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A STUDY ON N-POLICY $BMAP/G/1$ QUEUEING SYSTEM

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Abstract. This study presents a simple approach for analyzing the *BMAP/G/1* queueing system under *N* policy. In this policy, the server goes idle at the end of a busy period and the queue length is checked at every arrival moment. The idle server starts serving customers when the queue length reaches or above a predetermined number, namely *N*, and keeps doing so until the system is completely empty. We first use a simple sequential substitution approach to derive the system length distribution at departure epoch. A comparative study is also conducted to highlight the benefits and strength of our straightforward approach to that of the *RG*-factorization technique and the roots finding method. We extract the distribution of system length at a random time point by utilizing the remaining service time of a customer who is currently being served as the supplementary variable. The probability density function of the sojourn time distribution for an arbitrary customer of an incoming batch is also computed. We propose an expected linear cost function to estimate the optimal value of *N* at minimum cost. The validity of our analytic technique has been shown through a variety of numerical examples involving different service time distributions.

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1. Introduction

Queueing models with threshold policies have a major impact in analyzing the performance measures of inventory control systems, manufacturing industries, transportation systems, communication systems and computer networks, to name a few. In reality, it can be widely employed for practical purposes of power conservation schemes in wireless sensor networks to address the energy hole issue (EHP) in wireless sensor network (WSN), see Nageswari *et al.* [22]. The threshold control strategy is a significant power saving scheme in some extremely hostile areas such as wildlife tracking, security surveillance and forest fire monitoring since it is often difficult to recharge or replace the batteries in these not-easily-accessible environments. The threshold policy is an economically effective service policy to prolong the lifetime of the WSN because it can shorten the amount of setup power needed to switch between the sleeping and wake-up states. Many researchers have explored threshold policy (N-policy, T-policy, D-policy, F-policy, $\mathrm{Min}(N, D)$-policy, and so on) queueing models extensively throughout the last few decades. Readers who are interested may read the works of Choudhury and Paul [6], Jiang *et al.* [9], Kuo *et al.* [11], Lee and Song [13], Lee *et al.* [17], Wang *et al.* [28], Hao *et al.* [8], Lee *et al.* [14], Liu *et al.* [20], Yu and Alfa [30], Begum and Choudhury [3], and related references therein.

Keywords. Batch Markovian arrival process (BMAP), expected cost function, N-policy, queueing, Sojourn time distribution.

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Queueing models with N-policy have given much attention by numerous researchers since the pioneering work of Yadin and Naor [29]. The idea behind N-policy is widely used for modeling purpose of computer and telecommunication systems as well as production and manufacturing industries. The so-called N-policy specifies that when a busy period ends, the server becomes idle and inspects the queue length at every arrival moment. When the system reaches at least N $(N \geq 1)$ customers, the idle server starts service and serves them until the system is empty. An excellent and comprehensive studies on this topic related to the single or batch arrival $M/G/1$ queue together with the $GI/M/1$ queue can be found in Zhang and Tian [31], Artalejo [1], Lee *et al.* [15], Chang and Pearn [5], and Li and Liu [18]. For more references concerning N-policy queues can be found in their reference list. However, only a few authors have investigated the threshold policy queueing models with correlation in arrivals. Lee and Song [13] investigated the workload and waiting time analysis of the $MAP/G/1$ queue under the D-policy, which states that an idle server won't start serving customers until the sum of all pending customer wait times hits a specific threshold (D). Sreenivasan *et al.* [27] investigated the $MAP/PH/1$ queue under N-policy with working vacations, in which the server provides a decreased service rate during vacations. Kasahara *et al.* [10] carried out the stationary queue length and the waiting time distributions for $MAP/G/1$ queueing system with and without vacations under N-policy. The $MAP/G/1$ queueing system with a Min(N, D)-policy was taken into consideration by Lee *et al.* [17]. Under this system, the inactive server starts operation if either N customers build up in the system or the total amount of time needed to serve all of the waiting customers reaches D, whichever occurs first. Lee *et al.* [16] developed precise form of matrix decomposition of the queue length for the $MAP/G/1$ queue with multiple and single vacations under N-policy. To the best of our knowledge, the $BMAP/G/1$ queueing system under the N-policy is only briefly discussed in a few papers in the queueing literature. Lee and Park [12] examined the waiting time analysis of the $BMAP/G/1$ queue under N-policy with multiple vacations based on the factorization property. Nandi *et al.* [23] investigated the D-$BMAP/G/1$ queue under N-policy in discrete-time using the roots of the corresponding characteristic equation.

The preceding review of the literature inspires us to investigate continuous-time $BMAP/G/1$ queueing model with N-policy. A simple sequential substitution approach is used to first derive the system length distribution at departure epoch. A comparative study is also conducted to highlight the benefits and strength of our straightforward approach to that of the RG-factorization method and the roots finding method. The remaining service time of a customer in service is used as the additional variable to calculate the system length distribution at random epoch. The probability density function (PDF) of the sojourn time distribution for a randomly chosen customer from an incoming batch is also calculated. To do so, we need to first determine the Laplace-Stieltjes transform (L.-S.T.) of the sojourn time distribution of a randomly chosen customer from an arrival batch. We then express the L.-S.T. function into its partial fraction form using the zeros (with negative real part) of the denominator of L.-S.T. function. To get the optimal value of N at lowest cost, we consider an expected linear cost function. To highlight the validity of our analytic technique, we show a variety of numerical and graphical findings for various service time distributions. The model presented in this paper has potential to be used in the production unit to detect damaged products in a more optimistic manner.

The rest of the paper is laid out in the following manner. The description of the model is presented in Section 2. In Section 3, we get the system length distributions at departure and random epochs. The sojourn time distribution of a randomly chosen customer from an arrival batch is examined in Section 4. The cost function to determine the optimal value of N at lowest cost is established in Section 5. Section 6 provides computational experiences with a wide range of numerical and graphical outcomes. The paper is wrapped up in Section 7.

2. Description of the model

We investigate the $BMAP/G/1$ queueing system with N-policy, where users access the server using n-state batch Markovian arrival process (BMAP). The server is automatically shut off when the system is vacant at the end of an active period, and it remains inactive until the queue length hits or surpasses a pre-specified threshold number N. When the queue length reaches or surpasses N, the server is activated and continues to service customers until the system is completely empty. The parameter matrices $\mathbf{D}_r$, $r = 0, 1, 2, \ldots$, characterize the BMAP, where $\mathbf{D}_r$ is a $n \times n$ rate matrix with a batch arrival of size r if $r \geq 1$ and no arrival if $r = 0$. The n states of BMAP are phases of the underlying Markov chain (UMC); for more information, see Lucantoni [21]. The rate of change from state j to state k during a phase transition of the UMC, with a batch arrival of size r, is represented by the (j, k)th element $[D_r]_{jk}$ in the matrix $\mathbf{D}_r$. The stationary probability vector $\overline{\boldsymbol{\pi}}$ of an infinitesimal generator $\mathbf{D} = \sum_{r=0}^{\infty} \mathbf{D}_r$ exists such that $\overline{\boldsymbol{\pi}}\mathbf{D} = \mathbf{0}$ and $\overline{\boldsymbol{\pi}}\mathbf{e} = 1$, where $\mathbf{e}$ is a column vector with an appropriate order whose all elements are 1. The average arrival rate λ^* of the stationary BMAP is provided by $\lambda^* = \overline{\boldsymbol{\pi}} \sum_{r=1}^{\infty} r\mathbf{D}_r\mathbf{e}$. Let $b(u)$ and $B(u)$, $u \geq 0$, denote the probability density function (PDF) and cumulative distribution function (CDF) of service time distribution, respectively. Also, let $B^*(s)$, $\mathrm{Re}(s) \geq 0$, be the Laplace-Stieltjes transform (L.-S.T.) of $b(u)$. The mean service rate is $\mu = 1/E[B]$, where $E[B] = -\frac{\mathrm{d}}{\mathrm{d}s} B^*(s) \mid_{s=0}$. The utilization factor ρ is defined as $\rho = \lambda^*/\mu < 1$.

3. Analysis of the model

In this section, we perform an analytical investigation of the $BMAP/G/1$ queue under the N-policy, and obtain the system length distributions at departure and random epochs. Let us begin by establishing some preliminary results and introducing some relevant notations. Let $\mathcal{N}(u)$ denote the number of customers that arrive in the time interval $(0, u]$ and $\mathcal{J}(u)$ be the phase (state) of the underlying Markov chain corresponding to the BMAP at time u with state-space $\{i : 1 \leq i \leq n\}$. Then $\{(\mathcal{N}(u), \mathcal{J}(u))\}$ is a two-dimensional Markov process of BMAP with state-space $\{(k, i) : k \geq 0, 1 \leq i \leq n\}$. Let us consider $\{\mathbf{P}(k, u), k \geq 0, u \geq 0\}$ is a $n \times n$ matrix whose (i, j)th element is the conditional probability defined as

$$P_{ij}(k, u) = Pr\{\mathcal{N}(u) = k, \mathcal{J}(u) = j | \mathcal{N}(0) = 0, \mathcal{J}(0) = i\}, \quad 1 \leq i, j \leq n.$$

Using the property of BMAP and probability arguments, we have the following system of matrix differential-difference equations

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{P}(k, u) = \sum_{r=0}^{k} \mathbf{P}(r, u)\mathbf{D}_{k-r}, \quad k \geq 0, \tag{1}$$

with $\mathbf{P}(0, 0) = \mathbf{I}_n$ and $\mathbf{P}(k, 0) = \mathbf{0}$, for $k \geq 1$, where $\mathbf{I}_n$ is the identity matrix of order n.

Assume that $\mathbf{A}_k$ is the $n \times n$ matrix. The (i, j)th element of this matrix denotes the probability that k customers arrive in the course of the service time of the subsequent departure with batch arrival process being in phase j, provided a departure occurred with minimum one customer in the queue and the batch arrival process in phase i. Then, we have

$$\mathbf{A}_k = \int_0^\infty \mathbf{P}(k, u) \, \mathrm{d}B(u), \quad k \geq 0. \tag{2}$$

To evaluate the matrix $\mathbf{A}_k$ for arbitrary service-time distributions, we apply the uniformization argument given in Lucantoni [21] as

$$\mathbf{P}(k, u) = \sum_{r=0}^{\infty} e^{-\theta u} \frac{(\theta u)^r}{r!} \mathbf{X}_k^{(r)}, \quad k \geq 0, \quad u \geq 0,$$

where $\phi = \max_j [-D_0]_{jj}$, $1 \leq j \leq n$, and $\mathbf{X}_k^{(r)}$ are given by

$$\mathbf{X}_k^{(r+1)} = \mathbf{X}_k^{(r)} + \phi^{-1} \sum_{l=0}^{k} \mathbf{X}_l^{(r)} \mathbf{D}_{k-l}, \quad k \geq 0, \quad r \geq 0,$$

with $\mathbf{X}_0^{(0)} = \mathbf{I}_n$, and $\mathbf{X}_k^{(0)} = \mathbf{0}$, $k \geq 1$.

Now, substitute the value of $\mathbf{P}(k,u)$ in (2), we obtain

$$\mathbf{A}_k = \sum_{r=0}^{\infty} \eta_r \mathbf{X}_k^{(r)}, \quad k \geq 0, \tag{3}$$

where

$$\eta_r = \int_0^{\infty} e^{-\phi u} \frac{(\phi u)^r}{r!} \,\mathrm{d}B(u), \quad r \geq 0. \tag{4}$$

Now, multiplying both the sides of (4) by y^r and adding up all of the r values between 0 and ∞, we obtain

$$\begin{aligned}
\sum_{r=0}^{\infty} \eta_r y^r &= \int_0^{\infty} e^{-\phi u} \sum_{r=0}^{\infty} \frac{(\phi u y)^r}{r!} \,\mathrm{d}B(u) \\
&= \int_0^{\infty} e^{-(\phi - \phi y)u} \,\mathrm{d}B(u) \\
&= B^*(\phi - \phi y).
\end{aligned} \tag{5}$$

Since we assume that the L.-S.T. of service-time distribution is of rational form, *i.e.*, the degree of the polynomial of numerator should be less then or equal to the degree of the polynomial of denominator of the L.-S.T. of the service-time distribution, therefore we consider that $B^*(\phi - \phi y)$ will be in the following form

$$B^*(\phi - \phi y) = \frac{\sum_{r=0}^{m-1} \theta_r y^r}{\sum_{r=0}^{m} \varphi_r y^r}, \quad \text{with} \quad \varphi_0 = 1, \tag{6}$$

where m is the degree of denominator of $B^*(\phi - \phi y)$.

From (5) and (6), we obtain

$$\sum_{r=0}^{\infty} \eta_r y^r \sum_{r=0}^{m} \varphi_r y^r = \sum_{r=0}^{m-1} \theta_r y^r. \tag{7}$$

From both sides of (7), we extract the coefficients of y^r, where $r = 0, 1, 2, 3, \ldots,$ as

$$\eta_r = \begin{cases} \theta_0, & r = 0, \\ \theta_r - \sum_{i=0}^{r-1} \eta_i \varphi_{r-i}, & r = 1, 2, \ldots, m-1, \\ -\sum_{i=1}^{m} \eta_{r-i} \varphi_i, & r \geq m. \end{cases}$$

Using the above results of η_r, $r \geq 0$, in (3), we have determined $\mathbf{A}_k$ for $k \geq 0$.

Further, let $\mathbf{C}_k$ be the $n \times n$ matrix. The (i,j)th element represents the conditional probability that given a departure which leaves the system empty with batch arrival process in phase i, the next departure occurs with k customers left in the system and batch arrival process in phase j. Then, we have

$$\begin{aligned}
\mathbf{C}_k &= \int_0^{\infty} \sum_{l=0}^{N-1} \left[\int_0^{u} \mathbf{P}(l,x) \sum_{r=0}^{k-N+1} \mathbf{D}_{k-l-r+1} \mathbf{P}(r, u-x) b(u-x) \right] \mathrm{d}x \,\mathrm{d}u, \\
&= \sum_{l=0}^{N-1} \mathbf{K}(l) \sum_{r=0}^{k-N+1} \mathbf{D}_{k-l-r+1} \mathbf{A}_r, \quad k \geq N-1,
\end{aligned} \tag{8}$$

where $\mathbf{K}(l) = \int_0^{\infty} \mathbf{P}(l,u) \,\mathrm{d}u$, $l \geq 0$, according to the argument stated in Bank and Samanta [2], can be obtained as

$$\mathbf{K}(l) = \begin{cases} [-\mathbf{D}_0]^{-1}, & l = 0, \\ \sum_{r=0}^{l-1} \mathbf{K}(r) \mathbf{D}_{l-r} \mathbf{K}(0), & l \geq 1. \end{cases}$$

3.1. System length distribution at departure epoch

In this subsection, we determine the joint probability distribution of number of customers in the system and phase of the batch arrival process at departure epoch. To demonstrate the advantages and benefits of our simple sequential substitution approach discussed in Section 3.1.1 over the RG-factorization technique and roots method, we present here a complete investigation of the queue length distribution at departure epoch using these techniques, see Sections 3.1.2 and 3.1.3 for RG-factorization technique and roots method, respectively. Let $\{u_m^+ : m \geq 1\}$ represent the set of time epochs right after a customer's service completion. Let us define $\{(Y(u_m^+), J(u_m^+))\}$ is a Markov chain with state space $\{(k,i) : k \geq 0, 1 \leq i \leq n\}$, where $Y(u_m^+)$ is the number of customers in the system and $J(u_m^+)$ is the state of the UMC immediately after a customer departure at u_m^+. In the limiting case, let us define $\pi_i^+(k) = \lim_{m\to\infty} Pr\{Y(u_m^+) = k, J(u_m^+) = i\}, \quad k \geq 0,\ 1 \leq i \leq n$. Let $\boldsymbol{\pi}^+(k)$ be the row vector of order n whose ith element $\pi_i^+(k)$ represents the probability of k customers in the system and the batch arrival process in phase i right after a customer departure.

3.1.1. Our sequential substitution approach

We first derive the following system of vector-difference equations in the steady state by observing two consecutive departure epochs:

$$\boldsymbol{\pi}^+(k) = \sum_{r=0}^{k} \boldsymbol{\pi}^+(r+1)\mathbf{A}_{k-r}, \qquad 0 \leq k \leq N-2, \tag{9}$$

$$\boldsymbol{\pi}^+(k) = \boldsymbol{\pi}^+(0)\mathbf{C}_k + \sum_{r=0}^{k} \boldsymbol{\pi}^+(r+1)\mathbf{A}_{k-r}, \qquad k \geq N-1. \tag{10}$$

To compute the vectors $\boldsymbol{\pi}^+(k)$, $k \geq 0$, from (9) and (10), we first determine the vector $\boldsymbol{\pi}^+(0)$ by the following procedure. From the system of vector-difference equations (9) and (10), the vectors $\boldsymbol{\pi}^+(k)$, $k \geq 1$, can be rewritten in terms of $\boldsymbol{\pi}^+(0)$ using the following repeated substitution

$$\boldsymbol{\pi}^+(1) = \boldsymbol{\pi}^+(0)\mathbf{A}_0^{-1}, \tag{11}$$

$$\boldsymbol{\pi}^+(k) = \boldsymbol{\pi}^+(k-1)\mathbf{A}_0^{-1} - \sum_{r=1}^{k-1} \boldsymbol{\pi}^+(r)\mathbf{A}_{k-r}\mathbf{A}_0^{-1}, \qquad 2 \leq k \leq N-1, \tag{12}$$

$$\boldsymbol{\pi}^+(k) = \boldsymbol{\pi}^+(k-1)\mathbf{A}_0^{-1} - \boldsymbol{\pi}^+(0)\mathbf{C}_{k-1}\mathbf{A}_0^{-1} - \sum_{r=1}^{k-1} \boldsymbol{\pi}^+(r)\mathbf{A}_{k-r}\mathbf{A}_0^{-1}, \qquad k \geq N. \tag{13}$$

Further, multiplying (9) and (10) by proper powers of z and using $\boldsymbol{\Pi}^+(z) = \sum_{k=0}^{\infty} \boldsymbol{\pi}^+(k)z^k$, we have

$$\boldsymbol{\Pi}^+(z) = \boldsymbol{\pi}^+(0)\left[\sum_{l=0}^{N-1} \mathbf{K}(l) \sum_{m=N-l}^{\infty} \mathbf{D}_m z^{m+l} - \mathbf{I}_n\right]\mathbf{A}(z)[z\mathbf{I}_n - \mathbf{A}(z)]^{-1}, \tag{14}$$

where

$$[z\mathbf{I}_n - \mathbf{A}(z)]^{-1} = \frac{\operatorname{adj}[z\mathbf{I}_n - \mathbf{A}(z)]}{\det[z\mathbf{I}_n - \mathbf{A}(z)]}.$$

It is possible to rewrite the $\boldsymbol{\Pi}^+(z)$ stated in (14) as

$$\boldsymbol{\Pi}^+(z) = \left[\frac{U_1(z)}{U(z)}, \frac{U_2(z)}{U(z)}, \ldots, \frac{U_j(z)}{U(z)}, \ldots, \frac{U_n(z)}{U(z)}\right], \tag{15}$$

where $U(z) = \det[z\mathbf{I}_n - \mathbf{A}(z)]$ and $U_i(z)$ is the ith element of

$$\boldsymbol{\pi}^+(0)\left[\sum_{l=0}^{N-1}\mathbf{K}(l)\sum_{m=N-l}^{\infty}\mathbf{D}_m z^{m+l} - \mathbf{I}_n\right]\mathbf{A}(z)\,\mathrm{adj}\,[z\mathbf{I}_n - \mathbf{A}(z)].$$

Substitute $z = 1$ and apply the L'Hospital's rule in (15) as each component of $\mathbf{\Pi}^+(z)$ converges in $|z| \le 1$ and $U(z)$ has one zero at $z = 1$, we have

$$\sum_{k=0}^{\infty}\boldsymbol{\pi}^+(k) = \left[\frac{U_1'(1)}{U'(1)},\ldots,\frac{U_i'(1)}{U'(1)},\ldots,\frac{U_n'(1)}{U'(1)}\right], \tag{16}$$

where $U'(z) = \frac{\mathrm{d}U(z)}{\mathrm{d}z}$ and $U_i'(z) = \frac{\mathrm{d}U_i(z)}{\mathrm{d}z}$.

In order to determine $\boldsymbol{\pi}^+(0)$, we use the values of $\boldsymbol{\pi}^+(k)$, $k \ge 1$, from (11) to (13) in the left-hand side of (16) and compare component wise both the sides, we have a set of n linear equations in n unknowns $\pi_i^+(0)$, $1 \le i \le n$. Now, we must replace any one equation obtained from (16) with the normalization equation $\sum_{i=1}^{n} U_i'(1) = U'(1)$. This substitution enables us to solve n linearly independent equations for the n unique numerical values of $\pi_i^+(0)$, $1 \le i \le n$. The vector $\boldsymbol{\pi}^+(0)$ is thus determined. Once we have $\boldsymbol{\pi}^+(0)$, we can use repeated substitution to get the remaining vectors $\boldsymbol{\pi}^+(k)$, $k \ge 1$ from the recurrence relations (11) to (13).

Remark 1. The above procedure is valid only for continuous-time queueing models as $\mathbf{D}_0$ is invertible and therefore $\mathbf{A}_0$ is invertible but not for discrete-time queueing models as $\mathbf{D}_0$ may not be invertible.

3.1.2. RG-factorization technique

We describe the RG-factorization technique in order to compare it to our approach for determining the departure epoch probability for this queueing system. Now, using the RG-factorization technique, we want to find the solution of (9) and (10), for $\boldsymbol{\pi}^+(k)$, $k \ge 0$. It should be noted that the probability vectors $\boldsymbol{\pi}^+(k)$, $k \ge 0$, are evaluated using the $\boldsymbol{R}$- and $\mathbf{G}$-measures in a significant way. For further information on the RG-factorization technique, readers may read Li *et al.* [19] and Lee *et al.* [17]. Due to the $M/G/1$ type Markov chain is irreducible in the departure epoch, the matrix $\mathbf{G}$ exists as the smallest non-negative solution of

$$\mathbf{G} = \sum_{r=0}^{\infty}\mathbf{A}_r\mathbf{G}^r. \tag{17}$$

We simply need to obtain the stochastic matrix $\boldsymbol{G}$ before using the RG-factorization technique, which can be calculated using the above non-linear matrix equation (17), for more details on this topic, see Nishimura *et al.* [25], and Bini and Meini [4]. For the RG-factorization technique, the $\boldsymbol{G}$-measure is provided by

$$\boldsymbol{G}_r = \begin{cases}\boldsymbol{G}_1, & r = 1,\\ \boldsymbol{G}, & r \ge 2,\end{cases}$$

where

$$\boldsymbol{G}_1 = (\mathbf{I}_n - \boldsymbol{\Psi})^{-1}\mathbf{A}_0,$$

with the $\boldsymbol{U}$-measure

$$\boldsymbol{\Psi}_0 = \sum_{r=N-1}^{\infty}\mathbf{C}_r\boldsymbol{G}^{r-1}\boldsymbol{G}_1,$$

and, for $k \geq 1$,

$$\mathbf{\Psi} = \mathbf{\Psi}_k = \sum_{r=1}^{\infty} \mathbf{A}_r \boldsymbol{G}^{r-1}.$$

The $\boldsymbol{R}$-measure is given by

$$\boldsymbol{R}_{0,k} = \left[\sum_{j=k+1}^{\infty} \mathbf{C}_j \mathbf{G}^{j-k-1} \right] (\mathbf{I}_n - \mathbf{\Psi})^{-1}, \quad k \geq 1,$$

$$\boldsymbol{R}_{k} = \left[\sum_{j=k+1}^{\infty} \mathbf{A}_j \mathbf{G}^{j-k-1} \right] (\mathbf{I}_n - \mathbf{\Psi})^{-1}, \quad k \geq 1.$$

Thus, the stationary probability vectors $[\boldsymbol{\pi}^+(0), \boldsymbol{\pi}^+(1), \boldsymbol{\pi}^+(2), \ldots]$ are given by

$$\boldsymbol{\pi}^+(0) = \theta \mathbf{z}_0, \tag{18}$$

$$\boldsymbol{\pi}^+(k) = \boldsymbol{\pi}^+(0) \boldsymbol{R}_{0,k} + \sum_{j=1}^{k-1} \boldsymbol{\pi}^+(j) \boldsymbol{R}_{k-j}, \quad k \geq 1, \tag{19}$$

where $\mathbf{z}_0$ satisfies $\mathbf{z}_0 \mathbf{\Psi}_0 = \mathbf{z}_0$ with $\mathbf{z}_0 \mathbf{e} = 1$ and the scalar quantity θ is derived exclusively by the normalisation condition $\sum_{k=0}^{\infty} \boldsymbol{\pi}^+(k) \mathbf{e} = 1$.

3.1.3. Roots method

We describe the roots method which determines the departure epoch probability for this queueing system, before contrasting it with our approach. We generate the vectors $\boldsymbol{\pi}^+(k)$, $k \geq 0$, from (9) and (10) using the roots finding approach, which identifies the zeros of the so-called characteristic polynomial $U(z)$. Our initial goal is to correctly find $\boldsymbol{\pi}^+(0)$ from (9) and (10). We can see the interpretations given in Gail *et al.* [7] that $U(z)$ has exactly n zeros in $|z| \leq 1$ when $\mathbf{D}$ is irreducible and $\rho < 1$. We suppose that $U(z)$ has M zeros with absolute values greater than one (depending on the service time distribution). Assume that the zeros of $U(z)$ with absolute values less than one are $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$, whereas the zeros with absolute values larger than one are $\alpha_1, \alpha_2, \ldots, \alpha_M$. Now, since each element $\pi_m^+(z)$ of $\mathbf{\Pi}^+(z)$ is convergent in $|z| \leq 1$, we have

$$U_m(\gamma_i) = 0, \quad i = 1, 2, \ldots, n-1, \tag{20}$$

and using the normalize equation $\mathbf{\Pi}^+(1)\mathbf{e} = 1$, we have

$$1 = \lim_{z \to 1} \frac{\sum_{j=1}^{n} U_j(z)}{U(z)} = \frac{\sum_{j=1}^{n} U_j'(1)}{U'(1)}. \tag{21}$$

Equations (20) and (21) provide n independent simultaneous linear equations with decision variables $\pi_i^+(0)$, $1 \leq i \leq n$. The decision variables $\pi_i^+(0)$, $1 \leq i \leq n$, can be found by solving these n equations. This establishes the vector $\boldsymbol{\pi}^+(0)$. After determining $\boldsymbol{\pi}^+(0)$, our attention is now directed towards finding the vectors $\boldsymbol{\pi}^+(k)$, $k \geq 1$. Each element $\pi_m^+(z)$ of $\mathbf{\Pi}^+(z)$ is fully known polynomial once the value of $\boldsymbol{\pi}^+(0)$ has been substituted in (15). Let M_j, $j = 1, 2, \ldots, n$ and M_d be the degrees of $U_j(z)$ and $U(z)$, respectively. The jth element of $\mathbf{\Pi}^+(z)$ has been calculated using the partial-fraction method, and the result is

$$\pi_j^+(z) = \sum_{m=0}^{M_j - M_d} d_{m,j} z^m + \sum_{m=1}^{M} \frac{c_{m,j}}{\alpha_m - z}, \quad 1 \leq j \leq n, \tag{22}$$

where

$$c_{m,j} = -\frac{U_j(\alpha_m)}{U'(\alpha_m)}, \qquad m = 1, 2, \ldots, M,$$

$$d_{m,j} = \frac{\mathrm{d}^m}{m!\mathrm{d}z^m}\left[\frac{U_j(z)\prod_{m=1}^{M}(z-\alpha_m)}{U(z)} - \sum_{m=1}^{M}\frac{m!c_{m,j}}{(\alpha_m - z)^{m+1}}\right]_{z=0}, \qquad 0 \le m \le M_j - M_d.$$

By collecting the z^k coefficients from both the sides of (22), we have

$$\pi_j^+(k) = d_{k,j} + \sum_{m=1}^{M}\frac{c_{m,j}}{\alpha_m^{k+1}}, \qquad 1 \le j \le n, \quad 0 \le k \le M_j - M_d,$$

$$\pi_j^+(k) = \sum_{m=1}^{M}\frac{c_{m,j}}{\alpha_m^{k+1}}, \qquad 1 \le j \le n, \quad n \ge M_j - M_d + 1.$$

Hence, we have

$$\boldsymbol{\pi}^+(k) = \left[d_{k,1} + \sum_{m=1}^{M}\frac{c_{m,1}}{\alpha_m^{k+1}}, \ldots, d_{k,j} + \sum_{m=1}^{M}\frac{c_{m,j}}{\alpha_m^{k+1}}, \ldots, d_{k,n} + \sum_{m=1}^{M}\frac{c_{m,n}}{\alpha_m^{k+1}}\right], \qquad 0 \le k \le M_j - M_d,$$

$$\boldsymbol{\pi}^+(k) = \left[\sum_{m=1}^{M}\frac{c_{m,1}}{\alpha_m^{k+1}}, \ldots, \sum_{m=1}^{M}\frac{c_{m,j}}{\alpha_m^{k+1}}, \ldots, \sum_{m=1}^{M}\frac{c_{m,n}}{\alpha_m^{k+1}}\right], \qquad k \ge M_j - M_d + 1.$$

Hence, the CDF of system length at departure epoch is given by

$$\tau(k) = \sum_{i=0}^{k}\boldsymbol{\pi}^+(i), \quad k = 0, 1, 2, 3, \ldots.$$

3.2. System length distribution at random epoch

The remaining service time of the customer in service is used as a supplementary variable in the differential-difference equations which are developed to obtain the system length distribution at random epoch. The random variables mentioned below indicate the system's state at time t:

- Z_t = number of customers present in the system,
- J_t = state of the UMC of the $BMAP$,
- U_t = remaining service time of a customer in service,
- Λ_t = the state of the server, that is, $\Lambda_t = 1$ or 0 indicates that the server is active or inactive, respectively.

Let us define their joint probabilities, for $1 \le i \le n$, as

$$\omega_i(k) = \lim_{t\to\infty} Pr[Z_t = k, J_t = i, \Lambda_t = 0], \qquad 0 \le k \le N-1,$$

$$\pi_i(k,u)\,\mathrm{d}u = \lim_{t\to\infty} Pr[Z_t = k, J_t = i, u < U_t \le u + \mathrm{d}u, \Lambda_t = 1], \qquad k \ge 1, \quad u \ge 0.$$

Let us define $\boldsymbol{\pi}(k,u) = [\pi_1(k,u), \ldots, \pi_i(k,u), \ldots, \pi_n(k,u)]$, where $\pi_i(k,u)$ represents the joint probability of k customers in the system, i indicates the phase of the batch arrival process, and u reflects the amount of time a customer has left in service, when the server being busy. Similarly, $\boldsymbol{\omega}(k) = [\omega_1(k), \ldots, \omega_i(k), \ldots, \omega_n(k)]$, where $\omega_i(k)$ represents the probability of k customers in the system and the phase of the batch arrival process in i, when the server being idle. The steady-state system of vector differential-difference equations is now obtained by connecting the system's states at time u and $u+\,\mathrm{d}u$ as

$$0 = \boldsymbol{\omega}(0)\mathbf{D}_0 + \boldsymbol{\pi}(1,0), \tag{23}$$

$$0 = \sum_{r=0}^{k} \boldsymbol{\omega}(r)\mathbf{D}_{k-r}, \qquad 1 \le k \le N-1, \tag{24}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}\boldsymbol{\pi}(k,u) = \sum_{r=1}^{k} \boldsymbol{\pi}(r,u)\mathbf{D}_{k-r} + \boldsymbol{\pi}(k+1,0)b(u), \qquad 1 \le k \le N-1, \tag{25}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}\boldsymbol{\pi}(k,u) = \sum_{r=1}^{k} \boldsymbol{\pi}(r,u)\mathbf{D}_{k-r} + \boldsymbol{\pi}(k+1,0)b(u) + \sum_{r=0}^{N-1} \boldsymbol{\omega}(r)\mathbf{D}_{k-r}b(u), \qquad k \ge N. \tag{26}$$

Using $\boldsymbol{\pi}^*(k,s) = \int_0^\infty e^{-su}\boldsymbol{\pi}(k,u)\,\mathrm{d}u$ with $\mathrm{Re}(s) \ge 0$ in (25) and (26), we obtain

$$-s\boldsymbol{\pi}^*(k,s) + \boldsymbol{\pi}(k,0) = \sum_{r=1}^{k} \boldsymbol{\pi}^*(r,s)\mathbf{D}_{k-r} + \boldsymbol{\pi}(k+1,0)B^*(s), \qquad 1 \le k \le N-1, \tag{27}$$

$$-s\boldsymbol{\pi}^*(k,s) + \boldsymbol{\pi}(k,0) = \sum_{r=1}^{k} \boldsymbol{\pi}^*(r,s)\mathbf{D}_{k-r} + \boldsymbol{\pi}(k+1,0)B^*(s) + \sum_{r=0}^{N-1} \boldsymbol{\omega}(r)\mathbf{D}_{k-r}B^*(s), \qquad k \ge N. \tag{28}$$

After multiplying (23) and (24) by $\mathbf{e}$ and adding them together, we now get

$$\boldsymbol{\pi}(1,0)\mathbf{e} + \sum_{k=0}^{N-1} \boldsymbol{\omega}(k) \sum_{r=0}^{N-1-k} \mathbf{D}_r\mathbf{e} = 0. \tag{29}$$

Once more, after multiplying (27) and (28) by $\mathbf{e}$, putting them together, and using (29), we have

$$\sum_{k=1}^{\infty} \boldsymbol{\pi}^*(k,s)\mathbf{e} = \frac{1-B^*(s)}{s} \sum_{k=1}^{\infty} \boldsymbol{\pi}(k,0)\mathbf{e}. \tag{30}$$

Using the limit of $s \to 0$ in both sides of (30) and using the normalizing condition $\sum_{k=0}^{N-1} \boldsymbol{\omega}(k)\mathbf{e} + \sum_{k=1}^{\infty} \boldsymbol{\pi}(k)\mathbf{e} = 1$, we obtain

$$\sum_{k=1}^{\infty} \boldsymbol{\pi}(k,0)\mathbf{e} = \frac{1-\sum_{k=0}^{N-1} \boldsymbol{\omega}(k)\mathbf{e}}{E[B]}, \tag{31}$$

where $\boldsymbol{\pi}(k) = \boldsymbol{\pi}^*(k,0)$ is a row vector of order n and its ith element $\pi_i(k)$ indicates the probability of batch arrival process in phase i and k customers in the system, when the server being busy at random epoch.

The relation between $\boldsymbol{\pi}^+(k-1)$ and $\boldsymbol{\pi}(k,0)$, $k \ge 1$, is now developed as follows. Using the conditional probability argument, we come to the following

$$\begin{aligned}
\pi_j^+(k-1) &= \lim_{t\to\infty} Pr[Z_t = k,\ J_t = j,\ \Lambda_t = 1 \mid U_t = 0], \\
&= \lim_{t\to\infty} \frac{Pr[Z_t = k,\ J_t = j,\ U_t = 0,\ \Lambda_t = 1]}{Pr[U_t = 0]}, \\
&= \frac{\pi_j(k,0)}{\sum_{r=1}^{\infty} \boldsymbol{\pi}(r,0)\mathbf{e}}, \quad 1 \le j \le n, \quad k \ge 1.
\end{aligned}$$

We can write in vector notation as

$$\boldsymbol{\pi}^{+}(k-1) = E^{*}\boldsymbol{\pi}(k,0), \quad k \geq 1, \tag{32}$$

where $E^{*} = \frac{1}{\sum_{k=1}^{\infty}\boldsymbol{\pi}(k,0)\mathbf{e}}$ indicates the mean inter-departure time of customers, and hence $\frac{1}{E^{*}}$ represents the average number of departure per unit time.

Applying (32) in (23) and (24), after simplification, we have

$$\boldsymbol{\omega}(k) = \frac{1}{E^{*}}\boldsymbol{\pi}^{+}(0)\mathbf{K}(k), \quad 0 \leq k \leq N-1. \tag{33}$$

Now, post-multiplying (32) by $\mathbf{e}$, adding them and using (31), we have

$$\frac{E[B]}{E^{*}} = 1 - \sum_{k=0}^{N-1}\boldsymbol{\omega}(k)\mathbf{e}.$$

Using (33) in the above yields

$$E^{*} = E[B] + \boldsymbol{\pi}^{+}(0)\sum_{k=0}^{N-1}\mathbf{K}(k)\mathbf{e}. \tag{34}$$

Substitute $s = 0$ in (27) and (28), using (32), we have

$$\begin{aligned} \boldsymbol{\pi}(k) &= \sum_{r=1}^{k-1}\boldsymbol{\pi}(r)\mathbf{D}_{k-r}(-\mathbf{D}_0)^{-1} + \frac{1}{E^{*}}\left[\boldsymbol{\pi}^{+}(k) - \boldsymbol{\pi}^{+}(k-1)\right](-\mathbf{D}_0)^{-1}, && 1 \leq k \leq N-1, \\ \boldsymbol{\pi}(k) &= \sum_{r=1}^{k-1}\boldsymbol{\pi}(r)\mathbf{D}_{k-r}(-\mathbf{D}_0)^{-1} + \frac{1}{E^{*}}\left[\boldsymbol{\pi}^{+}(k) - \boldsymbol{\pi}^{+}(k-1)\right](-\mathbf{D}_0)^{-1} \\ &\quad + \sum_{r=0}^{N-1}\boldsymbol{\omega}(r)\mathbf{D}_{k-r}(-\mathbf{D}_0)^{-1}, && k \geq N. \end{aligned}$$

Hence, the CDF of system length at random epoch is given by

$$\Theta(k) = \begin{cases} \boldsymbol{\omega}(0) & \text{if } k = 0, \\ \boldsymbol{\omega}(0) + \sum_{i=1}^{k}(\boldsymbol{\omega}(i) + \boldsymbol{\pi}(i)) & \text{if } 1 \leq k \leq N-1, \\ \sum_{i=0}^{N-1}(\boldsymbol{\omega}(i) + \boldsymbol{\pi}(i)) + \sum_{i=N}^{k}\boldsymbol{\pi}(i) & \text{if } k \geq N. \end{cases}$$

The average number of customers in the system at random epoch is $L_s = \sum_{k=1}^{N-1}k\boldsymbol{\omega}(k)\mathbf{e} + \sum_{k=1}^{\infty}k\boldsymbol{\pi}(k)\mathbf{e}$. Let W_s be the mean sojourn time in the system of an arbitrary customer in a batch. Then, by Little's rule, we have $W_s = \frac{L_s}{\lambda^{*}}$.

4. Sojourn time distribution

The sojourn time of an arbitrary customer in a batch under the first-come-first-served queueing discipline is obtained in this section. The sojourn time is the amount of time from the time the customer enters the system until they leave. The number of customers in an incoming batch is generally finite size in real-world scenarios. So, M_b is taken into consideration as the BMAP's maximum batch size. In order to obtain the sojourn time distribution of an arbitrary customer, we consider three cases:

Case 1. Let $\mathbf{W}_1(s)$ be the L.-S.T. of the sojourn time distribution of an arbitrary customer of a batch which comes during an idle period of the system, when k $(0 \leq k \leq N-1)$ customers are already present in the queue and arriving batch size is at most $(N-1-k)$. In this scenario, the idle time is not terminated and a random customer from the arrival batch must wait until there are at least N customers in the queue before the idle period can be terminated. Therefore, the $\mathbf{W}_1(s)$ is given by

$$\begin{aligned}\mathbf{W}_1(s) &= \frac{1}{E^*}\boldsymbol{\pi}^+(0)\sum_{k=0}^{N-2}\mathbf{K}(k)\sum_{r=1}^{N-1-k}\frac{\mathbf{D}_r}{\lambda^*}\sum_{i=0}^{N-1-r-k}\mathbf{P}^*(i,s)\times\sum_{j=N-k-r-i}^{M_b}\mathbf{D}_j\sum_{l=1}^{r}[B^*(s)]^{k+l},\\ &= \frac{1}{E^*}\boldsymbol{\pi}^+(0)\sum_{k=0}^{N-2}\mathbf{K}(k)\sum_{r=1}^{N-1-k}\mathbf{D}_r\sum_{i=0}^{N-1-r-k}\mathbf{P}^*(i,s)\\ &\quad\times\sum_{j=N-k-r-i}^{M_b}\mathbf{D}_j\left(\frac{(B^*(s))^{k+1}-(B^*(s))^{k+r+1}}{\lambda^*(1-B^*(s))}\right),\end{aligned} \tag{35}$$

where $\mathbf{P}^*(k,s) = \int_0^\infty e^{-st}\mathbf{P}(k,t)\,\mathrm{d}t$, $k \geq 0$, with $\mathrm{Re}(s) \geq 0$, is given by

$$\begin{aligned}\mathbf{P}^*(0,s) &= (s\mathbf{I}_n - \mathbf{D}_0)^{-1},\\ \mathbf{P}^*(k,s) &= \sum_{i=0}^{k-1}\mathbf{P}^*(i,s)\mathbf{D}_{k-i}\mathbf{P}^*(0,s), \quad k \geq 1.\end{aligned}$$

To deal with different types of service time distributions, the L.-S.T. $B^*(s)$ can be represented in its rational form as $B^*(s) = \frac{Q(s)}{V(s)}$ in which both $Q(s)$ and $V(s)$ are polynomials in s of finite degrees, say Y_1 and Y_2 $(Y_2 \geq Y_1)$, respectively. In order to efficiently derive the sojourn time distribution from (35), we express the jth component $W_{1,j}(s)$ of $\mathbf{W}_1(s)$ in its rational form as

$$W_{1,j}(s) = \frac{F_{1,j}(s)}{H_1(s)}, \quad j = 1, 2, \ldots, n, \tag{36}$$

where

$$H_1(s) = \frac{1}{\upsilon_1}(V(s)-Q(s))(V(s))^{N-1}(\det[s\mathbf{I}_n - \mathbf{D}_0])^{N-1}$$

with υ_1 is the leading coefficient of the polynomial $(V(s)-Q(s))(V(s))^{N-1}(\det[s\mathbf{I}_n - \mathbf{D}_0])^{N-1}$, and $F_{1,j}(s)$ is the jth component of the vector $\mathbf{W}_1(s)H_1(s)$.

As each component $W_{1,j}(s)$ given in (36) is convergent for $\mathrm{Re}(s) \geq 0$, the zeros of $H_1(s)$ whose real part is grater than or equal to zero must be the zeros of $F_{1,j}(s)$ and they are automatically cancelled out from it. Thus, we have focused only the zeros with negative real part. Since $\frac{Q(s)}{V(s)}$ is a L.-S.T. of service time distribution, $1 - \frac{Q(s)}{V(s)}$ has exactly one zero at $s = 0$ and no other zeros with positive real part. This implies that $V(s)-Q(s) = 0$ has $Y_2 - 1$ roots with negative real part and we call these roots as ν_l, $l = 1, 2, \ldots Y_2 - 1$. By the same logic, $V(s)$ has Y_2 zeros with negative real part and therefore, the $V(s)^{N-1}$ has Y_2 zeros with negative real part, say ξ_k, $k = 1, 2, \ldots Y_2$ with each of multiplicity $N-1$. Using Rouché's theorem, it can be shown that $(\det[s\mathbf{I}_n - \mathbf{D}_0])^{N-1} = 0$ has n roots each with multiplicity $N-1$, and they have negative real part and no other roots with non-negative real part. We call these n roots with negative real part as β_i, $i = 1, 2, \ldots, n$, with each multiplicity $N-1$. Now, the jth component $W_{1,j}(s)$ can be rewritten into its partial-fractions as

$$W_{1,j}(s) = K_{1,j} + \sum_{i=1}^{Y_2-1}\frac{\Omega_{1,ij}}{s-\nu_i} + \sum_{i=1}^{N-1}\sum_{k=1}^{n}\frac{\Gamma_{1,ikj}}{(s-\beta_k)^i} + \sum_{i=1}^{N-1}\sum_{k=1}^{Y_2}\frac{\Delta_{1,ikj}}{(s-\xi_k)^i}, \tag{37}$$

where

$$\Omega_{1,ij} = \left[\frac{F_{1,j}(\nu_i)}{\prod\limits_{r=1;r\neq i}^{Y_2-1}(\nu_i-\nu_r)\prod\limits_{r=1}^{n}(\nu_i-\beta_r)^{N-1}\prod\limits_{r=1}^{Y_2}(\nu_i-\xi_r)^{N-1}}\right], \quad i=1,2,\ldots,Y_2-1,$$

$$\Gamma_{1,ikj} = \frac{1}{(N-1-i)!}\left[\frac{\mathrm{d}^{N-1-i}}{\mathrm{d}s^{N-1-i}}\left(\frac{F_{1,j}(s)}{\prod\limits_{r=1}^{Y_2-1}(s-\nu_r)\prod\limits_{r=1;r\neq k}^{n}(s-\beta_r)^{N-1}\prod\limits_{r=1}^{Y_2}(s-\xi_r)^{N-1}}\right)\right]_{s=\beta_k},$$
$$i=1,2,\ldots,N-1;\ k=1,2,\ldots,n,$$

$$\Delta_{1,ikj} = \frac{1}{(N-1-i)!}\left[\frac{\mathrm{d}^{N-1-i}}{\mathrm{d}s^{N-1-i}}\left(\frac{F_{1,j}(s)}{\prod\limits_{r=1}^{Y_2-1}(s-\nu_r)\prod\limits_{r=1}^{n}(s-\beta_r)^{N-1}\prod\limits_{r=1;r\neq k}^{Y_2}(s-\xi_r)^{N-1}}\right)\right]_{s=\xi_k},$$
$$i=1,2,\ldots,N-1;\ k=1,2,\ldots,Y_2,$$

$$K_{1,j} = \left[\frac{F'_{1,j}(0)}{H'_1(0)} + \sum_{i=1}^{Y_2-1}\frac{\Omega_{1,ij}}{\nu_i} - \sum_{i=1}^{N-1}\sum_{k=1}^{n}\frac{\Gamma_{1,kij}}{(-1)^i\beta_k^i} - \sum_{i=1}^{N-1}\sum_{k=1}^{Y_2}\frac{\Delta_{1,kij}}{(-1)^i\xi_k^i}\right],$$

with $F'_{1,j}(0)$ and $H'_1(0)$ are the first order derivatives of $F_{1,j}(s)$ and $H_1(s)$ at $s=0$, respectively.
Let $\mathbf{w}_1(t) = [w_{1,1}(t),\ldots,w_{1,j}(t),\ldots, w_{1,n}(t)]$, $t\geq 0$, denote the inverse Laplace transform of $\mathbf{W}_1(s)$. Taking the inverse L.-S.T. of the expression (37), we obtain

$$w_{1,j}(t) = K_{1,j}\delta(t) + \sum_{i=1}^{Y_2-1}\Omega_{1,ij}e^{\nu_i t} + \sum_{i=1}^{N-1}\sum_{k=1}^{n}\Gamma_{1,ikj}\frac{1}{(i-1)!}t^{i-1}e^{\beta_k t}$$
$$+\sum_{i=1}^{N-1}\sum_{k=1}^{Y_2}\Delta_{1,ikj}\frac{1}{(i-1)!}t^{i-1}e^{\xi_k t}, \quad t\geq 0, \tag{38}$$

where $\delta(t)$ is the Dirac delta function.

Case 2. Let $\mathbf{W}_2(s)$ be the L.-S.T. of the sojourn time distribution of an arbitrary customer of a batch which comes in an idle period of the system, when k customers are already in the queue and arriving batch size is at least $(N-k)$. In this scenario, the idle period ends since there are at least N customers in the queue. Therefore, the $\mathbf{W}_2(s)$ is given by

$$\mathbf{W}_2(s) = \frac{1}{E^*}\boldsymbol{\pi}^+(0)\sum_{k=0}^{N-1}\mathbf{K}(k)\sum_{r=N-k}^{M_b}\frac{\mathbf{D}_r}{\lambda^*}\sum_{l=1}^{r}[B^*(s)]^{k+l},$$
$$= \frac{1}{E^*}\boldsymbol{\pi}^+(0)\sum_{k=0}^{N-1}\mathbf{K}(k)\sum_{r=N-k}^{M_b}\mathbf{D}_r\left(\frac{(B^*(s))^{k+1}-(B^*(s))^{k+r+1}}{\lambda^*(1-B^*(s))}\right). \tag{39}$$

We now express the jth component $W_{2,j}(s)$ of $\mathbf{W}_2(s)$ in its rational form as

$$W_{2,j}(s) = \frac{F_{2,j}(s)}{H_2(s)}, \quad j=1,2,\ldots,n, \tag{40}$$

where $H_2(s) = \frac{1}{\upsilon_2}(V(s)-Q(s))(V(s))^{N-1+M_b}$ with υ_2 is the leading coefficient of the polynomial $(V(s)-Q(s))(V(s))^{N-1+M_b}$, and $F_{2,j}(s)$ is the jth element of the vector $\mathbf{W}_2(s)H_2(s)$. By the same arguments, $(V(s))^{N-1+M_b}$ has Y_2 zeros with negative real part, say $\xi_l,\ l=1,2,\dots Y_2$ with each of multiplicity $N-1+M_b$. Therefore, the jth element $W_{2,j}(s)$ can be rewritten into its partial-fractions as

$$W_{2,j}(s) = K_{2,j} + \sum_{i=1}^{Y_2-1}\frac{\Omega_{2,ij}}{s-\nu_i} + \sum_{i=1}^{N-1+M_b}\sum_{k=1}^{Y_2}\frac{\Delta_{2,ikj}}{(s-\xi_k)^i}, \tag{41}$$

where

$$\Omega_{2,ij} = \left[\frac{F_{2,j}(\nu_i)}{\prod_{r=1;r\neq i}^{Y_2-1}(\nu_i-\nu_r)\prod_{r=1}^{Y_2}(\nu_i-\xi_r)^{N-1+M_b}}\right], \quad i=1,2,\dots,Y_2-1,$$

$$\Delta_{2,ikj} = \frac{1}{(M_b+N-1-i)!}\left[\frac{\mathrm{d}^{M_b+N-1-i}}{\mathrm{d}s^{M_b+N-1-i}}\left(\frac{F_{2,j}(s)}{\prod_{r=1}^{Y_2-1}(s-\nu_r)\prod_{r=1;r\neq k}^{Y_2}(s-\xi_r)^{M_b+N-1}}\right)\right]_{s=\xi_l},$$
$$i=1,2,\dots,N-1+M_b;\ l=1,2,\dots,Y_2,$$

$$K_{2,j} = \left[\frac{F'_{2,j}(0)}{H'_2(0)} + \sum_{i=1}^{Y_2-1}\frac{\Omega_{2,ij}}{\nu_i} - \sum_{i=1}^{N-1+M_b}\sum_{k=1}^{Y_2}\frac{\Delta_{2,kij}}{(-1)^i\xi_k^i}\right]$$

with $F'_{2,j}(0)$ and $H'_2(0)$ are the first order derivatives of $F_{2,j}(s)$ and $H_2(s)$ at $s=0$, respectively.
Let $\mathbf{w}_2(t) = [w_{2,1}(t),\dots,w_{2,j}(t),\dots,w_{2,n}(t)]$, $t\geq 0$, denote the inverse Laplace transform of $\mathbf{W}_2(s)$. Taking the inverse L.-S.T. of the expression (41), we obtain

$$W_{2,j}(t) = K_{2,j}\delta(t) + \sum_{i=1}^{Y_2-1}\Omega_{2,ij}e^{\nu_i t} + \sum_{i=1}^{N-1+M_b}\sum_{k=1}^{Y_2}\Delta_{2,ikj}\frac{1}{(i-1)!}t^{i-1}e^{\xi_k t}, \qquad t\geq 0. \tag{42}$$

Case 3. Let $\mathbf{W}_3(s)$ be the L.-S.T. of the sojourn time distribution of an arbitrary customer of a batch which comes in a busy period of the system. In order to achieve this goal, we must consider the amount of time the customer still has left in his service during a busy period. Therefore, the $\mathbf{W}_3(s)$ is given by

$$\begin{aligned}\mathbf{W}_3(s) &= \sum_{k=1}^{\infty}\boldsymbol{\pi}^*(k,s)\sum_{r=1}^{M_b}\frac{\mathbf{D}_r}{\lambda^*}\sum_{l=1}^{r}[B^*(s)]^{k+l-1},\\ &= \frac{\sum_{k=1}^{\infty}\boldsymbol{\pi}^*(k,s)[B^*(s)]^k[\mathbf{D}-\mathbf{D}(B^*(s))]}{\lambda^*(1-B^*(s))}.\end{aligned} \tag{43}$$

Let us define

$$\Phi^*(z,s) = \sum_{k=1}^{\infty}\boldsymbol{\pi}^*(k,s)z^k, \quad \text{and} \quad \Psi^*(z,0) = \sum_{k=1}^{\infty}\boldsymbol{\pi}(k,0)z^k, \quad \text{for } \mid z\mid\leq 1.$$

By multiplying z^k on both the sides of (27) and (28), and then summing them over k from 1 to ∞, we obtain

$$\Phi^*(z,s)[-s\mathbf{I}_n-\mathbf{D}(z)] = \left[\frac{1}{E^*}\boldsymbol{\pi}^+(0)\sum_{r=0}^{N-1}\mathbf{K}(r)\sum_{k=N}^{M_b+r}\mathbf{D}_{k-r}z^k - \boldsymbol{\pi}(1,0)\right]B^*(s) + \Psi^*(z,0)\left[\frac{1}{z}B^*(s)-1\right]. \tag{44}$$

Substitute $z = B^*(s)$ in (44) and using (34), we have

$$\Phi^*(B^*(s), s) = \left[\frac{1}{E^*}\boldsymbol{\pi}^+(0)\sum_{r=0}^{N-1}\mathbf{K}(r)\sum_{k=N}^{M_b+r}\mathbf{D}_{k-r}[B^*(s)]^k - \frac{1}{E^*}\boldsymbol{\pi}^+(0)\right] \times B^*(s)[-s\mathbf{I}_n - \mathbf{D}(B^*(s))]^{-1}. \quad (45)$$

Using (45) in (43), we obtain

$$\mathbf{W}_3(s) = \frac{1}{E^*}\boldsymbol{\pi}^+(0)\left[\sum_{r=0}^{N-1}\mathbf{K}(r)\sum_{k=N}^{M_b+r}\mathbf{D}_{k-r}[B^*(s)]^k - \mathbf{I}_n\right] \times \frac{B^*(s)[-s\mathbf{I}_n - \mathbf{D}(B^*(s))]^{-1}[\mathbf{D} - \mathbf{D}(B^*(s))]}{\lambda^*(1 - B^*(s))}. \quad (46)$$

We now express the jth component $W_{3,j}(s)$ of $\mathbf{W}_3(s)$ in its rational form as

$$W_{3,j}(s) = \frac{F_{3,j}(s)}{H_3(s)}, \quad j = 1, 2, \ldots, n, \quad (47)$$

where $H_3(s) = \frac{1}{\upsilon_3}(V(s) - Q(s))(V(s))^{M_b}\det\left[-s\mathbf{I}_n - \mathbf{D}\left(\frac{Q(s)}{V(s)}\right)\right]$ with υ_3 is the leading coefficient of the polynomial $(V(s) - Q(s))(V(s))^{M_b}\det\left[-s\mathbf{I}_n - \mathbf{D}\left(\frac{Q(s)}{V(s)}\right)\right]$, and $F_{3,j}(s)$ is the jth component of the vector $\mathbf{W}_3(s)H_3(s)$.

It can be shown with the argument given in Gail *et al.* [7] that when $\mathbf{D}$ is irreducible and $\rho < 1$, then $\det[-s\mathbf{I}_n - \mathbf{D}(B^*(s))] = 0$ has exactly $(n-1)$ roots with positive real part and one root at $s = 0$. Except these n roots with non-negative real part, let $\det[-s\mathbf{I}_n - \mathbf{D}(B^*(s))] = 0$ has Y_3 roots each with negative real part, viz., χ_i, $i = 1, 2, \ldots, Y_3$. By the same sense, $V(s)^{M_b}$ has Y_2 zeros with negative real part, say ξ_l, $l = 1, 2, \ldots Y_2$ with each of multiplicity M_b. Therefore, the jth component $W_{3,j}(s)$ can be rewritten into its partial-fractions as

$$W_{3,j}(s) = K_{3,j} + \sum_{i=1}^{Y_3}\frac{C_{ij}}{s - \chi_i} + \sum_{i=1}^{Y_2-1}\frac{\Omega_{3,ij}}{s - \nu_i} + \sum_{i=1}^{M_b+N-1}\sum_{k=1}^{Y_2}\frac{\Delta_{3,ikj}}{(s - \xi_k)^i}, \quad (48)$$

where

$$C_{ij} = \left[\frac{F_{3,j}(\chi_i)}{\prod_{r=1;i\neq r}^{Y_3}(\chi_i - \chi_r)\prod_{r=1}^{Y_2-1}(\chi_i - \nu_r)\prod_{r=1}^{Y_2}(\chi_i - \xi_r)^{M_b}}\right], \qquad i = 1, 2, \ldots, Y_3,$$

$$\Omega_{3,ij} = \left[\frac{F_{3,j}(\nu_i)}{\prod_{r=1}^{Y_3}(\nu_i - \chi_r)\prod_{r=1;r\neq i}^{Y_2-1}(\nu_i - \nu_r)\prod_{r=1}^{Y_2}(\nu_i - \xi_r)^{M_b}}\right], \qquad i = 1, 2, \ldots, Y_2 - 1,$$

$$\Delta_{3,ikj} = \frac{1}{(M_b - i)!}\left[\frac{\mathrm{d}^{M_b-i}}{\mathrm{d}s^{M_b-i}}\left(\frac{F_{3,j}(s)}{\prod_{r=1}^{Y_3}(s - \chi_r)\prod_{r=1}^{Y_2-1}(s - \nu_r)\prod_{r=1;r\neq k}^{Y_2}(s - \xi_r)^{M_b}}\right)\right]_{s=\xi_l},$$
$$i = 1, 2, \ldots, M_b;\ l = 1, 2, \ldots, Y_2,$$

$$K_{3,j} = \left[\frac{F''_{3,j}(0)}{H''_3(0)} + \sum_{k=i}^{Y_3}\frac{C_{ij}}{\chi_i} + \sum_{i=1}^{Y_2-1}\frac{\Omega_{3,ij}}{\nu_i} - \sum_{i=1}^{M_b}\sum_{k=1}^{Y_2}\frac{\Delta_{3,kij}}{(-1)^i\xi_k^i}\right],$$

with $F''_{3,j}(0)$ and $H''_3(0)$ are the second order derivatives of $F_{3,j}(s)$ and $H_3(s)$ at $s = 0$, respectively.
Let $\mathbf{w}_3(t) = [w_{3,1}(t), \ldots, w_{3,j}(t), \ldots, w_{3,n}(t)]$, $t \geq 0$, denote the inverse Laplace transform of $\mathbf{W}_3(s)$. Taking the inverse L.-S.T. of the expression (48), we obtain

$$W_{3,j}(t) = K_{3,j}\delta(t) + \sum_{i=1}^{Y_3} C_{ij}e^{\chi_i t} + \sum_{i=1}^{Y_2-1} \Omega_{3,ij}e^{\nu_i t} + \sum_{i=1}^{M_b}\sum_{k=1}^{Y_2} \Delta_{3,ikj}\frac{1}{(i-1)!}t^{i-1}e^{\xi_k t}, t \geq 0. \tag{49}$$

Let $\mathbf{W}(s) = [W_1(s), \ldots, W_j(s), \ldots, W_n(s)]$ denote the L.-S.T. of the sojourn time distribution $\mathbf{w}(t) = [w_1(t), \ldots, w_j(t), \ldots, w_n(t)]$, $t \geq 0$, of an arbitrary customer in a batch, where $w_j(t)$ represents the joint probability that the sojourn time of an arbitrary customer in a batch is t time units and the batch arrival process being in phase j. Therefore, we have $\mathbf{W}(s) = \mathbf{W}_1(s) + \mathbf{W}_2(s) + \mathbf{W}_3(s)$ and $\mathbf{w}(t) = \mathbf{w}_1(t) + \mathbf{w}_2(t) + \mathbf{w}_3(t)$, $t \geq 0$.
Further, let $W = \int_0^\infty t\mathbf{w}(t)\mathbf{e}\,\mathrm{d}t$ denote the mean sojourn time in the system of an arbitrary customer in a batch. Then, we have $W = -\sum_{i=1}^3 \sum_{j=1}^n \frac{\mathrm{d}}{\mathrm{d}s}W_{i,j}(s)\mid_{s=0}$.

5. Optimal N-policy

A cost optimization analysis is an important characteristic from an economic perspective to design the system. The system managers are interested to estimate the ideal value of N in minimizing the expected operating cost per unit time. It is necessary to build the expected busy and idle periods in order to establish the expected operating cost function $\Lambda(N)$. Following are descriptions of the system's busy and idle periods:

(1) **Idle period** ($\mathcal{I}_N$): the interval between the system becoming vacant and when at least N users have built up in the queue is known as an idle period per cycle.
(2) **Busy period** ($\mathcal{B}_N$): the amount of time between when the number of users arriving during the idle period exceeds N and when the system is fully empty is known as a busy period per cycle.

Let $E[\mathcal{I}_N]$ and $E[\mathcal{B}_N]$ represent the expected length of idle period and busy period, respectively. The expected idle time for the $BMAP/G/1$ queueing system with the N-policy is given by

$$E[\mathcal{I}_N] = \frac{N}{\lambda^*}\cdot$$

The fraction of time that the server is busy in the long term is given by

$$\frac{E[\mathcal{B}_N]}{E[\mathcal{I}_N] + E[\mathcal{B}_N]} = \rho,$$

which yields

$$E[\mathcal{B}_N] = \frac{N}{\mu(1-\rho)}\cdot$$

As N is the main decision variable, we now build an estimated cost function per unit of time. To minimize the cost function, we want to identify the optimal N, say $N^\star$. We take into account the following cost factors:

C_h : cost of holding per customer in the queue per unit;
C_i : maintenance cost per idle period;
C_s : setup cost per busy period.

Under the linear cost structure, the long-run expected cost function per unit time is equal to

$$\Lambda(N) = \mathrm{C_h}L_s + \mathrm{C_i}\frac{1}{E[\mathcal{I}_N]} + \mathrm{C_s}\frac{1}{E[\mathcal{B}_N]},$$

$$= \mathrm{C_h}\left[\sum_{k=1}^{N-1} k\boldsymbol{\omega}(k)\mathbf{e} + \sum_{k=1}^{\infty} k\boldsymbol{\pi}(k)\mathbf{e}\right] + \mathrm{C_i}\frac{\lambda^*}{N} + \mathrm{C_s}\frac{\mu(1-\rho)}{N}\cdot \tag{50}$$

Due to the extreme complexity of the expected cost function shown in (50), we are unable to precisely derive the formula for N from $\Lambda(N)$. Determining the analytical solution for the ideal value of $N^\star$ is therefore a difficult task. We employ the direct search approach to determine the ideal value of N for the unconstrained expected cost function $\Lambda(N)$.

6. Numerical results

We provide here a few numerical findings for various BMAP arrival matrices and service time distributions based on the suggested analytical approach addressed in earlier sections to confirm the accuracy of our analytical results. The numerical work is performed for both high and low values of the model parameters. The model parameter values are selected in order to meet the stability constraint $\rho < 1$. Self-explanatory tables and graphs are employed to display some numerical information, which makes it easy to understand how system parameters behave. The numerical results provided in this study are expected to have the benefit to theorists and practitioners who wish to compare their findings with ours whenever they employ other techniques. All the digital operations are performed using MAPLE software on a PC having configurations as Intel(R) Core i7-6500U CPU processor @ 2.50 GHz with 8 GB of RAM in Windows 10 environment. We perform all of our numerical computations with the best precision, however, they are all provided here rounded to eight decimal places. The system length distributions at two different epochs (departure and random epochs) and the sojourn time distribution of a random customer in a batch are shown in Tables 1–3. The mean system length (L_s) and the mean sojourn time in the system (W_s) of a random customer are also presented at the bottom of the Table 2. One must also note that the mean sojourn time in the system (W_s) of a random customer of an arrival batch obtained from the Little's law matches with the one obtained from the sojourn time distribution, see the bottom of the Tables 2 and 3. One can observed in Table 2 that $\overline{\boldsymbol{\pi}} = \sum_{n=0}^{N-1} \boldsymbol{\omega}(n) + \sum_{n=0}^{\infty} \boldsymbol{\pi}(n)$, as desired. It may be noted here that $\sum_{n=0}^{N-1} \boldsymbol{\omega}(n)\mathbf{e} = 1-\rho$, which represents the probability that the server is idle. These facts confirm the correctness of our analytical and numerical results.

Example 1. For the $BMAP/ME/1$ queue under the N-policy with $N = 10$, the system length distributions at different epochs and the sojourn time distribution of a customer in a batch are provided in Tables 1–3, where matrix-exponential distribution is represented by the abbreviation ME. The ME service time distribution having representation $ME(\boldsymbol{\beta}, \mathbf{S}, \boldsymbol{\eta})$ as

$$\boldsymbol{\beta} = \begin{bmatrix} 1+16\pi^2, & 0, & 0 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1-16\pi^2 & -2-16\pi^2 & -2 \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

whose cumulative distribution function and probability density function, respectively, are given by $B(u) = 1+\boldsymbol{\beta}e^{\mathbf{S}u}\mathbf{S}^{-1}\boldsymbol{\eta}$ and $b(u) = \boldsymbol{\beta}e^{\mathbf{S}u}\boldsymbol{\eta}$, $u \geq 0$. Thus, $\mu = 0.99374662$ is obtained. We consider the parameter matrices of the BMAP as follows

$$\mathbf{D}_0 = \begin{bmatrix} -0.426 & 0.069 & 0.014 & 0.000 \\ 0.122 & -0.426 & 0.000 & 0.202 \\ 0.230 & 0.000 & -0.499 & 0.121 \\ 0.234 & 0.006 & 0.000 & -0.443 \end{bmatrix}, \qquad \mathbf{D}_1 = \begin{bmatrix} 0.000 & 0.005 & 0.007 & 0.012 \\ 0.015 & 0.000 & 0.000 & 0.034 \\ 0.000 & 0.051 & 0.018 & 0.000 \\ 0.029 & 0.000 & 0.036 & 0.027 \end{bmatrix},$$

$$\mathbf{D}_2 = \begin{bmatrix} 0.010 & 0.000 & 0.014 & 0.022 \\ 0.003 & 0.011 & 0.000 & 0.008 \\ 0.000 & 0.001 & 0.002 & 0.054 \\ 0.021 & 0.033 & 0.000 & 0.024 \end{bmatrix}, \qquad \mathbf{D}_3 = \begin{bmatrix} 0.000 & 0.011 & 0.032 & 0.014 \\ 0.003 & 0.005 & 0.000 & 0.009 \\ 0.008 & 0.000 & 0.004 & 0.001 \\ 0.000 & 0.007 & 0.003 & 0.000 \end{bmatrix},$$

$$\mathbf{D}_4 = \begin{bmatrix} 0.006 & 0.000 & 0.010 & 0.080 \\ 0.000 & 0.001 & 0.002 & 0.000 \\ 0.002 & 0.003 & 0.000 & 0.000 \\ 0.007 & 0.000 & 0.000 & 0.009 \end{bmatrix}, \qquad \mathbf{D}_6 = \begin{bmatrix} 0.020 & 0.080 & 0.020 & 0.000 \\ 0.000 & 0.001 & 0.003 & 0.007 \\ 0.003 & 0.000 & 0.001 & 0.000 \\ 0.000 & 0.002 & 0.000 & 0.005 \end{bmatrix},$$

with $\mathbf{D}_r = \mathbf{0},\ r \in \mathbb{N} \cup \{0\} - \{0,1,2,3,4,6\}$, where $\mathbb{N}$ is the set of natural numbers. Note that the input matrices for this BMAP are taken in which the maximum batch size is 6. This yields

$$\overline{\boldsymbol{\pi}} = \begin{bmatrix} 0.37573133 & 0.20328023 & 0.10503097 & 0.31595746 \end{bmatrix},$$

with $\lambda^* = 0.71703640$. Therefore, the utilization factor is $\rho = 0.72154852$. The matrix-generating function $\mathbf{A}(z)$ for the matrix-exponential distribution can be obtained from (2) by using (1) as

$$\begin{aligned} \mathbf{A}(z) = \sum_{k=0}^{\infty} \mathbf{A}_k z^k &= \int_0^{\infty} e^{\mathbf{D}(z)t}\,\mathrm{d}B(t), \\ &= \int_0^{\infty} e^{\mathbf{D}(z)t} \otimes \boldsymbol{\beta} e^{\mathbf{S}t} \boldsymbol{\eta}\,\mathrm{d}t, \\ &= (\mathbf{I}_n \otimes \boldsymbol{\beta}) \left(\int_0^{\infty} \left(e^{\mathbf{D}(z)t} \otimes e^{\mathbf{S}t} \right) \mathrm{d}t \right) (\mathbf{I}_n \otimes \boldsymbol{\eta}), \\ &= (\mathbf{I}_n \otimes \boldsymbol{\beta}) (\mathbf{D}(z) \oplus \mathbf{S})^{-1} (\mathbf{I}_n \otimes (-\boldsymbol{\eta})). \end{aligned}$$

The precise expression of $\mathbf{A}_k$, $k \geq 0$, is provided for the ME service time distribution as

$$\mathbf{A}_k = \mathbf{L}_k (\mathbf{I}_n \otimes \boldsymbol{\eta}), \quad n \geq 0, \tag{51}$$

where

$$\mathbf{L}_0 = -(\mathbf{I}_n \otimes \boldsymbol{\beta})(\mathbf{D}_0 \otimes \mathbf{I}_3 + \mathbf{I}_3 \otimes \mathbf{S})^{-1}, \tag{52}$$

$$\mathbf{L}_k = -\sum_{r=0}^{k-1} \mathbf{L}_r (\mathbf{D}_{k-r} \otimes \mathbf{I}_3)(\mathbf{D}_0 \otimes \mathbf{I}_3 + \mathbf{I}_3 \otimes \mathbf{S})^{-1}, \quad k \geq 1. \tag{53}$$

We have omitted the proof from this section because it is a simple extension of Neuts ([24], pp. 67–70) work.

Example 2. The system length distributions at departure and random time epochs as well as the sojourn time distribution of an arbitrary customer in a batch for the $BMAP/WB/1$ queue under N-policy with $N = 5$ have been presented graphically in Figures 1 to 6, where WB is an abbreviation for Weibull distribution. We consider the BMAP as follows

$$\mathbf{D}_0 = \begin{bmatrix} -0.477 & 0.072 & 0.014 & 0.000 \\ 0.112 & -0.316 & 0.000 & 0.102 \\ 0.130 & 0.000 & -0.439 & 0.111 \\ 0.264 & 0.005 & 0.000 & -0.392 \end{bmatrix}, \qquad \mathbf{D}_1 = \begin{bmatrix} 0.010 & 0.000 & 0.014 & 0.012 \\ 0.000 & 0.017 & 0.000 & 0.023 \\ 0.061 & 0.000 & 0.028 & 0.000 \\ 0.019 & 0.016 & 0.000 & 0.017 \end{bmatrix},$$

$$\mathbf{D}_3 = \begin{bmatrix} 0.012 & 0.000 & 0.015 & 0.023 \\ 0.002 & 0.021 & 0.000 & 0.009 \\ 0.011 & 0.000 & 0.002 & 0.074 \\ 0.011 & 0.013 & 0.000 & 0.014 \end{bmatrix}, \qquad \mathbf{D}_5 = \begin{bmatrix} 0.011 & 0.032 & 0.000 & 0.014 \\ 0.000 & 0.003 & 0.005 & 0.009 \\ 0.008 & 0.000 & 0.004 & 0.001 \\ 0.000 & 0.007 & 0.003 & 0.000 \end{bmatrix},$$

TABLE 1. System length distribution at departure epoch.

n	$\pi_1^+(n)$	$\pi_2^+(n)$	$\pi_3^+(n)$	$\pi_4^+(n)$	$\boldsymbol{\pi}^+(n)\mathbf{e}$
0	0.01178538	0.00445077	0.00189148	0.00723620	0.02536384
1	0.01415099‘	0.00550072	0.00268082	0.00933584	0.03166836
2	0.01644424	0.00666742	0.00345181	0.01156940	0.03813287
3	0.01850833	0.00781658	0.00430087	0.01376891	0.04439469
4	0.02041804	0.00896400	0.00498979	0.01612197	0.05049380
5	0.02200134	0.01025980	0.00564052	0.01786111	0.05576278
6	0.02334544	0.01181063	0.00634533	0.01949828	0.06099968
7	0.02409815	0.01276892	0.00691497	0.02089628	0.06467833
8	0.02437599	0.01371970	0.00750973	0.02217462	0.06778005
9	0.02401861	0.01471770	0.00818309	0.02332792	0.07024732
10	0.02143260	0.01372414	0.00734550	0.02089233	0.06339457
15	0.01067957	0.00730442	0.00368375	0.01036538	0.03203312
20	0.00518900	0.00359465	0.00179806	0.00502830	0.01561001
30	0.00121136	0.00083863	0.00041956	0.00117388	0.00364344
40	0.00028284	0.00019581	0.00009796	0.00027409	0.00085069
50	0.00006604	0.00004572	0.00002287	0.00006400	0.00019863
60	0.00001542	0.00001067	0.00000534	0.00001494	0.00004638
70	0.00000360	0.00000249	0.00000125	0.00000349	0.00001083
80	0.00000084	0.00000058	0.00000029	0.00000081	0.00000253
100	0.00000004	0.00000003	0.00000002	0.00000005	0.00000014
120	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.36254610	0.20873614	0.10841870	0.32029906	1.00000000

$$\mathbf{D}_7 = \begin{bmatrix} 0.000 & 0.007 & 0.011 & 0.090 \\ 0.002 & 0.000 & 0.003 & 0.000 \\ 0.000 & 0.004 & 0.000 & 0.002 \\ 0.008 & 0.000 & 0.006 & 0.000 \end{bmatrix}, \qquad \mathbf{D}_8 = \begin{bmatrix} 0.030 & 0.070 & 0.040 & 0.000 \\ 0.000 & 0.003 & 0.001 & 0.004 \\ 0.001 & 0.000 & 0.002 & 0.000 \\ 0.000 & 0.002 & 0.000 & 0.007 \end{bmatrix},$$

with $\mathbf{D}_r = \mathbf{0}$, $r \in \mathbb{N} \cup \{0\} - \{0, 1, 3, 5, 7, 8\}$. The input matrices for this BMAP are taken with a maximum batch size of 8. This yields

$$\overline{\boldsymbol{\pi}} = \begin{bmatrix} 0.33902856 & 0.27367027 & 0.09178987 & 0.29551129 \end{bmatrix},$$

with $\lambda^* = 1.04149281$. With $\beta = 0.184$ and $c = 0.411$, Weibull service time distribution is assumed to have the following PDF and CDF $b(u) = \frac{c}{\beta}(\frac{u}{\beta})^{c-1} e^{-(\frac{u}{\beta})^c}$, $u \geq 0$ and $B(u) = 1 - e^{-(\frac{u}{\beta})^c}$, $u \geq 0$, respectively. Due to this, $\mu = \frac{1}{\beta \Gamma(1+\frac{1}{c})} = 1.75929497$ and the traffic intensity $\rho = 0.59199442$ are produced. Our goal is to find the explicit expressions of $\mathbf{A}(z)$ and $\mathbf{A}_k$, $k \geq 0$. To determine the approximate L.-S.T. of the Weibull distribution, we employ the geometric transform approximation approach represented by Shortle *et al.* [26]. In accordance with Shortle *et al.* [26], we take into account $M = 100$ probabilities with the condition that $r_i = 1 - \sigma^i$, where $r_i = B(u_i)$, $1 \leq i \leq M$, and for some σ in $(0, 1)$. Consequently, it follows that $u_i = \beta\big[i \log(\frac{1}{\sigma})\big]^{\frac{1}{c}}$ when $1 - \Big(\frac{u_i}{\beta}\Big)^c = 1 - \sigma^i$. Each point u_i should have the probability r_i as follows:

$$g_i = \frac{r_{i+1} - r_{i-1}}{2}, \quad i = 2, 3, \dots, M-1,$$
$$g_1 = \frac{r_1 + r_2}{2}, \quad g_M = 1 - \frac{r_{M-1} + r_M}{2}\cdot$$

TABLE 2. System length distribution at random epoch.

n	$\omega_1(n)$	$\omega_2(n)$	$\omega_3(n)$	$\omega_4(n)$	$\boldsymbol{\omega}(n)\mathbf{e}$
0	0.03615450	0.01361424	0.00373232	0.01893973	0.07244079
1	0.00604579	0.00191587	0.00217782	0.00464703	0.01478652
2	0.00759012	0.00347553	0.00174089	0.00617669	0.01898323
3	0.00697836	0.00332563	0.00345133	0.00558758	0.01934290
4	0.01011624	0.00317417	0.00234146	0.01100698	0.02663886
5	0.00636670	0.00239234	0.00207537	0.00547686	0.01631128
6	0.01391707	0.01090401	0.00345618	0.01026830	0.03854556
7	0.00871912	0.00425559	0.00263732	0.00725551	0.02286754
8	0.00953766	0.00493257	0.00241469	0.00820896	0.02509389
9	0.00880479	0.00462846	0.00274105	0.00726660	0.02344090
n	$\pi_1(n)$	$\pi_2(n)$	$\pi_3(n)$	$\pi_4(n)$	$\boldsymbol{\pi}(n)\mathbf{e}$
1	0.00852587	0.00322204	0.00137343	0.00524280	0.01836414
2	0.01023147	0.00398068	0.00194232	0.00675863	0.02291311
3	0.01188331	0.00482205	0.00249934	0.00836917	0.02757388
4	0.01337094	0.00565101	0.00310985	0.00995780	0.03208960
5	0.01474579	0.00648067	0.00360644	0.01164796	0.03648086
6	0.01588561	0.00741848	0.00407674	0.01290161	0.04028244
7	0.01684958	0.00853050	0.00458367	0.01408098	0.04404474
8	0.01738777	0.00922211	0.00499514	0.01508861	0.04669363
9	0.01758151	0.00990900	0.00542534	0.01600987	0.04892572
10	0.01731413	0.01063109	0.00591319	0.01684067	0.05069907
20	0.00432477	0.00299839	0.00149876	0.00418884	0.01301076
30	0.00100933	0.00069876	0.00034958	0.00097810	0.00303577
50	0.00005502	0.00003809	0.00001906	0.00005332	0.00016550
80	0.00000070	0.00000048	0.00000024	0.00000068	0.00000211
100	0.00000004	0.00000003	0.00000001	0.00000004	0.00000012
120	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Sum	0.37573133	0.20328023	0.10503097	0.31595746	1.00000000
$L_s = 9.59125995$, $W_s = 13.37625244$					

The aforementioned equations can be expressed using σ as follows:

$$g_i = \frac{\sigma^{i-1} - \sigma^{i+1}}{2}, \quad i = 2, 3, \ldots, M-1,$$
$$g_1 = \frac{2 - \sigma - \sigma^2}{2}, \quad g_M = \frac{\sigma^{M-1} + \sigma^M}{2}.$$

A binary search on σ meets the condition $\sum_{i=1}^{M} g_i x_i = \frac{1}{\mu}$ yields $\sigma = 0.90851423$. Consequently, the estimated L.-S.T. of the Weibull service time distribution is represented as

$$B^*(s) = \sum_{i=1}^{M} u_i e^{-s u_i}. \tag{54}$$

We use Padé approximation approach to roughly convert $B^*(s)$ to a rational function. We use $Padé(3,4)$ of (54) to accomplish this, and it is provided by

$$B^*(s) \approx \frac{1.00000000 + 13.22418111s + 59.20066387s^2 + 79.33794747s^3}{1.00000000 + 13.79259062s + 65.47312216s^2 + 102.78543525s^3 + 15.33529857s^4}. \tag{55}$$

TABLE 3. Sojourn time distribution of an arbitrary customer of a batch.

t	$w_1(t)$	$w_2(t)$	$w_3(t)$	$w_4(t)$	$\mathbf{w}(t)\mathbf{e}$
0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.5	0.00011764	0.00021121	0.00016298	0.00029177	0.00078360
1.0	0.00048968	0.00091208	0.00067692	0.00122219	0.00330086
1.5	0.00112457	0.00216167	0.00154919	0.00280895	0.00764438
2.0	0.00203593	0.00402694	0.00279014	0.00507000	0.01392301
2.5	0.00324065	0.00657608	0.00441201	0.00802544	0.02225418
3.0	0.00475661	0.00987152	0.00642809	0.01169815	0.03275437
3.5	0.00660024	0.01396415	0.00885141	0.01611217	0.04552798
4.0	0.00878436	0.01888880	0.01169269	0.02128878	0.06065464
4.5	0.01131604	0.02466108	0.01495791	0.02724121	0.07817624
5.0	0.01419493	0.03127542	0.01864580	0.03396920	0.09808535
10.5	0.06111388	0.14049444	0.07784734	0.14185482	0.42131048
20.0	0.12122607	0.28253167	0.15316616	0.27660378	0.83352769
30.5	0.13995768	0.32720909	0.17655449	0.31801492	0.96173618
40.0	0.14406177	0.33704952	0.18164600	0.32701431	0.98977161
50.5	0.14520733	0.33981158	0.18305779	0.32950651	0.99758322
70.0	0.14553717	0.34061089	0.18346194	0.33021928	0.99982929
100.5	0.14556185	0.34067094	0.18349206	0.33027236	0.99999722
130.0	0.14556225	0.34067192	0.18349255	0.33027322	0.99999995
150.5	0.14556226	0.34067194	0.18349256	0.33027324	1.00000000
⋮	⋮	⋮	⋮	⋮	⋮
$W = 13.37625244$					

By substituting $s\mathbf{I}_n = -\mathbf{D}(z)$ in (55) and applying the relation $\mathbf{A}(z) \approx B^*(-\mathbf{D}(z))$, we are able to derive

$$\mathbf{A}(z) \approx \left[\mathbf{I}_n + 13.22418111(-\mathbf{D}(z)) + 59.20066387(-\mathbf{D}(z))^2 + 79.33794747(-\mathbf{D}(z))^3\right]$$
$$\times\left[\mathbf{I}_n + 13.79259062(-\mathbf{D}(z)) + 65.47312216(-\mathbf{D}(z))^2 + 102.78543525(-\mathbf{D}(z))^3 + 15.33529857(-\mathbf{D}(z))^4\right]^{-1}.$$

We are now focusing on evaluating the matrices $\mathbf{A}_k$ from (3). In order to do this, we take into account (4) which is pretty much equivalent to $B^*(\phi - \phi z)$, *i.e.*, $\sum_{k=0}^{\infty} \eta_k z^k \approx B^*(\phi - \phi z)$. Thus, from (55), we obtain

$$\sum_{k=0}^{\infty} \eta_k z^k \approx \frac{0.85368424 - 1.71616194z + 1.14165069z^2 - 0.25012471z^3}{1.00000000 - 2.12096212z + 1.54324112z^2 + 0.41629214z^3 + 0.02306143z^4}.$$

The computational results of the probability mass function (PMF) and the cumulative distribution function (CDF) of the system length at departure epoch are presented in Figures 1 and 2, respectively, to provide a straightforward overview of the behavior of system characteristics. The computational results of the probability mass function of the system length at random epoch for idle and busy modes are plotted in Figures 3 and 4, respectively. From Figures 1 and 4, it is observed that the graph for probability of system length is concave in nature, *i.e.*, the probability is increasing upto some number of customers in the system and then the probability is decreasing as the system length increases. Figure 3 shows that the probability of system length at random epoch in idle mode is unstable. Figure 5 graphically displays the result of the cumulative distribution function of the system length at random epoch. Figures 2 and 5 shows that the cumulative distribution for the queue length at departure epoch and random epoch goes to 1 for large value of n. Figure 6 graphically displays the result of the cumulative distribution function of sojourn time distribution. Figure 6 shows that the cumulative distribution for sojourn time goes to 1 when t becomes large.

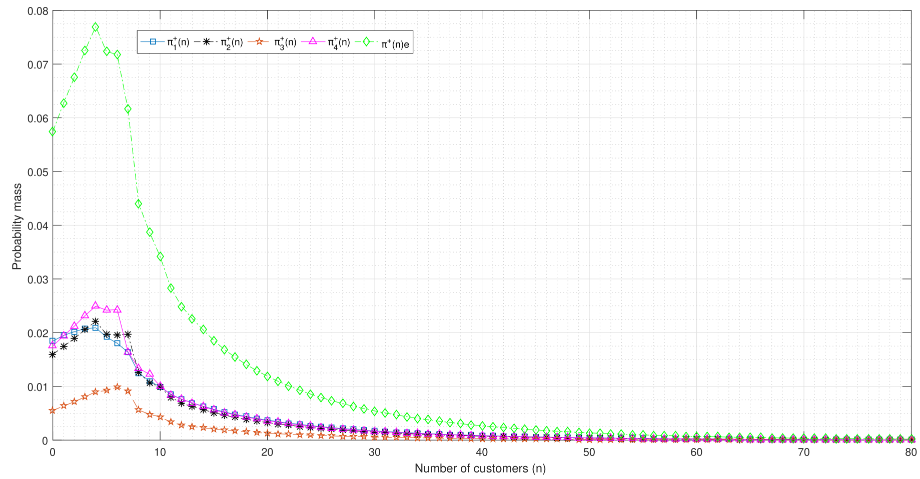

FIGURE 1. The PMF of system length at departure epoch.

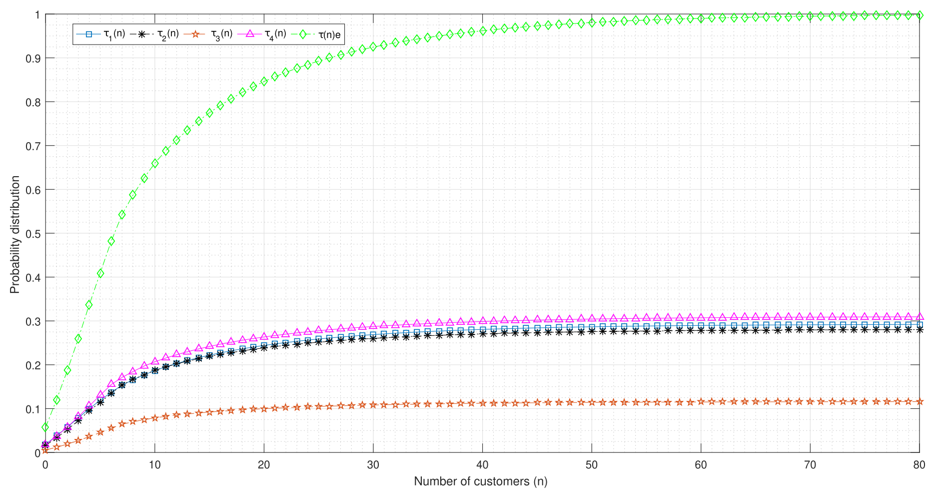

FIGURE 2. The CDF of system length at departure epoch.

Example 3. Three service time distributions such as matrix-exponential, Weibull and Pareto are taken into consideration in order to examine the impact of the service time distribution with the same mean but distinct coefficient of variation (CV) on the optimum value of N with minimum cost. The PDF and CDF of ME service time distribution are taken as $b(u) = \boldsymbol{\beta} e^{\mathbf{S}u}\boldsymbol{\eta}$, $u \geq 0$ and $B(u) = 1 + \boldsymbol{\beta} e^{\mathbf{S}u}\mathbf{S}^{-1}\boldsymbol{\eta}$, respectively. The PDF and CDF of WB service time distribution are taken as $b(u) = \frac{c}{\beta}\left(\frac{u}{\beta}\right)^{c-1} e^{-(\frac{u}{\beta})^c}$, $u \geq 0$ and $B(u) = 1 - e^{-(\frac{u}{\beta})^c}$, $u \geq 0$, respectively. The PDF and CDF of the Pareto (PE) service time distribution with shape parameter α and scale parameter β are taken as $b(u) = \frac{\alpha\beta^\alpha}{u^{\alpha+1}}$, and $B(u) = 1 - (\frac{\beta}{u})^\alpha$, $\alpha > 0, \beta > 0, u > 0$, respectively. Now, we consider two cases

Case 1. Same mean arrival rate and same mean of the service time distributions with different coefficient of variations.

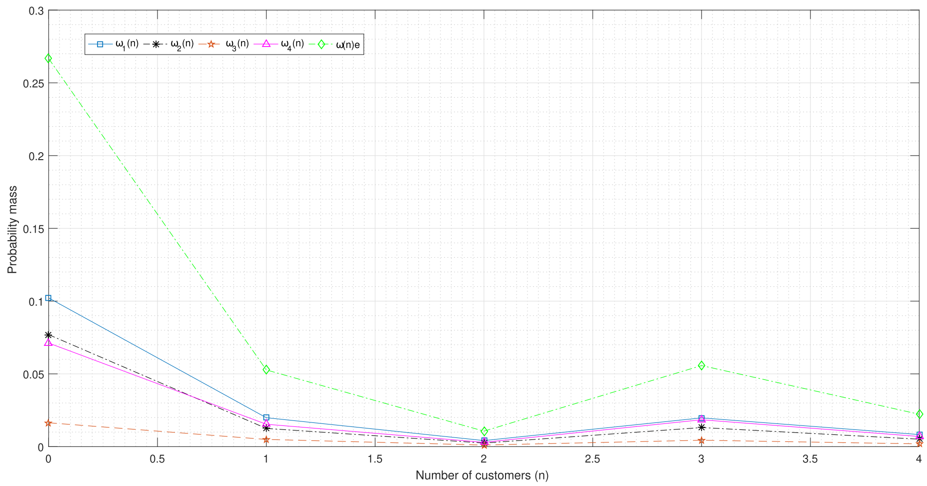

FIGURE 3. The PMF of system length at random epoch in idle mode.

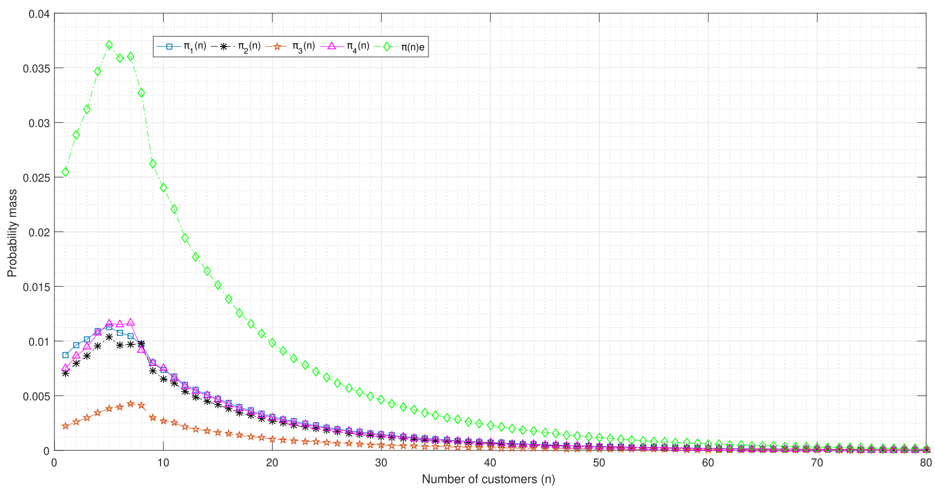

FIGURE 4. The PMF of system length at random epoch in busy mode.

In this case, we consider the BMAP as

$$\mathbf{D}_0 = \begin{bmatrix} -0.3110 & 0.0690 & 0.0140 & 0.0000 \\ 0.1220 & -0.4540 & 0.0000 & 0.2020 \\ 0.2300 & 0.0000 & -0.4764 & 0.1210 \\ 0.2340 & 0.0060 & 0.0000 & -0.4210 \end{bmatrix}, \qquad \mathbf{D}_2 = \begin{bmatrix} 0.0000 & 0.0260 & 0.0170 & 0.0120 \\ 0.0150 & 0.0000 & 0.0110 & 0.0340 \\ 0.0000 & 0.0510 & 0.0280 & 0.0000 \\ 0.0190 & 0.0000 & 0.0360 & 0.0370 \end{bmatrix},$$

$$\mathbf{D}_3 = \begin{bmatrix} 0.0600 & 0.0000 & 0.0240 & 0.0220 \\ 0.0080 & 0.0210 & 0.0000 & 0.0180 \\ 0.0000 & 0.0110 & 0.0020 & 0.0240 \\ 0.0210 & 0.0430 & 0.0000 & 0.0140 \end{bmatrix}, \qquad \mathbf{D}_5 = \begin{bmatrix} 0.0000 & 0.0210 & 0.0420 & 0.0040 \\ 0.0050 & 0.0080 & 0.0000 & 0.0100 \\ 0.0060 & 0.0000 & 0.0020 & 0.0014 \\ 0.0000 & 0.0060 & 0.0050 & 0.0000 \end{bmatrix},$$

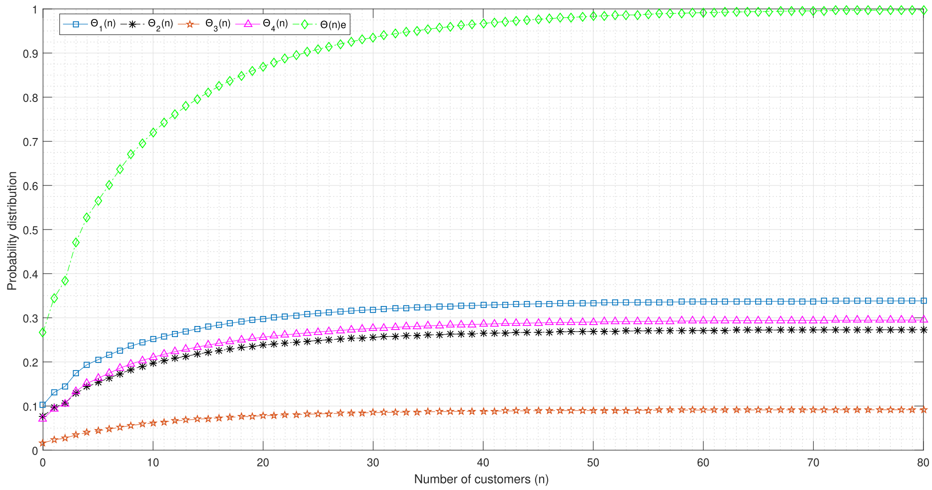

FIGURE 5. The CDF of system length at random epoch.

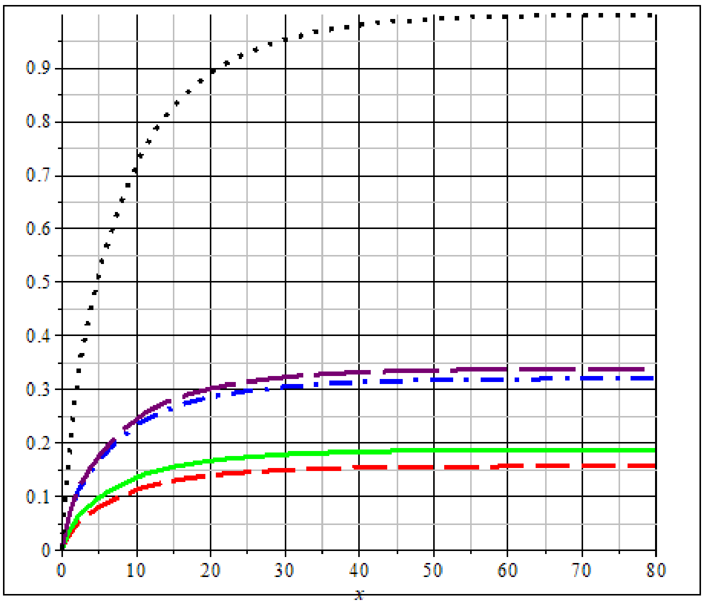

FIGURE 6. The CDF of sojourn time of a random customer $\boxed{--w_1(t), \cdot - w_2(t), —w_3(t), ——w_4(t), \cdots \mathbf{w(t)e}}$.

with $\mathbf{D}_r = \mathbf{0},\ r \in \mathbb{N} \cup \{0\} - \{0, 2, 3, 5\}$.
In case of matrix-exponential service time, we take $\boldsymbol{\beta} = [1 + 6\pi^2, 0, 0]$,

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 - 6\pi^2 & -2 - 6\pi^2 & -2 \end{bmatrix}, \text{ and } \boldsymbol{\eta} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \text{ Thus, we have } \mu = 0.98366483.$$

Here, we consider $\beta = 0.184$ and $c = 0.34084929$ for Weibull service time distribution whereas we consider $\alpha = 2.95$ and $\beta = 0.67199408$ for Pareto service time distribution.

Case 2. Different mean arrival rate and same mean of the service time distributions with different coefficient of variations. In this case, we consider the parameter matrices of the BMAP where each entry of matrices is a function of t as follows

$$\mathbf{D}_0 = \begin{bmatrix} -5t & \frac{t}{5} & \frac{t}{3} \\ t & -7t & \frac{t}{4} \\ \frac{t}{2} & \frac{t}{2} & -6t \end{bmatrix}, \qquad \mathbf{D}_1 = \begin{bmatrix} \frac{t}{3} & \frac{t}{5} & \frac{t}{5} \\ \frac{t}{4} & \frac{t}{2} & \frac{t}{2} \\ \frac{t}{3} & \frac{t}{3} & \frac{t}{2} \end{bmatrix},$$
$$\mathbf{D}_4 = \begin{bmatrix} \frac{t}{5} & \frac{t}{2} & \frac{t}{3} \\ \frac{t}{4} & t & \frac{t}{4} \\ \frac{t}{3} & \frac{t}{2} & t \end{bmatrix}, \qquad \mathbf{D}_6 = \begin{bmatrix} \frac{t}{5} & \frac{t}{2} & 2t \\ \frac{t}{2} & 2t & \frac{t}{2} \\ t & \frac{t}{2} & \frac{t}{2} \end{bmatrix},$$

with $\mathbf{D}_r = \mathbf{0},\ r \in \mathbb{N} \cup \{0\} - \{0, 1, 4, 6\}$.

So, for different values of t, we generate different BMAP. In case of matrix-exponential service time, we take $\boldsymbol{\beta} = [1 + 3\pi^2, 0, 0]$,

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 - 3\pi^2 & -2 - 3\pi^2 & -2 \end{bmatrix}, \text{ and } \boldsymbol{\eta} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ with } t = 0.0276 \text{ for BMAP.}$$

In case of Weibull service time distribution, we take $\beta = 0.15500000$ and $c = 0.32440126$ with $t = 0.012$ for BMAP. In case of Pareto service time distribution, we take $\alpha = 3.95000000$ and $\beta = 0.77123480$ with $t = 0.021$ for BMAP.

In this numerical experiment, we adopt cost values as $C_h = 21, C_i = 121, C_b = 175$. It is observed from Figure 7 that CV effects on the optimum cost but not on the optimum threshold value of N. This demonstrates that the expected cost function is convex. For all three service time distributions with same mean $\frac{1}{\mu}$ along with same arrival rate λ^*, we find out the optimum threshold value of N at $N^* = 5$ and optimum cost decreases as CV decreases, where $\Lambda(5) = 110.2117618, 115.5186807$ and 248.1123803 for PE, ME and WB, respectively, are the lowest values of the long-run expected operational cost per unit time.

In this numerical experiment, we adopt cost values as $C_h = 5, C_i = 250, C_b = 34$. It is observed from Figure 8 that CV does not play any major role on the optimum cost as well as on optimum N. The arrival rate λ^* is crucial in this situation since it reduces the optimal cost and yields two distinct optimum N values. For matrix-exponential service time distribution, we see that optimum threshold value is $N^* = 10$. In case of Weibull and Pareto service time distributions, we see that optimum threshold value is $N^* = 6$, where $\Lambda(10) = 69.73440670$ for ME, $\Lambda(6) = 42.01135721$ and 32.19090807 for PE and WB, respectively, are the lowest values of the long-run expected operational cost per unit time.

Example 4. In this example, we show a graphical representation of the computational time to compute the departure epoch stationary probability vectors using various methods with respect to different utilization factor ρ. To do this, we take into account the BMAP as follows.

$$\mathbf{D}_0 = \begin{bmatrix} -79 & 2 & 6 & 1 & 3 \\ 2 & -85 & 4 & 8 & 1 \\ 1 & 3 & -80 & 2 & 5 \\ 2 & 3 & 1 & -84 & 4 \\ 1 & 3 & 5 & 7 & -105 \end{bmatrix}, \qquad \mathbf{D}_3 = \begin{bmatrix} 1 & 2 & 1 & 4 & 3 \\ 2 & 3 & 1 & 1 & 4 \\ 5 & 2 & 3 & 1 & 0 \\ 3 & 1 & 1 & 1 & 2 \\ 1 & 4 & 3 & 2 & 2 \end{bmatrix},$$

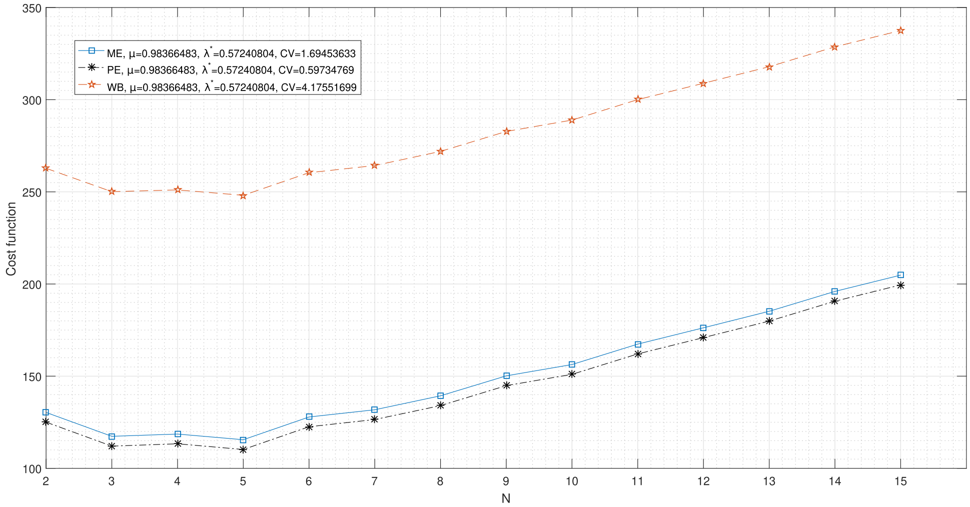

FIGURE 7. Cost per unit time over N for different service time distributions (Case 1).

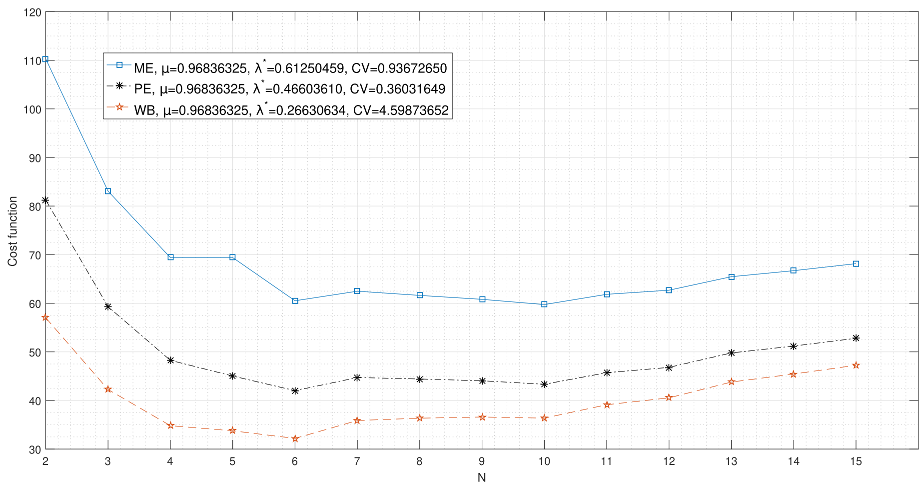

FIGURE 8. Cost per unit time over N for different service time distributions (Case 2).

$$\mathbf{D}_5 = \begin{bmatrix} 3 & 4 & 5 & 1 & 2 \\ 3 & 5 & 6 & 1 & 1 \\ 2 & 5 & 2 & 3 & 4 \\ 3 & 4 & 1 & 1 & 2 \\ 4 & 4 & 3 & 1 & 2 \end{bmatrix}, \qquad \mathbf{D}_9 = \begin{bmatrix} 2 & 3 & 5 & 5 & 5 \\ 1 & 3 & 3 & 3 & 4 \\ 2 & 2 & 2 & 3 & 5 \\ 3 & 4 & 7 & 3 & 2 \\ 4 & 3 & 5 & 6 & 7 \end{bmatrix},$$

$$\mathbf{D}_{12} = \begin{bmatrix} 2 & 1 & 1 & 1 & 4 \\ 8 & 2 & 2 & 1 & 2 \\ 3 & 2 & 2 & 2 & 4 \\ 4 & 3 & 6 & 6 & 2 \\ 2 & 6 & 4 & 4 & 5 \end{bmatrix}, \qquad \mathbf{D}_{15} = \begin{bmatrix} 4 & 3 & 3 & 1 & 1 \\ 4 & 2 & 1 & 5 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \\ 2 & 3 & 6 & 4 & 2 \end{bmatrix},$$

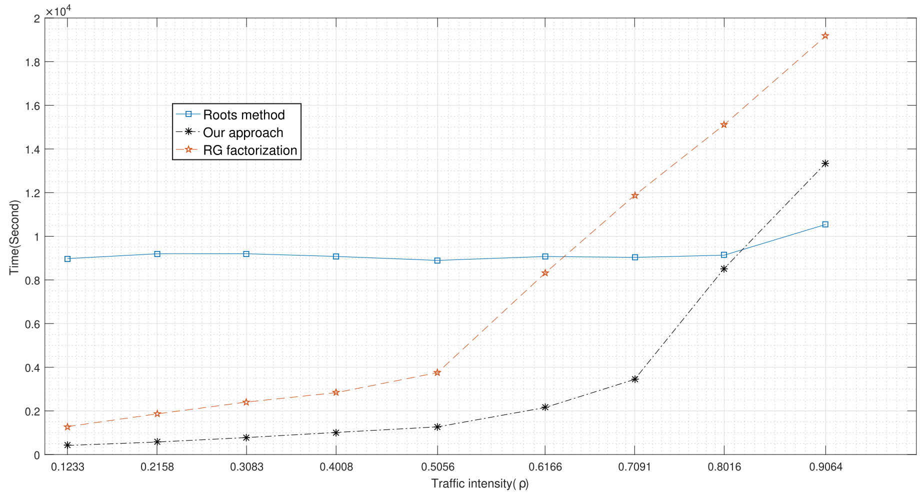

FIGURE 9. Computational time *versus* traffic intensity.

with $\mathbf{D}_r = \mathbf{0},\ r \in \mathbb{N} \cup \{0\} - \{0, 3, 5, 9, 12, 15\}$. Note that the input matrices for this BMAP are taken with a maximum batch size of 15. This yields

$$\overline{\boldsymbol{\pi}} = \begin{bmatrix} 0.19755265 & 0.19955886 & 0.23628396 & 0.20110291 & 0.16550162 \end{bmatrix},$$

with $\lambda^* = 670.70107729$. We choose $N = 14$ and use the same Weibull service time distribution as in Example 2 with the shape parameter $\beta = 0.0002, 0.00035, 0.0005, 0.00065,\ 0.00082, 0.00100, 0.00115, 0.00130, 0.00147$, and scale parameter $c = 5.11$.

Due to this, we have $\rho = 0.12331816,\ 0.21580679,\ 0.30829541,\ 0.40078403,\ 0.50560447,\ 0.61659082,\ 0.70907944,\ 0.80156806,\ 0.90638850$.

Every stationary probability at the departure epoch determined by the RG-factorization and roots approaches agrees with the stationary probabilities established by our simpler approach. Roots approach has avoided evaluating $\mathbf{A}_k$'s. Once $\mathbf{A}(z)$ is known, it is possible to compute any number of probabilities; however, the number of roots considerably rises when BMAP's dimension and maximum batch size are both large. So, in this situation, it is obvious that the computation requires more time than our approach. Calculating $\mathbf{A}_k$ is a crucial step in both our strategy and RG-factorization method. However, there are two more crucial steps in the RG-factorization process: computing $\mathbf{G}$ and $\mathbf{R}$-measures. Consequently, our method ultimately requires less calculation time than RG-factorization. The case of high traffic intensity necessitates a time-consuming evaluation of $\mathbf{A}_k$. If so, the roots method might be computationally more efficient than both our approach and the RG-factorization strategy. The results are plotted in Figure 9.

7. Conclusion

The main objective of this paper is to provide a simpler repeated substitution approach to obtain the system length distribution at departure epoch. Next, using the supplementary variable technique, we have established a relationship between departure- and random-epoch probabilities in order to determine the system length distribution at random epoch. For a randomly chosen customer from an incoming batch, the probability density function of the sojourn time distribution has also been established. In order to determine the best value of N at the lowest cost, we have also offered an expected linear cost function. Further, the validation of our analytic procedure has demonstrated with numerical results with distinct service time distributions. The analytical results reported in this study will be useful to practitioners and system designers in modelling a variety of real-world quality and service systems.

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