0$.

Then

$$\overline{D}_R^l J_+(\hat{v}, \hat{m})(v) = \overline{D}_R^l J\,(\hat{v}, \hat{m})\,(v) + Q$$

and

$$\mathrm{BPMin}_Q \overline{D}_R^l J_+(\hat{v}, \hat{m})(v) = \mathrm{BPMin}_Q \overline{D}_R^l J\,(\hat{v}, \hat{m})\,(v).$$

Proof. " $\supset$" Let $v \in X$, $m \in \overline{D}_R^l J(\hat{v}, \hat{m})(v)$ and $q \in Q$. Then there are $t_n > 0, (v_n, m_n) \to (v, m)$ such that

$$\hat{m} + t_n^l m_n \in J(\hat{v} + t_n v_n).$$

We can get from $q \in Q$ and $t_n > 0$ that

$$\hat{m} + t_n^l (m_n + q) \in J(\hat{v} + t_n v_n) + Q,$$

which leads to $m + q \in \overline{D}_R^l J_+\,(\hat{v}, \hat{m})\,(v)$.

" $\subset$" Let (i) hold. Since $\overline{D}_R^l J_+(\hat{v}, \hat{m})(v)$ fulfills the Q-domination property for all $v \in H$, one can get from Proposition 5.4 that

$$\begin{aligned} \overline{D}_R^l J_+(\hat{v}, \hat{m})(v) &\subset \mathrm{Min}_Q \overline{D}_R^l J_+(\hat{v}, \hat{m})(v) + Q \\ &\subset \overline{D}_R^l J(\hat{v}, \hat{m})(v) + Q. \end{aligned}$$

Let (ii) hold and $m \in \overline{D}_R^l J_+(\hat{v}, \hat{m})(v)$. If $(v, m) = (0, 0)$, then we have $0 \in \overline{D}_R^l J(\hat{v}, \hat{m})(v) + Q$. For $(v, m) \neq (0, 0)$, there exist $t_n > 0$ and $(v_n, m_n) \to (v, m)$ such that $\hat{m} + t_n^l m_n \in J\,(\hat{v} + t_n v_n) + Q$. Then, there is $q_n \in Q$ such that

$$\hat{m} + t_n^l \left(m_n - \frac{q_n}{t_n^l} \right) \in J\,(\hat{v} + t_n v_n)\,. \tag{24}$$

Since J is higher-order v-directional radial compact at $(\hat{v}, \hat{m})$, there exists some $\bar{m} \in Y$ such that

$$m_n - \frac{q_n}{t_n^l} \to \bar{m}, \tag{25}$$

combining this with closed cone Q, we have

$$\frac{q_n}{t_n^l} \to m - \bar{m} \in Q. \tag{26}$$

It follows from (24), (25) and (26) that $\bar{m} \in \overline{D}^l_R J(\hat{v},\hat{m})(v)$ and $m \in \overline{D}^l_R J(\hat{v},\hat{m})(v) + Q$.

Let (iii) hold and $m \in \overline{D}^l_R J_+(\hat{v},\hat{m})(v)$. For $(v,m) = (0,0)$ the assertion is trivial. Thus, we only need to consider $(v,m) \neq (0,0)$. Then there exist $t_n > 0, (v_n,m_n) \to (v,m)$ and $q_n \in Q$ such that $\hat{m} + t_n^l m_n \in J(\hat{v} + t_n v_n) + q_n$. If $q_n = 0$ for many infinitely n, then $m \in \overline{D}^l_R J(\hat{v},\hat{m})(v) + Q$. We now assume that $q_n \neq 0$. Since Y is finite dimensional, $\frac{q_n}{\|q_n\|} \to \bar{q}$ for some $\bar{q} \in Q$ with $\|\bar{q}\| = 1$. Let $w_n := \sqrt[m]{\|q_n\|}$ and suppose that $\frac{w_n}{t_n} \to +\infty$. Then, we have

$$\hat{m} + w_n^l \left((\frac{t_n}{w_n})^l m_n - \frac{q_n}{w_n^l} \right) \in J\left(\hat{v} + w_n (\frac{t_n}{w_n} v_n) \right),$$

$(\frac{t_n}{w_n})^l m_n - \frac{q_n}{w_n^l} \to -\bar{q}$ and $\frac{t_n}{w_n} v_n \to 0$, which leads to $-\bar{q} \in \overline{D}^l_R J(\hat{v},\hat{m})(0)$, this contradicts condition (iii), so for some $\alpha > 0$, we have $\frac{w_n}{t_n} \to \alpha$. And then, from

$$\hat{m} + t_n^l \left(m_n - (\frac{w_n}{t_n})^l \frac{q_n}{w_n^l} \right) \in J(\hat{v} + t_n v_n),$$

$m_n - (\frac{w_n}{t_n})^l \frac{q_n}{w_n^l} \to m - \alpha^l \bar{q}$ and $v_n \to v$, it follows that $m - \alpha^l \bar{q} \in \overline{D}^l_R J(\hat{v},\hat{m})(v)$, and hence $m \in \overline{D}^l_R J(\hat{v},\hat{m})(v) + Q$.

Consequently, we get

$$\overline{D}^l_R J_+(\hat{v},\hat{m})(v) = \overline{D}^l_R J(\hat{v},\hat{m})(v) + Q.$$

Thus,

$$\begin{aligned} \text{BPMin}_Q \overline{D}^l_R J_+(\hat{v},\hat{m})(v) &= \text{BPMin}_Q (\overline{D}^l_R J(\hat{v},\hat{m})(v) + Q) \\ &= \text{BPMin}_Q \overline{D}^l_R J(\hat{v},\hat{m})(v). \end{aligned}$$

□

Remark 5.6. For condition (i), since the Q-domination property of $\overline{D}^l_R J_+(\hat{v},\hat{m})(v)$ holds and the corresponding derivatives $D^l J_+(\hat{v},\hat{m},u_1,v_1,\ldots,u_{m-1},v_{m-1})(v)$ in [23] may not exist, Proposition 5.5 improves Theorem 3.1 of [23]. Here is an example to show the case.

Example 5.7. Let $Q = \mathbb{R}_+$ and $J : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$J(v) = \{m \in Y : m \geq v^2\}.$$

Take $(\hat{v},\hat{m}) = (0,0), (u_1,v_1) = (0,-1)$. By calculating, we get

$$\overline{D}^2_R J_+(\hat{v},\hat{m})(v) = \{m \in Y : m \geq v^2\}.$$

Therefore, $\overline{D}^2_R J_+(\hat{v},\hat{m})(v)$ fulfills the Q-domination property for all $v \in X$ and

$$\overline{D}^2_R J_+(\hat{v},\hat{m})(v) = \overline{D}^2_R J(\hat{v},\hat{m})(v) + Q, \quad v \in X.$$

However, $D^2 J_+(\hat{v},\hat{m},u_1,v_1)(v) = \emptyset$ and the Q-domination property for $D^2 J_+(\hat{v},\hat{m},u_1,v_1)(v)$ does not hold, which leads to Theorem 3.1 of [23] is unusable here.

Proposition 5.8. Let $(\hat{v},\hat{m}) \in \text{gr}J$, $v \in X$ and $\hat{Q} \subset \text{int}Q \cup \{0\}$ be a closed convex cone. Suppose that one of the following conditions holds:

(i) $\hat{Q}$ has a compact base B, $\overline{D}^l_R J_+(\hat{v},\hat{m})(v)$ fulfills the $\hat{Q}$-domination property for all $v \in H$;

(ii) for $v \in X$, if J is higher-order v-directionally radial compact at $(\hat{v}, \hat{m})$;

(iii) Y is finite dimensional and $\overline{D}^l_R J(\hat{v}, \hat{m})(0) \cap (-\hat{Q}) = 0$.

Then for all $v \in \mathrm{dom}\overline{D}^l_R J\,(\hat{v}, \hat{m})$

$$\mathrm{WMin}_Q \overline{D}^l_R (J + \hat{Q})(\hat{v}, \hat{m})(v) = \mathrm{WMin}_Q \overline{D}^l_R J\,(\hat{v}, \hat{m})\,(v).$$

Proof. The proof is similar to that of Proposition 5.5. □

An example is provided to illustrate a case where the cone $\hat{Q}$ can not be instead by Q.

Example 5.9. Let $Q = \mathbb{R}^2_+$ and $J : \mathbb{R} \to 2^{\mathbb{R}^2}$ be defined by

$$J(v) = \{(v^2, v^2)\}.$$

Take $(\hat{v}, \hat{m}) = (0, (0, 0))$. Then for any $v \in \mathbb{R}$, we have

$$\overline{D}^l_R J(\hat{v}, \hat{m})(v) = \{(v^2, v^2)\},$$

$$\overline{D}^l_R J_+(\hat{v}, \hat{m})(v) = \{(m_1, m_2) \in \mathbb{R}^2_+ : m_1 \geq v^2, m_2 \geq v^2\},$$

$$\mathrm{WMin}_Q \overline{D}^l_R J(\hat{v}, \hat{m})(v) = \{(v^2, v^2)\}$$

and

$$\begin{aligned}&\mathrm{WMin}_Q \overline{D}^l_R J_+(\hat{v}, \hat{m})(v)\\ &\quad = \{(m_1, m_2) \in \mathbb{R}^2_+ : m_1 = v^2, m_2 \geq v^2\} \cup \{(m_1, m_2) \in \mathbb{R}^2_+ : m_1 \geq v^2, m_2 = v^2\}.\end{aligned}$$

It is obvious that $\overline{D}^l_R J(\hat{v}, \hat{m})(0) \cap (-Q) = 0$, but

$$\mathrm{WMin}_Q \overline{D}^l_R J_+(\hat{v}, \hat{m})(v) \neq \mathrm{WMin}_Q \overline{D}^l_R J\,(\hat{v}, \hat{m})\,(v).$$

Next, we apply above-mentioned results to investigate relationships between the higher outer radial derivatives of J and that of $\mathcal{W}$ and $\mathcal{B}$. We denote $\bar{H} := \mathrm{dom}\overline{D}^l_R \mathcal{W}(\hat{v}, \hat{m})$ and $\hat{H} := \mathrm{dom}\overline{D}^l_R \mathcal{B}(\hat{v}, \hat{m})$.

Definition 5.10. [22] J is said to be Q-minicomplete by $\mathcal{W}$, if, for any $v \in X$,

$$J(v) \subset \mathcal{W}(v) + Q.$$

Remark 5.11. If J is Q-minicomplete by $\mathcal{W}$, then

$$\mathcal{W}_+(v) = J_+(v), \quad \forall v \in X,$$

which leads to

$$\overline{D}^l_R \mathcal{W}_+(\hat{v}, \hat{m}) = \overline{D}^l_R J_+(\hat{v}, \hat{m}).$$

Proposition 5.12. Let $(\hat{v}, \hat{m}) \in \mathrm{gr}J$. If $\overline{D}^l_R J(\hat{v}, \hat{m}) = \underline{D}^l_R J(\hat{v}, \hat{m})$, then for $v \in \bar{H}$,

$$\overline{D}^l_R \mathcal{W}(\hat{v}, \hat{m})(v) \subset \mathrm{WMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v). \tag{27}$$

Proof. Let $m \in \overline{D}^l_R \mathcal{W}(\hat{v}, \hat{m})(v)$. Suppose to the contrary, if $m \notin \mathrm{WMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v)$, then there exists $m' \in \overline{D}^l_R J(\hat{v}, \hat{m})(v)$ such that $m' - m \in -\mathrm{int}Q$. By assumption, there exist $t_n > 0$ and $(v_n, m_n, m'_n) \to (v, m, m')$ such that

$$\bar{m}_n := \hat{m} + t^l_n m_n \in \mathcal{W}\,(\hat{v} + t_n v_n) = \mathrm{WMin}_Q J\,(\hat{v} + t_n v_n)\,,$$
$$\bar{m}'_n := \hat{m} + t^l_n m'_n \in J\,(\hat{v} + t_n v_n)\,,$$

which implies that $\bar{m}'_n - \bar{m}_n = t^l_n\,(m'_n - m_n) \notin -\mathrm{int}Q$. Since Q is a cone, $m'_n - m_n \notin -\mathrm{int}Q$. Taking $n \to \infty$, we have $m' - m \notin -\mathrm{int}Q$, a contradiction. This completes the proof. □

Proposition 5.13. Let $(\hat{v}, \hat{m}) \in \mathrm{gr}J$. Suppose that the following conditions hold:

(i) $\overline{D}^l_R J(\hat{v}, \hat{m}) = \underline{D}^l_R J(\hat{v}, \hat{m})$.

(ii) For all $v \in \hat{H}, m - z \in Q \cup (-Q), \forall m, z \in J(v), m \neq z$.

(iii) J is Q-minicomplete by $\mathcal{B}$.

Then for all $v \in \hat{H}$,

$$\overline{D}^l_R \mathcal{B}(\hat{v}, \hat{m})(v) \subset \mathrm{BPMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v). \tag{28}$$

Proof. Let $m, z \in \mathcal{B}(v) \subset J(v)$ with $z \neq m$. We claim that $\mathcal{B}(v)$ is a single point set for all $v \in \hat{H}$. Suppose to the contrary, by assumption (ii), it yields that $m - z \in Q \cup (-Q)$, that is, $m - z \in Q$ or $m - z \in -Q$. Obviously, $m - z \neq 0$. If $m - z \in Q$. then we have

$$\mathrm{clcone}(J(v) + Q - m) \cap ((-Q) \setminus \{0\}) \neq \emptyset,$$

which contradicts $m \in \mathcal{B}(v)$. If $m - z \in -Q$, we have

$$\mathrm{clcone}(J(v) + Q - z) \cap ((-Q) \setminus \{0\}) \neq \emptyset,$$

which contradicts $z \in \mathcal{B}(v)$. So, $\mathcal{B}(v)$ is singleton for all $v \in \hat{H}$.

Let $m \in \overline{D}^l_R \mathcal{B}\left(\hat{v}, \hat{m}\right)(v) \subset \overline{D}^l_R J\left(\hat{v}, \hat{m}\right)(v)$ and suppose that $m \notin \mathrm{BPMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v)$. Then there exist $\hat{t} \geq 0, \hat{q} \in Q$ and $\hat{m} \in \overline{D}^l_R J\left(\hat{v}, \hat{m}\right)(v)$ such that

$$\hat{t}\left(\hat{m} + \hat{q} - m\right) \in -Q \setminus \{0\}\,. \tag{29}$$

By assumption (i), there exist $t_n > 0$ and $(v_n, m_n, \hat{m}_n) \to (v, m, \hat{m})$ such that

$$\hat{m} + t^l_n m_n \in \mathcal{B}(\hat{v} + t^l_n x_n) \subset J(\hat{v} + t^l_n x_n) \tag{30}$$

and

$$\hat{m} + t^l_n \hat{m}_n \in J(\hat{v} + t^l_n x_n). \tag{31}$$

It follows from the single point set $\mathcal{B}(v)$, condition (iii), (30) and (31) that $\hat{m}_n - m_n \in Q$. Since Q is a closed cone, $\hat{m} - m \in Q$, which implies that $\hat{t}(\hat{m} + \hat{q} - m) \in Q$, this contradicts (29). Therefore, the proof is complete. ☐

Remark 5.14. Assumption (ii) is essential in Proposition 5.13. We give an example to show this case.

Example 5.15. Let $Q = \{0\} \times \mathbb{R}_+$ and

$$J(v) = \left\{(m_1, m_2) \in \mathbb{R}^2 : m_2 \geq 2m_1^2\right\}, \quad \forall v \in \mathbb{R}.$$

Take $(\hat{v}, \hat{m}) = (0, (0, 0))$. By a simply calculation, for any $v \in \mathbb{R}$, one gets

$$\mathcal{B}(v) = \left\{(m_1, m_2) \in \mathbb{R}^2 : m_2 = 2m_1^2\right\},$$

$$\overline{D}^2_R J\left(\hat{v}, \hat{m}\right)(v) = \underline{D}^2_R J\left(\hat{v}, \hat{m}\right)(v) = \left\{(m_1, m_2) \in \mathbb{R}^2 : m_2 \geq 0\right\},$$

$$\overline{D}^2_R \mathcal{B}\left(\hat{v}, \hat{m}\right)(v) = \left\{(m_1, m_2) \in \mathbb{R}^2 : m_2 = 2m_1^2\right\}$$

and

$$\mathrm{BPMin}_Q\, \overline{D}^2_R J\left(\hat{v}, \hat{m}\right)(v) = \left\{(m_1, m_2) \in \mathbb{R}^2 : m_2 = 0\right\},$$

which implies that

$$\overline{D}^2_R \mathcal{B}\left(\hat{v}, \hat{m}\right)(v) \not\subset \mathrm{BPMin}_Q \overline{D}^2_R J\left(\hat{v}, \hat{m}\right)(v),$$

this means that Proposition 5.13 is unusable here. Considering pairs $(1, 10), (2, 9) \in J(v)$, we observe that $(2, 9) - (1, 10) = (1, -1) \notin Q \cup (-Q)$, indicating the failure of assumption (ii) in this case. Nevertheless, we can check assumptions (i) and (iii) in Proposition 5.13 hold. Hence, assumption (ii) is essential in Proposition 5.13.

Proposition 5.16. Let $(\hat{v}, \hat{m}) \in \mathrm{gr}J$ and $\hat{Q} \subset \mathrm{int}Q \cup \{0\}$ be a closed convex cone. Assume that one of the conditions of Proposition 5.8 holds and J is $\hat{Q}$-minicomplete by $\mathcal{W}$. Then for all $v \in \bar{H}$,

$$\mathrm{WMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v) \subset \overline{D}^l_R \mathcal{W}(\hat{v}, \hat{m})(v). \tag{32}$$

Proof. Since J is $\hat{Q}$-minicomplete, $J(v) + \hat{Q} = \mathcal{W}(v) + \hat{Q}$, which implies that

$$\overline{D}^l_R (J + \hat{Q})(\hat{v}, \hat{m})(v) = \overline{D}^l_R (\mathcal{W} + \hat{Q})(\hat{v}, \hat{m})(v).$$

If one of the conditions of Proposition 5.8 holds, then it follows from $\mathrm{gr}\mathcal{W} \subset \mathrm{gr}J$ that

$$\mathrm{WMin}_Q\, \overline{D}^l_R (J + \hat{Q})(\hat{v}, \hat{m})(v) = \mathrm{WMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v)$$

and

$$\mathrm{WMin}_Q\, \overline{D}^l_R (\mathcal{W} + \hat{Q})(\hat{v}, \hat{m})(v) = \mathrm{WMin}_Q\, \overline{D}^l_R \mathcal{W}(\hat{v}, \hat{m})(v).$$

Thus, for all $v \in \bar{H}$, we have

$$\begin{aligned}
\mathrm{WMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v) &= \mathrm{WMin}_Q\, \overline{D}^l_R (J + \hat{Q})(\hat{v}, \hat{m})(v) \\
&= \mathrm{WMin}_Q\, \overline{D}^l_R (\mathcal{W} + \hat{Q})(\hat{v}, \hat{m})(v) \\
&= \mathrm{WMin}_Q\, \overline{D}^l_R \mathcal{W}(\hat{v}, \hat{m})(v) \\
&\subset \overline{D}^l_R \mathcal{W}(\hat{v}, \hat{m})(v).
\end{aligned}$$

□

Proposition 5.17. Let $(\hat{v}, \hat{m}) \in \mathrm{gr}J$. Assume that one of the conditions of Proposition 5.5 holds and J is Q-minicomplete by $\mathcal{B}$. Then for all $v \in \hat{H}$,

$$\mathrm{BPMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v) \subset \overline{D}^l_R \mathcal{B}(\hat{v}, \hat{m})(v). \tag{33}$$

Proof. The proof is similar to that of Proposition 5.16. □

According to the above results, we can conclude the following two corollaries.

Corollary 5.18. Let $(\hat{v}, \hat{m}) \in \mathrm{gr}J$ and the following conditions be satisfied:
(i) Either conditions of Proposition 5.5 holds.
(ii) $\overline{D}^l_R J(\hat{v}, \hat{m}) = \underline{D}^l_R J(\hat{v}, \hat{m})$.
(iii) J is Q-minicomplete by $\mathcal{B}$.
(iv) For all $v \in \hat{H}, m - z \in Q \cup (-Q), \forall m, z \in J(v), m \neq z$.
Then

$$\mathrm{BPMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v) = \overline{D}^l_R \mathcal{B}(\hat{v}, \hat{m})(v).$$

Corollary 5.19. Let $(\hat{v}, \hat{m}) \in \mathrm{gr}J$, $\hat{Q} \subset \mathrm{int}Q \cup \{0\}$ be a closed convex cone and the following conditions be satisfied:
(i) Either conditions of Proposition 5.8 holds.
(iii) $\overline{D}^l_R J(\hat{v}, \hat{m}) = \underline{D}^l_R J(\hat{v}, \hat{m})$.
(ii) J is $\hat{Q}$-minicomplete by $\mathcal{W}$.
Then

$$\overline{D}^l_R \mathcal{W}(\hat{v}, \hat{m})(v) = \mathrm{WMin}_Q\, \overline{D}^l_R J(\hat{v}, \hat{m})(v).$$

6. Conclusion

In the paper, we have discussed several applications to set-valued maps in terms of higher-order radial derivatives. More specifically, via these derivatives, the higher-order optimality conditions for the derivative separation type of a set-valued optimization problem have been established. Then, the weak, strong and converse duality for a mixed type of primal-dual set-valued optimization problems have been presented. Moreover, some results of higher-order sensitivity analysis have been obtained. In this case, our obtained results improve some existed ones in the literature [14,16,23], the corresponding results have been summarized in Remarks 3.6 and 5.6.

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