RAIRO-Oper. Res. 58 (2024) 5381–5402
https://doi.org/10.1051/ro/2024199

RAIRO Operations Research
www.rairo-ro.org

GROUP DECISION-MAKING WITH HESITANT FUZZY LINGUISTIC PREFERENCE RELATIONS IN VIEW OF WORST AND AVERAGE INDEXES

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Abstract. To address the multi-criteria group decision-making (MCGDM) problems with hesitant fuzzy linguistic preference relations (HFLPRs), this study introduces a group decision-making (GDM) method in view of worst additive consistency index (WACI) and average additive consistency index (AACI) simultaneously. First, several optimization models are constructed for deriving the WACI and AACI. The main characteristic of the constructed models is that it takes into accounted the personalized individual semantics (PISs). Based on this, the concept of acceptable additive consistent HFLPRs is developed. Second, to improve the consistency of HFLPRs, several optimization models are constructed. Two predefined thresholds for the WACI and AACI are considered in the proposed models. It requires the consistency levels of all the linguistic preference relations (LPRs) associated with an HFLPR meets the threshold of WACI, and the average consistency level of all LPRs reaches the threshold of AACI. Third, an algorithm is designed for deriving priority weights from acceptable consistent HFLPRs. Finally, the presented models are validated for 3D visualization management system selection problem and extensive comparative analyses.

Mathematics Subject Classification. 35L05, 35L70.

Received December 25, 2022. Accepted October 22, 2024.

1. Introduction

Nowadays, group decision-making (GDM) problems are becoming increasingly common in daily life. In GDM process, a collective solution can be derived by considering more than one DMs' opinions [1]. Due to the ambiguity of human being's perception and the uncertainty of decision-making environments, it is natural that DMs use linguistic expressions when they provide their evaluation opinions [2]. For that, linguistic GDM (LGDM) has aroused the interest of scholars and many GDM methods have been developed [3]. Among these LGDM methods [4], linguistic preference relations (LPRs) are usually used to express DMs' opinions when evaluate alternatives through pairwise comparisons. LGDM involves a more complicated and uncertain decision-making circumstance, and has become an important component of multi-criteria GDM (MCGDM) [5,6]. With the rapid development of society, MCGDM issues are becoming challenging and increasingly complex, particularly when

Keywords. Multi-criteria group decision-making, hesitant fuzzy linguistic preference relations, worst additive consistency index, average additive consistency index, personalized individual semantics.

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they relate to more than one criterion. In these circumstances, DMs may feel hesitant among several linguistic terms when expressing their evaluation information. For instance, in electric vehicles selection problems, several DMs are invited to take part in the selection process and expressed their evaluation opinions. Due to the lack of historical data of the products, DMs are likely to use "between good and very good" as the degree to which one alternative is preferred to another. Under this scenario, hesitant fuzzy linguistic term sets (HFLTSs), which are developed by Rodriguez *et al.* [7] can be a valid tool for DMs to express such opinions more flexibly.

In recent years, research focusing on MCGDM with HFLTSs has become a hot topic [8–10]. Among these studies, hesitant fuzzy linguistic preference relations (HFLPRs) were defined and utilized for dealing with practical GDM problems. Generally speaking, based on the priority weights derivation process whether considers the consistency and consensus, these researches can be roughly divided into four categories: (1) the priority weights derivation process without considers consistency and consensus. Ren *et al.* [11] introduced a priority weights deriving method by introducing DM's satisfaction degree to measure the differences between the incompletely HFLPRs (IHFLPRs) and its corresponding weight vector. (2) The priority weights derivation process considers consistency only. Li *et al.* [12] proposed the interval consistency index to estimate the consistency range of an HFLPR, the consistency checking process by measuring the best consistency index (BCI) and the worst consistency index (WCI). (3) The priority weights derivation process considers consensus only. Zhang *et al.* [13] focused on clustering and consensus reaching process in large-scale GDM environments. (4) The priority weights derivation process considers consistency and consensus simultaneously. Ren *et al.* [14] proposed an GDM method that considering the individual acceptable consistency and group consensus.

Consistency is one of the challenging and key issues that needs to be resolved in MCGDM processes. Inconsistent preferences may derive unreasonable decision results [15, 16]. Thus, plenty of approaches have been developed to deal with the consistencies of various preference relations [17–20]. The consistency of HFLPRs can help DMs to derive reasonable priority weights, has also received great attention recently with regard to the following two aspects: (1) the consistency measurement process. Several approaches worth mentioning when checking the consistency level of HFLPRs, such as optimistic consistency [21], normalized consistency [22], average consistency [23] and partial average consistency [24], etc. The difference between these approaches is the number of LPRs are used, which associated with an HFLPR. And whether other values have been added into the original HFLPRs. (2) Consistency improvement process. The improving process guarantees the revised HFLPR remains the original evaluation information as much as possible, and meets the consistency level. As the consistency improvement process of others' preference relations, optimization model-based [23], interactive-based [25], and the combination above two algorithms [26] are three general approaches.

To estimate the consistency level of an HFLPR, Li *et al.* [12] first introduced the BCI and WCI. The BCI is determined by its LPR with the best consistency degree, and the WCI is determined by its LPR with the worst consistency degree. Thus, the larger the value of BCI and WCI, the more consistent HFLPR is. Afterwards, the BCI and WCI have aroused the interest of some scholars. Li *et al.* [23] proposed the BCI and the average consistency index (ACI) and utilized them to measure the additive consistency level of an IHFLPR. Chen *et al.* [27] proposed controlled the WCI of HFLPRs in consensus optimization models. Liu *et al.* [26] presented a method to calculate the scopes of HFLPRs based on BCI and WCI. Liu *et al.* [28] developed a method to calculate missing elements of an IHFLPR in view of BCI and WCI.

Although significant progress has been made on HFLPRs consistency measurement and improvement, there are some gaps still exist. (1) The consistency measurement and improvement methods in view of the BCI, WCI and ACI. These methods developed the consistency measurement and improvement processes based on the LPRs associated with HFLPRs. It is natural that the WCI and ACI of the LPRs associated with HFLPR play an important role in analyzing the consistency of an HFLPR. However, current researches have not yet studied the measurement the combination of these two consistency indexes. (2) The priority weights deriving methods based on satisfaction degree functions. These methods developed the priority weights deriving process without consistency measurement and improvement. These methods directly handled the original HFLPRs and obtained the priority weights under the consistent conditions, which reduced the complexities of decision-making processes. However, the BCI, WCI and ACI of HFLPRs were disregarded. (3) Since different DMs may have different understandings of the linguistic terms. In other words, words mean different things for different people [29]. Unfortunately, the existing consistency measurement and improvement methods have seldom concerned this aspect.

The above analysis highlights some research achievements related to the consistency measurement and improvement of HFLPRs. However, there are still some shortcomings that remain to be addressed. The research motivations of this study are summarized as follows:

(1) It is natural that the WCI and ACI of the LPRs associated with an HFLPR play an important role in analyzing the consistency of HFLPRs. However, the existing consistency measurement process seldom taken into accounted both of them. Therefore, it is necessary to develop a reasonable consistency measurement method for HFLPRs.
(2) It is reasonable that the priority weights are showed the hesitant values since they are obtained from the hesitant fuzzy linguistic information. Unfortunately, most of the existing methods for determining priority weights only shows the accurate values. For that, it is necessary to introduce a reasonable priority weights determination method for HFLPRs.
(3) Since different DMs may have different understandings of the linguistic terms. Unfortunately, the existing methods seldom concerned this aspect. Hence, the consistency improvement process integrating personalized individual semantics (PISs) seems necessary.

To achieve these goals, this paper further studies MCGDM with HFLPRs, focusing on the processes of consistency measurement and improvement. And then, an algorithm is designed for deriving priority weights from acceptable consistent HFLPRs. Finally, a new GDM procedure is developed. The primary contributions of this study are summarized as follows.

(1) Several optimization models are proposed for deriving the worst additive consistency index (WACI) and average additive consistency index (AACI) of HFLPRs. The main characteristic of the constructed models is they taken into accounted the PISs model.
(2) The concept of acceptable additive consistent HFLPRs is developed, which considers two predefined thresholds for the WACI and AACI respectively.
(3) Several optimization models are constructed for improving the consistency of HFLPRs. The main characteristic of the constructed models is that they require the consistency level of all the LPRs associated with an HFLPR to meet the threshold of WACI, and the average consistency level of all LPRs to reach the threshold of AACI.

The remainder of the study is organized as follows. In Section 2, the numerical scale model, the PISs model, and the related concepts of HFLTSs and HFLPRs are reviewed. In Section 3, the concepts of BCI, WCI and ACI of HFLPRs are reviewed. Several optimization models for deriving the BACI, WACI and AACI of HFLPRs are proposed. The concept of acceptable additive consistent HFLPRs is developed. Moreover, several optimization models are constructed for improving the consistency of HFLPRs. In Section 4, an algorithm is designed for deriving priority weights from acceptable consistent HFLPRs. In Section 5, the proposed method is illustrated by an example, and a comparative analysis is provided. Finally, conclusions are presented in Section 6.

2. Preliminaries

In this section, several relevant preliminaries are reviewed. To be specific, Section 2.1 introduces the numerical scale model, Section 2.2 reviews the PISs model, and the concepts of HFLTSs and HFLPRs are presented in Section 2.3.

2.1. Numerical scale model

As an extension of the 2-tuples linguistic representation, Dong *et al.* [30] developed the concept of numerical scale.

Definition 1 ([30]). Let $S=\{s_0,s_1,\cdots,s_g\}$ and R respectively be the set of linguistic terms and real numbers. A function $\mathrm{NS}:S->R$ is called a numerical scale of S, and $\mathrm{NS}(s_i)$ is the numerical index of s_i.

For a linguistic representation model with 2-tuple linguistic model (s_i,α), Dong *et al.* [30] developed the concept of numerical scale NS as follows.

Definition 2 ([30]). The numerical scale NS for (s_i,α) is defined as follows:

$$\mathrm{NS}(s_i,\alpha)=\begin{cases}\mathrm{NS}(s_i)+\alpha\times(\mathrm{NS}(s_{i+1})-\mathrm{NS}(s_i)), & \alpha\geq 0\\ \mathrm{NS}(s_i)+\alpha\times(\mathrm{NS}(s_i)-\mathrm{NS}(s_{i-1})), & \alpha<0.\end{cases}\tag{1}$$

Obviously, the numerical scale defines a one-to-one mapping between an LTS and a numerical scale. Afterwards, Dong *et al.* [31] proposed the inverse operator of numerical scale NS, which defines a one-to-one mapping between a numerical scale and an LTS.

Definition 3 ([31]). The inverse operator of numerical scale NS is defined as follows:

$$\mathrm{NS}^{-1}(\gamma)=\begin{cases}\left(s_i,\frac{\gamma-\mathrm{NS}(s_i)}{\mathrm{NS}(s_{i+1})-\mathrm{NS}(s_i)}\right), & \mathrm{NS}(s_i)<\gamma<\frac{\mathrm{NS}(s_i)+\mathrm{NS}(s_{i+1})}{2}\\ \left(s_i,\frac{\gamma-\mathrm{NS}(s_i)}{\mathrm{NS}(s_i)-\mathrm{NS}(s_{i-1})}\right), & \frac{\mathrm{NS}(s_{i-1})+\mathrm{NS}(s_i)}{2}\leq\gamma\leq\mathrm{NS}(s_i).\end{cases}\tag{2}$$

To demonstrate the validity of proposed numerical scale model, Dong *et al.* [32] provided a unified framework of numerical scale connected to the 2-tuple linguistic model, the unbalanced linguistic model, and the proportional 2-tuple linguistic model.

2.2. Personalized individual semantics model

Since different DMs might have different understandings of the linguistic terms. In other words, words mean different things for different people. To characterize such situations, Li *et al.* [29] developed the concept of PISs model. Afterwards, Li *et al.* [33] proposed a data-driven learning model to analyze the PISs of DMs to support an MCGDM model. Different from the previous studies, Li *et al.* [33]'s study investigated PISs in the context of MCGDM problems. Owing to its practicability and effectiveness, the use of PISs has been developed in various linguistic forms. Such as, comparative linguistic expression [34], flexible linguistic expression [35], probabilistic linguistic term sets [36], and distribution linguistic term sets [37].

The PISs of linguistic terms have different formats, such as exact values [29], and unit intervals [38]. In the following section, the method of PISs model presented in LPRs is reviewed.

Let $A=\{a_1,a_2,\cdots,a_n\}$ and $E=\{e_1,e_2,\cdots,e_m\}$ respectively be the set of alternatives and DMs. The DMs $\{e_1,e_2,\cdots,e_m\}$ provide the pairwise comparisons in the form of LPRs $L^k=(l_{ij}^k)_{n\times n}$, where $l_{ij}^k+l_{ji}^k=s_g, l_{ii}^k=s_{g/2}, l_{ij}^k\in S, i,j=1,2,\cdots,n, k=1,2,\cdots,m$. To derive the PISs of linguistic terms, Li *et al.* [39] developed the following model:

$$\max\ \mathrm{CI}\big(L^k\big)=1-\frac{4}{n(n-1)(n-2)}\sum_{i<j<z}\left|\mathrm{NS}\big(l_{ij}^k\big)+\mathrm{NS}\big(l_{jz}^k\big)-\mathrm{NS}\big(l_{iz}^k\big)-0.5\right|$$

$$\text{s.t.}\begin{cases}\mathrm{NS}(s_0)=0\\ \mathrm{NS}(s_g)=1\\ \mathrm{NS}\big(s_{g/2}\big)=0.5\\ \mathrm{NS}(s_i)\in[(i-1)/g,(i+1)/g], & i=1,2,\cdots,g-1, i\neq g/2\\ \mathrm{NS}(s_{i+1})-\mathrm{NS}(s_i)\geq\lambda, & i=0,1,\cdots,g-1.\end{cases}\tag{3}$$

In equation (3), the objective function indicates the larger the consistency level $\mathrm{CI}(L^k)$, the more consistent LPRs L^k is. The first to fourth constraints determine the range of numerical scales, and the order of numerical scales is presented in the fifth constraint. The constraint value $\lambda\in(0,1)$ is a small positive number, it can be set priori, such as $\lambda=0.01$.

2.3. HFLTSs and HFLPRs

Due to the complicated and uncertain decision environment, DMs may feel hesitant among several linguistic terms when expressing their evaluation opinions. To characterize the situations, Rodríguez *et al.* [7] developed the concept of HFLTSs.

Definition 4 ([7])**.** Let $S = \{s_0, s_1, \cdots, s_g\}$ be an LTS. An HFLTS H on an LTS S is an ordered finite subset of consecutive linguistic terms in S.

In what follows, the concept of HFLPRs is presented.

Definition 5 ([40])**.** Let $X = \{x_1, x_2, \cdots, x_n\}$ be the set of alternatives. A matrix $H = (h_{ij})_{n\times n}$ can be denoted as an HFLPR defined on X, where $h_{ij} = \{h_{ij}^1, h_{ij}^2, \cdots, h_{ij}^{\#h_{ij}}\}$ is an HFLTS and $\#h_{ij}$ is the cardinal of h_{ij}. For $\forall i, j \in \{1, 2, \cdots, n\}, h_{ij}$ satisfies the following conditions:

$$\begin{cases} h_{ji} = \mathrm{Neg}(h_{ij}) = \left\{\Delta\left(g - \Delta^{-1}\left(h_{ij}^{\#h_{ij}}\right)\right), \cdots, \Delta\left(g - \Delta^{-1}(h_{ij}^1)\right)\right\} \\ \#h_{ji} = \#h_{ij} \\ h_{ii} = s_{g/2} \\ \Delta^{-1}(h_{ij}^1) \leq \Delta^{-1}(h_{ij}^2) \leq \cdots \leq \Delta^{-1}\left(h_{ij}^{\#h_{ij}}\right) \end{cases} \tag{4}$$

where Δ is a function $\Delta : [0, g] \to X \times [0.5, 0.5)$ to transform numerical value $\beta \in [0, g]$ into 2-tuple (s_k, α). On the contrary, Δ^{-1} is a transformation function $\Delta^{-1}(s_k, \alpha) = k + \alpha = \beta$ to transform 2-tuple (s_k, α) into numerical value $\beta \in [0, g]$.

3. Acceptable additive consistent HFLPRs

In this section, the concepts of BCI, WCI and ACI of HFLPRs proposed by Li *et al.* [12] are reviewed. And then, several models integrating PISs model are developed respectively to derive the BACI, WACI and AACI. Based on with, the concepts of completely and acceptable additive consistent HFLPRs are developed. Finally, several optimization models are constructed for improving the consistency of HFLPRs.

3.1. Additive consistency indexes of HFLPRs

Let $H = (h_{ij})_{n\times n}$ be an HFLPR of $X = \{x_1, x_2, \cdots, x_n\}$, where

$$h_{ij} = \left\{h_{ij}^s \mid s = 1, 2, \cdots, \#h_{ij}\right\}, \qquad i, j \in N.$$

And $\Omega_H = \{L = (l_{ij})_{n\times n} \mid l_{ij} \in h_{ij}, l_{ji} = \mathrm{Neg}(l_{ij}), i < j, i, j \in N\}$ be a set of LPRs associate with H. By calculating the consistency level of HFLPR, Li *et al.* [12] developed the concepts of BCI, WCI and ACI.

Definition 6 ([12])**.** Let $H = (h_{ij})_{n\times n}$ and Ω_H be as defined. Then, the BCI of H is developed as follows:

$$\mathrm{BCI}(H) = \max_{L\in\Omega_H} 1 - \frac{4}{gn(n-1)(n-2)} \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \left|\Delta^{-1}(l_{ij}) + \Delta^{-1}(l_{jk}) - \Delta^{-1}(l_{ik}) - \frac{g}{2}\right|. \tag{5}$$

Definition 7 ([12])**.** Let $H = (h_{ij})_{n\times n}$ and Ω_H be as defined. Then, the WCI of H is developed as follows:

$$\mathrm{WCI}(H) = \min_{L\in\Omega_H} 1 - \frac{4}{gn(n-1)(n-2)} \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \left|\Delta^{-1}(l_{ij}) + \Delta^{-1}(l_{jk}) - \Delta^{-1}(l_{ik}) - \frac{g}{2}\right|. \tag{6}$$

Definition 8 ([12]). Let $H = (h_{ij})_{n\times n}$ and Ω_H be as defined. Then, the ACI of H is defined as follows:

$$\mathrm{ACI}(H) = 1 - \frac{4}{gn(n-1)(n-2)\#\Omega_H} \sum_{L\in\Omega_H} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left|\Delta^{-1}(l_{ij}) + \Delta^{-1}(l_{jk}) - \Delta^{-1}(l_{ik}) - \frac{g}{2}\right|, \tag{7}$$

where $\#\Omega_H = \prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \#h_{ij}$ associated the number of LPRs including in Ω_H.

To derive the BCI, Li *et al.* [12] constructed the following programming model:

$$\begin{aligned} \max\ \mathrm{BCI}(H) = 1 - \frac{4}{gn(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \Bigg| \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \Delta^{-1}\big(h_{ij}^s\big) \\ + \sum_{s=1}^{\#h_{jk}} \alpha_{jk}^s \Delta^{-1}\big(h_{jk}^s\big) - \sum_{s=1}^{\#h_{ik}} \alpha_{ik}^s \Delta^{-1}\big(h_{ki}^s\big) - \frac{g}{2} \Bigg| \\ \text{s.t.} \quad \begin{cases} \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s = 1, & i,j = 1,2,\cdots,n, i<j \\ \alpha_{ij}^s = 0 \vee 1, & i,j = 1,2,\cdots,n, i<j, s = 1,2,\cdots,\#h_{ij}. \end{cases} \end{aligned} \tag{8}$$

In a similar manner, the WCI is developed as follows [12]:

$$\begin{aligned} \min\ \mathrm{WCI}(H) = 1 - \frac{4}{gn(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \Bigg| \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \Delta^{-1}\big(h_{ij}^s\big) \\ + \sum_{s=1}^{\#h_{jk}} \alpha_{jk}^s \Delta^{-1}\big(h_{jk}^s\big) - \sum_{s=1}^{\#h_{ik}} \alpha_{ik}^s \Delta^{-1}\big(h_{ik}^s\big) - \frac{g}{2} \Bigg| \\ \text{s.t.} \quad \begin{cases} \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s = 1, & i,j = 1,2,\cdots,n, i<j \\ \alpha_{ij}^s = 0 \vee 1, & i,j = 1,2,\cdots,n, i<j, s = 1,2,\cdots,\#h_{ij}. \end{cases} \end{aligned} \tag{9}$$

Since different DMs may have different understandings of the linguistic terms. In other words, words mean different things for different people. It is reasonable to consider the PISs of DMs in MCGDM process. Unfortunately, the methods developed by Li *et al.* [12] to derive the BCI, WCI and ACI of HFLPRs fail to this. Considering the advantages of PISs, in what follows, the models integrate PISs model are developed to obtain the BACI, WACI and AACI.

For deriving the BACI, the programming model is constructed as follows:

$$\begin{aligned} &\max\ \mathrm{BACI}(H) = \theta \\ &\text{s.t.} \begin{cases} 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \Big| \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \mathrm{NS}\big(h_{ij}^s\big) \\ + \sum_{s=1}^{\#h_{jk}} \alpha_{jk}^s \mathrm{NS}\big(h_{jk}^s\big) - \sum_{s=1}^{\#h_{ik}} \alpha_{ik}^s \mathrm{NS}\big(h_{ik}^s\big) - 0.5 \Big| \geq \theta \\ \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s = 1, & i,j = 1,2,\cdots,n, i<j \\ \alpha_{ij}^s = 0 \vee 1, & i,j = 1,2,\cdots,n, i<j, s = 1,2,\cdots,\#h_{ij} \\ \mathrm{NS}(s_0) = 0 \\ \mathrm{NS}(s_g) = 1 \\ \mathrm{NS}\big(s_{g/2}\big) = 0.5 \\ \mathrm{NS}(s_i) \in [(i-1)/g, (i+1)/g], & i = 1,2,\cdots,g-1, i \neq g/2 \\ \mathrm{NS}(s_{i+1}) - \mathrm{NS}(s_i) \geq \lambda, & i = 0,1,\cdots,g-1. \end{cases} \end{aligned} \tag{10}$$

In equation (10), the first constraint ensures to derive the maximal consistency level of LPR associated with Ω_H, θ is the parameter that determined in objective function. It can be easily found that the first constraint is constructed based on minimax theorem [41]. The second to third constraints are the same as those presented in equation (8), and fourth to eighth constraints are the same as those showed in equation (3).

In a similar manner, the WACI is developed as follows:

$$\min\ \mathrm{WACI}(H)=\theta$$
$$\text{s.t.}\begin{cases}1-\frac{4}{n(n-1)(n-2)}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\Big|\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s}\mathrm{NS}\big(h_{ij}^{s}\big) \\ +\sum_{s=1}^{\#h_{jk}}\alpha_{jk}^{s}\mathrm{NS}\big(h_{jk}^{s}\big)-\sum_{s=1}^{\#h_{ik}}\alpha_{ik}^{s}\mathrm{NS}(h_{ik}^{s})-0.5\Big|\le\theta & \\ \sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s}=1, & i,j=1,2,\cdots,n,i<j \\ \alpha_{ij}^{s}=0\vee 1, & i,j=1,2,\cdots,n,i<j,s=1,2,\cdots,\#h_{ij} \\ \mathrm{NS}(s_0)=0 & \\ \mathrm{NS}(s_g)=1 & \\ \mathrm{NS}\big(s_{g/2}\big)=0.5 & \\ \mathrm{NS}(s_i)\in[(i-1)/g,(i+1)/g], & i=1,2,\cdots,g-1,i\ne g/2 \\ \mathrm{NS}(s_{i+1})-\mathrm{NS}(s_i)\ge\lambda, & i=0,1,\cdots,g-1.\end{cases} \tag{11}$$

In order to derive the AACI, the following indication variables are developed. Let $\alpha_{ij}^{s,t}=\begin{cases}1, & h_{ij}^{t}=h_{ij}^{s}\\ 0, & h_{ij}^{t}\ne h_{ij}^{s}\end{cases}, t=1,2,\cdots,\#\Omega_H, s=1,2,\cdots,\#h_{ij}, i<j, i,j\in N$ be a list of binary variables such that $\alpha_{ij}^{s,t}=1$ denotes that $\mathrm{NS}(h_{ij}^{s})$ is selected, while $\alpha_{ij}^{s,t}=0$ denotes that $\mathrm{NS}(h_{ij}^{s})$ is not selected. The symbol $\#\Omega_H$ be the number of LPRs including in Ω_H. Then $\mathrm{NS}(h_{ij}^{s})$ can be denoted as $\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,t}\mathrm{NS}(h_{ij}^{s}), t=1,2,\cdots,\#\Omega_H, i<j, i,j\in N$. Then, the AACI is developed as follows:

$$\max\ \mathrm{AACI}(H)=\frac{1}{\#\Omega_H}\sum_{t=1}^{\#\Omega_H}\theta_t$$
$$\text{s.t.}\begin{cases}1-\frac{4}{n(n-1)(n-2)}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\Big|\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,t}\mathrm{NS}\big(h_{ij}^{s}\big) \\ +\sum_{s=1}^{\#h_{jk}}\alpha_{jk}^{s,t}\mathrm{NS}\big(h_{jk}^{s}\big)-\sum_{s=1}^{\#h_{ik}}\alpha_{ik}^{s,t}\mathrm{NS}(h_{ik}^{s})-0.5\Big|=\theta_t & \\ \sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,t}=1, & i,j=1,2,\cdots,n,i<j,t=1,2,\cdots,\#\Omega_H \\ \alpha_{ij}^{s,t}=0\vee 1, & i,j=1,2,\cdots,n,i<j, \\ & s=1,2,\cdots,\#h_{ij},t=1,2,\cdots,\#\Omega_H \\ \mathrm{NS}(s_0)=0 & \\ \mathrm{NS}(s_g)=1 & \\ \mathrm{NS}\big(s_{g/2}\big)=0.5 & \\ \mathrm{NS}(s_i)\in[(i-1)/g,(i+1)/g], & i=1,2,\cdots,g-1,i\ne g/2 \\ \mathrm{NS}(s_{i+1})-\mathrm{NS}(s_i)\ge\lambda, & i=0,1,\cdots,g-1.\end{cases} \tag{12}$$

Solving equations (10)–(12) by using many available optimization software packages, such as LINGO, CPLEX or MATLAB. After solving them, one can derive the upper bound, lower bound and average value of the additive consistency levels of an HFLPR.

In what follows, an example derived from [12] is presented to show the difference between Li *et al.* [12]'s method and the proposed method.

Example 1 ([12])**.** Consider the HFLPR as follows

$$H=\begin{pmatrix}\{s_4\} & \{s_2,s_3,s_4\} & \{s_5,s_6\} & \{s_4\}\\ \{s_6,s_5,s_4\} & \{s_4\} & \{s_1,s_2,s_3\} & \{s_6,s_7\}\\ \{s_3,s_2\} & \{s_7,s_6,s_5\} & \{s_4\} & \{s_4,s_5\}\\ \{s_4\} & \{s_2,s_1\} & \{s_4,s_3\} & \{s_4\}\end{pmatrix}.$$

By equations (8) and (9), it is calculated that $\mathrm{BCI}(H) = 0.833$ and $\mathrm{WCI}(H) = 0.667$. If we utilize equations (10) and (11), it is calculated that $\mathrm{BACI}(H) = 0.9550$ and $\mathrm{WACI}(H) = 0.6075$. Obviously, the results of BACI and WACI of HFLPRs are different from different methods. Moreover, the corresponding LPRs of BCI and WCI derived from Li *et al.* [12]'s method are $L_B = \begin{pmatrix} s_4 & s_4 & s_5 & s_4 \\ s_4 & s_4 & s_3 & s_6 \\ s_3 & s_5 & s_4 & s_4 \\ s_4 & s_2 & s_4 & s_4 \end{pmatrix}$ and $L_W = \begin{pmatrix} s_4 & s_2 & s_6 & s_4 \\ s_6 & s_4 & s_1 & s_7 \\ s_2 & s_7 & s_4 & s_4 \\ s_4 & s_1 & s_4 & s_4 \end{pmatrix}$, while obtained from the proposed method are $L_B = \begin{pmatrix} s_4 & s_4 & s_5 & s_4 \\ s_4 & s_4 & s_3 & s_6 \\ s_3 & s_5 & s_4 & s_5 \\ s_4 & s_2 & s_3 & s_4 \end{pmatrix}$ and $L_W = \begin{pmatrix} s_4 & s_2 & s_6 & s_4 \\ s_6 & s_4 & s_1 & s_7 \\ s_2 & s_7 & s_4 & s_4 \\ s_4 & s_1 & s_4 & s_4 \end{pmatrix}$. It can be easily found that corresponding LPRs of BCI derived from Li *et al.* [12]'s method and the proposed method are different, while the corresponding LPRs of WCI are the same.

Based on the preceding analyses, three issues require attention: (1) when the PISs model is taken into accounted in measuring the consistency indexes, the results may be different from 2-tuples linguistic representation model, it will obtain the bigger the BACI and smaller the WACI. (2) The corresponding LPRs of BACI and WACI derived from proposed model may be different from 2-tuples linguistic representation model. (3) Although the same corresponding LPRs of BACI and WACI have been obtained, the values of BACI and WACI may be different from 2-tuples linguistic representation model.

3.2. Acceptable additive consistent HFLPRs

In most of existing studies, the concepts of completely additive consistent HFLPRs and acceptable additive consistent HFLPRs are developed based on ACI [42]. Based on the preceding analyses, ACI demonstrates the average consistency of all possible LPRs associated with Ω_H, it permits compensation of the BCI to the WCI, and ignores the worst circumstance. In contrast, the WCI reflects the worst consistency state of HFLPRs, and it provides the lower bound of the additive consistency index. As discussed in the introduction, it is necessary to take into accounted the WCI and ACI to develop consistency level in some decision-making environments. Keeping the above ideas in mind, the concepts of completely and acceptable additive consistent HFLPRs are developed based on the proposed WACI and AACI.

Definition 9. Let $H = (h_{ij})_{n\times n}$ be an HFLPR. Then, H is called completely additive consistent HFLPR if and only if the following equation establish $\mathrm{WACI}(H) = 1$.

Remark 1. Actually, when $\mathrm{WACI}(H) = 1$, it can be found that $\mathrm{AACI}(H) = 1$. One can check that the Definition 9 is in fact the same as most studies that the completely additive consistent HFLPRs are developed in ACI.

Definition 9 indicates that it is nearly impossible for an HFLPR is completely additive consistency, only in the case that an HFLPR is reduced to an LPR. For that, the acceptable consistency index is suitable for some decision environments. Keeping the above ideas in mind, if two thresholds are defined for the WACI and AACI respectively, one can develop the concept of acceptable additive consistent HFLPRs.

Definition 10. Let $H = (h_{ij})_{n\times n}$ be an HFLPR, $\overline{\mathrm{WCI}}$ and $\overline{\mathrm{ACI}}$ be two predefined thresholds for the WACI and AACI respectively. Then, H is called acceptable additive consistent HFLPR if the relationships $\mathrm{AACI}(H) \geq \overline{\mathrm{ACI}}$ and $\mathrm{WACI}(H) \geq \overline{\mathrm{WCI}}$ both set up simultaneously.

Remark 2. How to determine the thresholds and $\overline{\mathrm{ACI}}$ and $\overline{\mathrm{WCI}}$ for WACI and AACI is a meaningful topic, it will conduct in our further study. In this study, the values of them are determined based on different decision-making environments.

Example 2 ([12])**.** Now we reconsider Example 1. By equation (11), it is calculated that $\mathrm{AACI}(H) = 0.89$. If we set $\overline{\mathrm{ACI}} = 0.9$ and $\overline{\mathrm{WCI}} = 0.8$. Since $\mathrm{AACI}(H) < \overline{\mathrm{ACI}}$ and $\mathrm{WACI}(H) < \overline{\mathrm{WCI}}$, it can be observed that the consistency of H is unacceptable, and its consistency needs to be further improved.

3.3. Optimization models to improve the consistency of HFLPRs

Inconsistent HFLPRs often lead to misleading solutions. Therefore, developing an efficient method to obtain the expected consistency level becomes necessary. In what follows, several optimization models are constructed to improve the consistency level of HFLPRs.

Let $H = (h_{ij})_{n\times n}$, where $h_{ij} = \{h_{ij}^s \mid s = 1, 2, \cdots, \#h_{ij}\}$ be a given HFLPR. According to Definition 10, if H has been certified an unacceptable additive consistent HFLPR, then there is at least one of the conditions $\mathrm{AACI}(H) \geq \overline{\mathrm{AC}}\mathrm{I}$ and $\mathrm{WACI}(H) \geq \overline{\mathrm{WC}}\mathrm{I}$ does not hold. Let $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$, where $\tilde{h}_{ij} = \{\tilde{h}_{ij}^s \mid s = 1, 2, \cdots, \#h_{ij}\}$ be the revised HFLPR with acceptable consistency. The DMs usually want to preserve their initial opinions as much as possible when adjusting the evaluation values. Therefore, the objective function is developed to minimize the distance of adjusted value and original value as follows: $\min \frac{2}{n(n-1)\#h_{ij}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{s=1}^{\#h_{ij}} |\mathrm{NS}(h_{ij}^s) - \mathrm{NS}(\tilde{h}_{ij}^s)|$. Meanwhile, the consistency of revised HFLPR $\tilde{H}$ meets two thresholds $\overline{\mathrm{AC}}\mathrm{I}$ and $\overline{\mathrm{WC}}\mathrm{I}$ for WACI and AACI. Let $\mathrm{CI}(\tilde{L}) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} |\sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \mathrm{NS}(\tilde{h}_{ij}^s) + \sum_{s=1}^{\#h_{jk}} \alpha_{jk}^s \mathrm{NS}(\tilde{h}_{jk}^s) - \sum_{s=1}^{\#h_{ik}} \alpha_{ik}^s \mathrm{NS}(\tilde{h}_{ik}^s) - 0.5|$, then the optimization model is constructed as follows:

$$
\begin{aligned}
&\min z = \frac{2}{n(n-1)\#h_{ij}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{s=1}^{\#h_{ij}} \left| \mathrm{NS}\left(h_{ij}^s\right) - \mathrm{NS}\left(\tilde{h}_{ij}^s\right) \right| \\
&\text{s.t.} \begin{cases} \frac{1}{\#\Omega_{\tilde{H}}} \sum_{\tilde{L}\in\Omega_{\tilde{H}}} \mathrm{CI}(\tilde{L}) \geq \overline{\mathrm{AC}}\mathrm{I} \\ \min_{\tilde{L}\in\Omega_{\tilde{H}}} \mathrm{CI}(\tilde{L}) \geq \overline{\mathrm{WC}}\mathrm{I} \end{cases},
\end{aligned} \tag{13}
$$

where $\Omega_{\tilde{H}} = \{\tilde{L} = (\tilde{l}_{ij})_{n\times n} \mid \tilde{l}_{ij} \in \tilde{h}_{ij}, \tilde{l}_{ji} = \mathrm{Neg}(\tilde{l}_{ij}), \tilde{l}_{ii} = \frac{g}{2}, i < j, i, j \in N\}$ and $\tilde{l}_{ij}, i < j, i, j \in N$ are $\frac{n(n-1)}{2}$ decision variables. In equation (13), the first constraint controls the AACI of $\tilde{H}$ and the second constraint controls the WACI.

In order to make the first constraint to be true, the following indicator variables are developed. Let $\alpha_{ij}^{s,t} = \begin{cases} 1, \tilde{h}_{ij}^t = \tilde{h}_{ij}^s \\ 0, \tilde{h}_{ij}^t \neq \tilde{h}_{ij}^s \end{cases}, t = 1, 2, \cdots, \#\Omega_{\tilde{H}}, s = 1, 2, \cdots, \#h_{ij},\ i < j, i, j \in N$, be a list of binary variables such that $\alpha_{ij}^{s,t} = 1$ denotes that $\mathrm{NS}(\tilde{h}_{ij}^s)$ is selected, while $\alpha_{ij}^{s,t} = 0$ denotes that $\mathrm{NS}(\tilde{h}_{ij}^s)$ is not selected. Then $\mathrm{NS}(\tilde{h}_{ij}^s)$ are denoted as $\sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,t} \mathrm{NS}(\tilde{h}_{ij}^s), t = 1, 2, \cdots, \#\Omega_{\tilde{H}}, i < j, i, j \in N$. The first constraint $\frac{1}{\#\Omega_{\tilde{H}}} \sum_{\tilde{L}\in\Omega_{\tilde{H}}} \mathrm{CI}(\tilde{L}) \geq \overline{\mathrm{AC}}\mathrm{I}$ in equation (13) can be further converted into $1 - \alpha_0 \sum_{t=1}^{\#\Omega_{\tilde{H}}} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} |\sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,t} \mathrm{NS}(\tilde{h}_{ij}^s) + \sum_{s=1}^{\#h_{jk}} \alpha_{jk}^{s,t} \mathrm{NS}(\tilde{h}_{jk}^s) - \sum_{s=1}^{\#h_{ik}} \alpha_{ik}^{s,t} \mathrm{NS}(\tilde{h}_{ik}^s) - 0.5| \geq \overline{\mathrm{AC}}\mathrm{I}$, where $\alpha_0 = \frac{4}{n(n-1)(n-2)\#\Omega_{\tilde{H}}}$. On the other hand, the second constraint holds when the consistency levels for any LPRs associated with $\Omega_{\tilde{H}}$ is greater than or equal to threshold $\overline{\mathrm{WC}}\mathrm{I}$. The constraint $\min_{\tilde{L}\in\Omega_{\tilde{H}}} \mathrm{CI}(\tilde{L}) \geq \overline{\mathrm{WC}}\mathrm{I}$ can be converted into $1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} |\sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,t} \mathrm{NS}(\tilde{h}_{ij}^s) + \sum_{s=1}^{\#h_{jk}} \alpha_{jk}^{s,t} \mathrm{NS}(\tilde{h}_{jk}^s) - \sum_{s=1}^{\#h_{ik}} \alpha_{ik}^{s,t} \mathrm{NS}(\tilde{h}_{ik}^s) - 0.5| \geq \overline{\mathrm{WC}}\mathrm{I}, t = 1, 2, \cdots, \#\Omega_{\tilde{H}}$. The equivalent model to equation (13) can be constructed as follows:

$$\min\ z=\frac{2}{n(n-1)\#h_{ij}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{\#h_{ij}}\left|\mathrm{NS}\big(h_{ij}^{s}\big)-\mathrm{NS}\Big(\tilde{h}_{ij}^{s}\Big)\right|$$

$$\text{s.t.}\begin{cases}1-\frac{4}{n(n-1)(n-2)\#\Omega_{\tilde{H}}}\sum_{t=1}^{\#\Omega_{\tilde{H}}}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\Big|\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,t}\mathrm{NS}\Big(\tilde{h}_{ij}^{s}\Big) & \\ \quad+\sum_{s=1}^{\#h_{jk}}\alpha_{jk}^{s,t}\mathrm{NS}\Big(\tilde{h}_{jk}^{s}\Big)-\sum_{s=1}^{\#h_{ik}}\alpha_{ik}^{s,t}\mathrm{NS}\Big(\tilde{h}_{ik}^{s}\Big)-0.5\Big|\geq\overline{\mathrm{ACI}} & \\ 1-\frac{4}{n(n-1)(n-2)}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\Big|\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,t}\mathrm{NS}\Big(\tilde{h}_{ij}^{s}\Big) & \\ \quad+\sum_{s=1}^{\#h_{jk}}\alpha_{ij}^{s,t}\mathrm{NS}\Big(\tilde{h}_{jk}^{s}\Big)-\sum_{s=1}^{\#h_{ik}}\alpha_{ij}^{s,t}\mathrm{NS}\Big(\tilde{h}_{ik}^{s}\Big)-0.5\ \Big|\geq\overline{\mathrm{WCI}} & t=1,2,\cdots,\#\Omega_{\tilde{H}}\\ \mathrm{NS}\Big(\tilde{h}_{ij}^{1}\Big)\leq\mathrm{NS}\Big(\tilde{h}_{ij}^{2}\Big)\leq\cdots\leq\mathrm{NS}\Big(\tilde{h}_{ij}^{\#h_{ij}}\Big), & i,j=1,2,\cdots,n,i<j\\ 0\leq\mathrm{NS}\Big(\tilde{h}_{ij}^{s}\Big)\leq1, & i,j=1,2,\cdots,n,i<j,\\ & s=1,2,\cdots,\#h_{ij}\\ \sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,t}=1, & i,j=1,2,\cdots,n,i<j,\\ & t=1,2,\cdots,\#\Omega_{H}\\ \alpha_{ij}^{s,t}=0\vee1, & i,j=1,2,\cdots,n,i<j,\\ & s=1,2,\cdots,\#h_{ij},\\ & t=1,2,\cdots,\#\Omega_{\tilde{H}}\\ \mathrm{NS}(s_0)=0 & \\ \mathrm{NS}(s_g)=1 & \\ \mathrm{NS}\big(s_{g/2}\big)=0.5 & \\ \mathrm{NS}(s_i)\in[(i-1)/g,(i+1)/g], & i=1,2,\cdots,g-1,i\neq g/2\\ \mathrm{NS}(s_{i+1})-\mathrm{NS}(s_i)\geq\lambda, & i=0,1,\cdots,g-1.\end{cases}\tag{14}$$

In equation (14), the first constraint guarantees that revised $\tilde{H}$ satisfies $\frac{1}{\#\Omega_{\tilde{H}}}\sum_{\tilde{L}\in\Omega_{\tilde{H}}}\mathrm{CI}(\tilde{L})\geq\overline{\mathrm{ACI}}$. The second constraint guarantees that revised $\tilde{H}$ satisfies $\min_{\tilde{L}\in\Omega_{\tilde{H}}}\mathrm{CI}(\tilde{L})\geq\overline{\mathrm{WCI}}$. It essentially needs to be satisfied for any LPRs associated with $\Omega_{\tilde{H}}$. Thus it follows that t takes values from 1 to $\#\Omega_{\tilde{H}}$, where $\#\Omega_{\tilde{H}}=\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\#h_{ij}$ be the number of LPRs associated with $\Omega_{\tilde{H}}$. The fourth to fifth constraints guarantee that revised $\tilde{H}$ is an HFLPR. Furthermore, $\tilde{h}_{ij}^{s}, i,j=1,2,\cdots,n,i<j$ and $s=1,2,\cdots,\#h_{ij}$ are $\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\#h_{ij}$ non-negative decision variables.

Example 3. Consider the HFLPR as follows

$$H=\begin{bmatrix}\{s_3\} & \{s_4\} & \{s_4\} & \{s_3\}\\ \{s_2\} & \{s_3\} & \{s_3\} & \{s_1\}\\ \{s_2\} & \{s_3\} & \{s_3\} & \{s_4,s_5,s_6\}\\ \{s_3\} & \{s_5\} & \{s_2,s_1,s_0\} & \{s_3\}\end{bmatrix}.$$

By equations (11) and (12), it is calculated that $\mathrm{WACI}(H)=0.76$ and $\mathrm{AACI}(H)=0.93$. Let $\overline{\mathrm{ACI}}=0.90$ and $\overline{\mathrm{WCI}}=0.80$. Since $\mathrm{WACI}(H)<\overline{\mathrm{WCI}}$, we have H is an unacceptable additive consistent HFLPR. According to equation (14), we have $\mathrm{NS}(\tilde{H})=\begin{bmatrix}\{0.50\} & \{0.51\} & \{0.51\} & \{0.00\}\\ \{0.49\} & \{0.50\} & \{0.50\} & \{0.20\}\\ \{0.49\} & \{0.50\} & \{0.50\} & \{0.00\}\\ \{1.00\} & \{0.80\} & \{1.00\} & \{0.50\}\end{bmatrix}$, which the revised HFLPR $\tilde{H}$ is an acceptable additive consistent HFLPR.

After solving equation (14), one can derive the values of each personalized numerical scales $\mathrm{NS}(\tilde{h}_{ij}^{s}), i,j=1,2,\cdots,n,i<j$ and $s=1,2,\cdots,\#h_{ij}$. However, sometimes the solution of the equation (14) is not unique, and it becomes necessary to further derive the optimal solutions. From the viewpoint of an DM, it is hoped that

the number of the adjusted elements of an HFLPR should be as small as possible. Based on this consideration, another optimization model is developed.

Let z^* be the objective function value derive from the equation (14), and $\delta_{ij}^s = \begin{cases} 1, & \text{if } h_{ij}^s \neq \tilde{h}_{ij}^s \\ 0, & \text{if } h_{ij}^s = \tilde{h}_{ij}^s \end{cases}, i,j = 1,2,\cdots,n,\ i < j$ and $s = 1,2,\cdots,\#h_{ij}$ be a list of binary variables such that $\delta_{ij}^s = 1$ denotes that h_{ij}^s is adjusted, while $\delta_{ij}^s = 0$ denotes that h_{ij}^s is not adjusted. Motivated by Zhang *et al.* [42]'s work, an optimization model is constructed as follows:

$$
\min \pi = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{\#h_{ij}} \delta_{ij}^s
$$
$$
\text{s.t.}\begin{cases}
\frac{2}{n(n-1)\#h_{ij}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{\#h_{ij}}\left|\mathrm{NS}(h_{ij}^s)-\mathrm{NS}\left(\tilde{h}_{ij}^s\right)\right| = z^* & \\
1-\frac{4}{n(n-1)(n-2)\#\Omega_{\tilde{H}}}\sum_{t=1}^{\#\Omega_{\tilde{H}}}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,t}\mathrm{NS}\left(\tilde{h}_{ij}^s\right)\right. & \\
\quad \left.+\sum_{s=1}^{\#h_{jk}}\alpha_{jk}^{s,t}\mathrm{NS}\left(\tilde{h}_{jk}^s\right)-\sum_{s=1}^{\#h_{ik}}\alpha_{ik}^{s,t}\mathrm{NS}\left(\tilde{h}_{ik}^s\right)-0.5\right| \geq \overline{\mathrm{ACI}} & \\
1-\frac{4}{n(n-1)(n-2)}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,t}\mathrm{NS}\left(\tilde{h}_{ij}^s\right)\right. & \\
\quad \left.+\sum_{s=1}^{\#h_{jk}}\alpha_{ij}^{s,t}\mathrm{NS}\left(\tilde{h}_{jk}^s\right)-\sum_{s=1}^{\#h_{ik}}\alpha_{ij}^{s,t}\mathrm{NS}\left(\tilde{h}_{ik}^s\right)-0.5\right| \geq \overline{\mathrm{WCI}} & t=1,2,\cdots,\#\Omega_{\tilde{H}} \\
\mathrm{NS}\left(\tilde{h}_{ij}^1\right) \leq \mathrm{NS}\left(\tilde{h}_{ij}^2\right) \leq \cdots \leq \mathrm{NS}\left(\tilde{h}_{ij}^{\#h_{ij}}\right), & i,j=1,2,\cdots,n, i<j \\
0 \leq \mathrm{NS}\left(\tilde{h}_{ij}^s\right) \leq 1, & i,j=1,2,\cdots,n, i<j, \\
 & s=1,2,\cdots,\#h_{ij} \\
\left(\mathrm{NS}(h_{ij}^s)-\mathrm{NS}\left(\tilde{h}_{ij}^s\right)\right)(1-\delta_{ij}^s)=0, & i,j=1,2,\cdots,n, i<j, \\
 & s=1,2,\cdots,\#h_{ij} \\
\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,t}=1, & i,j=1,2,\cdots,n, i<j, \\
 & t=1,2,\cdots,\#\Omega_{\tilde{H}} \\
\alpha_{ij}^{s,t}=0\vee 1, & i,j=1,2,\cdots,n, i<j, \\
 & s=1,2,\cdots,\#h_{ij}, \\
 & t=1,2,\cdots,\#\Omega_{\tilde{H}} \\
\mathrm{NS}(s_0)=0 & \\
\mathrm{NS}(s_g)=1 & \\
\mathrm{NS}(s_{g/2})=0.5 & \\
\mathrm{NS}(s_i)\in[(i-1)/g,(i+1)/g], & i=1,2,\cdots,g-1, i\neq g/2 \\
\mathrm{NS}(s_{i+1})-\mathrm{NS}(s_i)\geq\lambda, & i=0,1,\cdots,g-1.
\end{cases} \tag{15}
$$

One can easily found that the main difference between equations (14) and (15) is that two equality constraints $\frac{2}{n(n-1)\#h_{ij}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{\#h_{ij}}|\mathrm{NS}(h_{ij}^s)-\mathrm{NS}(\tilde{h}_{ij}^s)| = z^*$ and $(\mathrm{NS}(h_{ij}^s)-\mathrm{NS}(\tilde{h}_{ij}^s))(1-\delta_{ij}^s)=0$ are considered in equation (15). The first equality constraint means that equation (15) is developed based on the optimal solution of equation (14). The second equality constraint means that at least one of two relationships $\mathrm{NS}(h_{ij}^s)=\mathrm{NS}(\tilde{h}_{ij}^s)$ and $\delta_{ij}^s=1$ is hold. After solving equation (15), one can derive the values of each personalized numerical scales $\mathrm{NS}(\tilde{h}_{ij}^s), i,j=1,2,\cdots,n, i<j$ and $s=1,2,\cdots,\#h_{ij}$.

Remark 3. From equations (14) and (15), we only obtain the personalized numerical scales $\mathrm{NS}(\tilde{h_{ij}}^s)$, it needs to be further conducted the inverse function of personalized numerical scales to derive the linguistic variables.

4. A framework of MCGDM procedure with HFLPRs

In this section, a framework of MCGDM procedure with HFLPRs is introduced. There are three subsections concerned, including the MCGDM problems is described in Section 4.1. In Section 4.2, an algorithm is designed for deriving priority weights from acceptable additive consistent HFLPRs. And the framework of MCGDM procedure with HFLPRs is introduced in Section 4.3.

4.1. The MCGDM problems with HFLPRs

Hesitant linguistic MCGDM problems involve n alternatives denoted as $A = \{a_1, a_2, \cdots, a_n\}$. Each alternative is assessed in view of several feature criteria. Let $E = \{e_1, e_2, \cdots, e_m\}$ be a set of DMs and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ be their corresponding weight vector. Assuming that the weights of the DMs are determined advance, which is completely known. The evaluation of the alternatives according to the feature criterion is provided by DM e_p with HFLPRs, and denoted as $H^p = (h_{ij}^p)_{n\times n}$, where $h_{ij}^p = \{h_{ij}^{s,p} \mid s = 1, 2, \cdots, \#h^p\}$. This study aims to derive priority weights from $H^p, p = 1, 2, \cdots, m$ considering the individual consistency of HFLPRs, and applying a selection process to obtain a best alternative from A.

4.2. An algorithm for deriving priority weights from acceptable additive consistent HFLPRs

After obtaining acceptable additive consistent HFLPRs, an essential process is to derive priority weights. In this section, an algorithm for deriving the priority weights from acceptable additive consistent HFLPRs is developed. The flowchart of the algorithm is depicted in Figure 1.

Algorithm 1.

Input: An HFLPR H.
Output: Priority weights $\bar{w}_i, i = 1, 2, \cdots, n$.
Step 1: Compute the WACI and AACI.
The WACI and AACI respectively derive from equations (11) and (12). And the corresponding LPRs with personalized numerical scales are denoted as NS(A) and NS(B).
Step 2: Derive the acceptable additive consistent HFLPR.
If H meets consistency levels, then go to next step, otherwise, the acceptable additive consistent HFLPR derived from equation (14). If the solution is not unique, then conducts equation (15).
When the acceptable additive consistent HFLPR has been derived, then return to step 1.
Step 3: Obtain the priority weights.
First, the elements in NS(A) and NS(B) are respectively put into the following formula to derive the worst and average priority weights:

$$w_i = \frac{\sum_{k=1}^{n} \mathrm{NS}(h_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n} \mathrm{NS}(h_{ik})}, \qquad i = 1, 2, \cdots, n. \tag{16}$$

Obviously, equation (16) is called normalizing rank aggregation method presented in [43]. The worst and average priority weights are respectively obtained and denoted as w_{Wi} and w_{Ai}.
Then, the priority weights are obtained according to the following formula:

$$\bar{w}_i = [\min\{w_{Ai}, w_{W_i}\}, \max\{w_{Ai}, w_{W_i}\}], \qquad i = 1, 2, \cdots, n. \tag{17}$$

Step 4: Output $\bar{w}_i, i = 1, 2, \cdots, n$.
Step 5: End.

Remark 4. In step 1, there are may be more than one corresponding LPRs are derived. In this case, the worst and average priority weights are developed as average of multiple priority weight vectors:

$$w = (w_1, w_2, \cdots, w_n) = \left(\frac{1}{l_0}\sum_{k=1}^{l_0} w_1^k, \frac{1}{l_0}\sum_{k=1}^{l_0} w_2^k, \cdots, \frac{1}{l_0}\sum_{k=1}^{l_0} w_n^k\right), \tag{18}$$

where $w_i^k, k = 1, 2, \cdots, l_0$, and it indicates that the number of corresponding LPRs including in A or B is l_0.

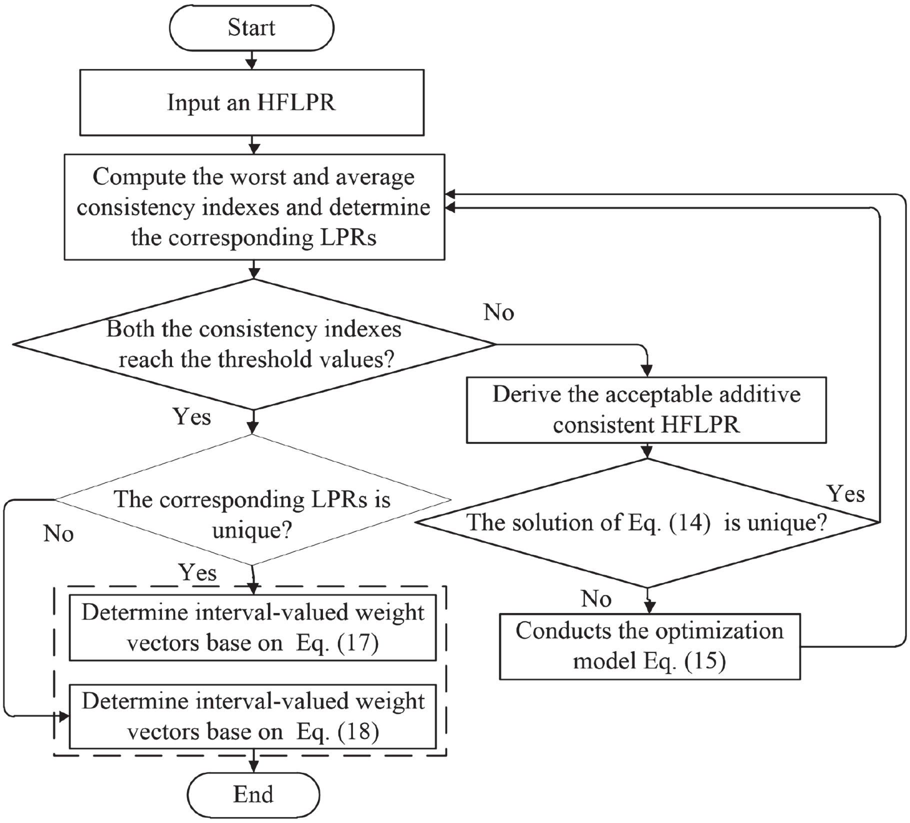

FIGURE 1. A framework for deriving priority weights from HFLPRs.

4.3. A framework of MCGDM procedure with HFLPRs

The proposed decision-making procedure is summarized in the following steps.

Step 1: Form individual HFLPRs.

According to the determine criteria, the DMs respectively provide their preference values of alternative a_i compare to alternative a_j, and denotes as $H^p = \left(h_{ij}^p\right)_{m\times m}, p = 1, 2, \cdots, m$.

Step 2: Derive priority weights of individual HFLPRs.

The priority weights of individual HFLPRs are determined with respective to Algorithm 1.

Step 3: Obtain the collective priority weights.

The collective priority weights are derived based on the following formula:

$$\bar{w}_i = \left[\sum_{p=1}^{m} \lambda_p w_i^{p-}, \sum_{p=1}^{m} \lambda_p w_i^{p+}\right], \qquad i = 1, 2, \cdots, n \tag{19}$$

where λ_p is the weight vector of DM e_p, w_i^{p-} and w_i^{p+} are derived from step 2.

Step 4: Calculate the dominance relation.

The dominance relation of interval-valued weight vector is determined according to the following formula [44]:

$$P(\bar{w}_i \geq \bar{w}_j) = \min\left\{\max\left\{\frac{w_i^+ - w_j^-}{w_i^+ - w_i^- + w_j^+ - w_j^-}, 0\right\}, 1\right\}. \tag{20}$$

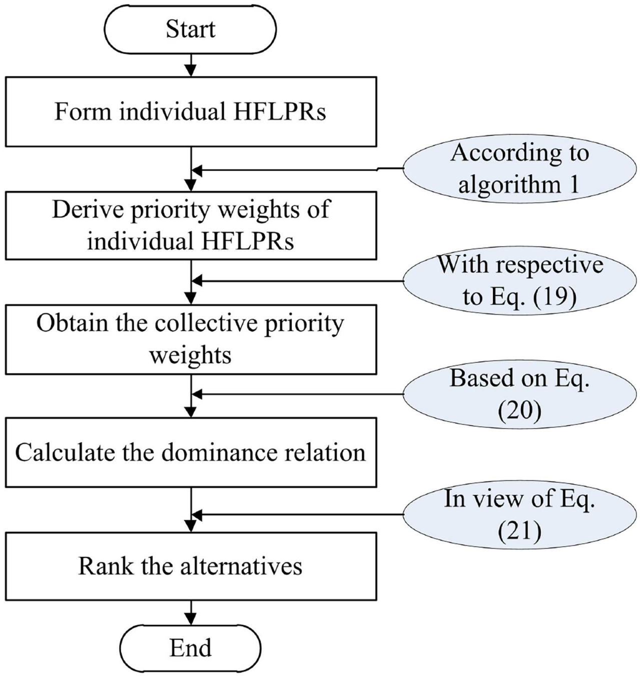

FIGURE 2. A framework of MCGDM process with HFLPRs.

Step 5: Rank the alternatives.

The ranking order of all alternatives is obtained based on the sum of the dominance relation:

$$\vartheta_i = \sum_{j=1}^{n} P(\bar{w}_i \geq \bar{w}_j), \qquad i = 1, 2, \cdots, n. \tag{21}$$

The proposed decision-making procedure is depicted in Figure 2.

5. An illustrative example

This section begins with a numerical example (revised from [23]) for 3D visualization management system selection of an automobile manufacturing enterprise to illustrate the implementation of the proposed method, and then provides detailed comparative analysis to justify the algorithm.

5.1. The descriptions of illustrative example

With the proposal of "Made in China 2025" and Industry 4.0 plan, manufacturing industry has gained in new development opportunities. Regarding to promote advanced manufacturing, the guideline urges the building of industrial bases focused on sectors including intelligent manufacturing, new materials, new energy vehicles and electronic information. So as to make the automobile production process transparent and the management more refined and efficient, the manufacturing industry has entered the era of intelligent manufacturing. Since the use of information technology in the intelligent manufacturing industry can improve the production and management capabilities of enterprises. In the whole process of manufacturing, more and more manufacturing enterprises are widely applying information technology.

To realize the visualization of the automobile production line and promote enterprise to enter the era of intelligent manufacturing. An automobile manufacturing enterprise wants to introduce a 3D visualization technology in the production process. After preliminary investigation and market research, four possible 3D visualization management systems developed from four information technology enterprises are determined, which are denoted as $A = \{a_1, a_2, a_3, a_4\}$. To select the best one, a decision-making team comes from the core departments of the automobile manufacturing enterprise is set up, which is composed of four DMs $E = \{e_1, e_2, e_3, e_4\}$. Without loss of generality, the four DMs are assumed to be of equal importance, that is, their weight vector is set as $\lambda = (0.25, 0.25, 0.25, 0.25)$.

By considering several characteristic factors such as automation level, cost and practicality, the four DMs provide their evaluation information for the four 3D visualization management systems through pairwise comparisons after further discussion and investigation, and the evaluation information is expressed with HFLPRs, which indicates the preference degree of alternative a_i compared with alternative a_j. To be specific, the LTS $S = \{s_0$: very poor, s_1: poor, s_2: slightly poor, s_3: fair, s_4: slightly good, s_5: good, s_6: very good$\}$ is used and the initial HFLPRs provided by the four DMs are respectively demonstrated in H^1, H^2, H^3 and H^4.

$$H^1 = \begin{bmatrix} \{s_3\} & \{s_1\} & \{s_3, s_4\} & \{s_3\} \\ \{s_5\} & \{s_3\} & \{s_6\} & \{s_5, s_6\} \\ \{s_3, s_2\} & \{s_3\} & \{s_3\} & \{s_2\} \\ \{s_3\} & \{s_1, s_0\} & \{s_4\} & \{s_3\} \end{bmatrix},$$

$$H^2 = \begin{bmatrix} \{s_3\} & \{s_4\} & \{s_4\} & \{s_3\} \\ \{s_2\} & \{s_3\} & \{s_3\} & \{s_2\} \\ \{s_2\} & \{s_3\} & \{s_3\} & \{s_2, s_3, s_4\} \\ \{s_3\} & \{s_4\} & \{s_4, s_3, s_2\} & \{s_3\} \end{bmatrix},$$

$$H^3 = \begin{bmatrix} \{s_3\} & \{s_4\} & \{s_6\} & \{s_4, s_5\} \\ \{s_2\} & \{s_3\} & \{s_5\} & \{s_3\} \\ \{s_0\} & \{s_1\} & \{s_3\} & \{s_1\} \\ \{s_2, s_1\} & \{s_3\} & \{s_5\} & \{s_3\} \end{bmatrix},$$

and

$$H^4 = \begin{bmatrix} \{s_3\} & \{s_4\} & \{s_3\} & \{s_4, s_5\} \\ \{s_2\} & \{s_3\} & \{s_2\} & \{s_4\} \\ \{s_3\} & \{s_4\} & \{s_3\} & \{s_4, s_5\} \\ \{s_2, s_1\} & \{s_2\} & \{s_2, s_1\} & \{s_3\} \end{bmatrix}.$$

5.2. Illustration of the proposed method

The procedure for selecting a potential supplier using the proposed method is presented as follows.

Step 1: Form individual HFLPRs.

All individual HFLPR matrices have been provided, as demonstrated in matrices H^1, H^2, H^3 and H^4.

Step 2: Derive priority weights of individual HFLPRs.

First, compute the WACI and AACI.

Since the individual evaluation information is completely HFLPRs. According to equation (11), the values of WACI for H^1, H^2, H^3 and H^4 are calculated as follows: $\mathrm{WACI}(H^1) = 0.86, \mathrm{WACI}(H^2) = 0.89, \mathrm{WACI}(H^3) = 0.87$ and $\mathrm{WACI}(H^4) = 0.81$.

In a similar way, according to equation (12), the values of AACI for H^1, H^2, H^3 and H^4 are calculated as follows: $\mathrm{AACI}(H^1) = 0.99, \mathrm{AACI}(H^2) = 0.97, \mathrm{AACI}(H^3) = 0.99$ and $\mathrm{AACI}(H^4) = 0.99$.

Second, derive the acceptable additive consistent HFLPR.

Let $\overline{\mathrm{ACI}} = 0.90$ and $\overline{\mathrm{WCI}} = 0.80$. For H^1, H^2, H^3 and H^4. Since $\mathrm{WACI}\big(H^1\big) > \overline{\mathrm{WCI}}, \mathrm{WACI}\big(H^2\big) > \overline{\mathrm{WCI}}, \mathrm{WACI}\big(H^3\big) > \overline{\mathrm{WCI}}, \mathrm{WACI}\big(H^4\big) > \overline{\mathrm{WCI}}$ and $\mathrm{AACI}\big(H^1\big) > \overline{\mathrm{ACI}}, \mathrm{AACI}\big(H^2\big) > \overline{\mathrm{ACI}}, \mathrm{AACI}\big(H^3\big) > \overline{\mathrm{ACI}}, \mathrm{AACI}\big(H^4\big) > \overline{\mathrm{ACI}}$, we have H^1, H^2, H^3 and H^4 are acceptable additive consistent HFLPRs, their corresponding personalized numerical scales for WACI and AACI are listed as follows:

$$\mathrm{NS}(A^1) = \begin{bmatrix} \{0.50\} & \{0.33\} & \{0.50\} & \{0.50\} \\ \{0.67\} & \{0.50\} & \{1.00\} & \{1.00\} \\ \{0.50\} & \{0.00\} & \{0.50\} & \{0.34\} \\ \{0.50\} & \{0.00\} & \{0.66\} & \{0.50\} \end{bmatrix},$$

$$\mathrm{NS}(A^2) = \begin{bmatrix} \{0.50\} & \{0.51\} & \{0.51\} & \{0.50\} \\ \{0.49\} & \{0.50\} & \{0.50\} & \{0.17\} \\ \{0.49\} & \{0.50\} & \{0.50\} & \{0.51\} \\ \{0.50\} & \{0.83\} & \{0.49\} & \{0.50\} \end{bmatrix},$$

$$\mathrm{NS}(A^3) = \begin{bmatrix} \{0.50\} & \{0.51\} & \{1.00\} & \{0.67\} \\ \{0.49\} & \{0.50\} & \{0.67\} & \{0.50\} \\ \{0.00\} & \{0.33\} & \{0.50\} & \{0.01\} \\ \{0.33\} & \{0.50\} & \{0.99\} & \{0.50\} \end{bmatrix},$$

$$\mathrm{NS}(A^4) = \begin{bmatrix} \{0.50\} & \{0.51\} & \{0.50\} & \{0.99\} \\ \{0.49\} & \{0.50\} & \{0.17\} & \{0.99\} \\ \{0.50\} & \{0.83\} & \{0.50\} & \{0.51\} \\ \{0.01\} & \{0.01\} & \{0.49\} & \{0.50\} \end{bmatrix},$$

$$\mathrm{NS}(B^1) = \begin{bmatrix} \{0.50\} & \{0.07\} & \{0.56\} & \{0.50\} \\ \{0.93\} & \{0.50\} & \{1.00\} & \{0.94\} \\ \{0.44\} & \{0.00\} & \{0.50\} & \{0.43\} \\ \{0.50\} & \{0.06\} & \{0.57\} & \{0.50\} \end{bmatrix},$$

$$\mathrm{NS}(B^2) = \begin{bmatrix} \{0.50\} & \{0.51\} & \{0.51\} & \{0.50\} \\ \{0.49\} & \{0.50\} & \{0.50\} & \{0.49\} \\ \{0.49\} & \{0.50\} & \{0.50\} & \{0.50\} \\ \{0.50\} & \{0.51\} & \{0.50\} & \{0.50\} \end{bmatrix},$$

$$\mathrm{NS}(B^3) = \begin{bmatrix} \{0.50\} & \{0.63\} & \{1.00\} & \{0.63\} \\ \{0.37\} & \{0.50\} & \{0.87\} & \{0.50\} \\ \{0.00\} & \{0.13\} & \{0.50\} & \{0.13\} \\ \{0.37\} & \{0.50\} & \{0.87\} & \{0.50\} \end{bmatrix},$$

and

$$\mathrm{NS}(B^4) = \begin{bmatrix} \{0.50\} & \{0.57\} & \{0.50\} & \{0.61\} \\ \{0.43\} & \{0.50\} & \{0.44\} & \{0.59\} \\ \{0.50\} & \{0.56\} & \{0.50\} & \{0.61\} \\ \{0.39\} & \{0.41\} & \{0.39\} & \{0.50\} \end{bmatrix}.$$

Third, obtain the priority weights.

The worst and average priority weights are obtained from equation (16) as follows:
$w_{W1}^1 = 0.23, w_{W2}^1 = 0.40, w_{W3}^1 = 0.17, w_{W4}^1 = 0.21; w_{W1}^2 = 0.25, w_{W2}^2 = 0.21, w_{W3}^2 = 0.25, w_{W4}^2 = 0.29; w_{W1}^3 = 0.34, w_{W2}^3 = 0.27, w_{W3}^3 = 0.11, w_{W4}^3 = 0.29$ and $w_{W1}^4 = 0.31, w_{W2}^4 = 0.27, w_{W3}^4 = 0.29, w_{W4}^4 = 0.13$. Besides, $w_{A1}^1 = 0.21, w_{A2}^1 = 0.40, w_{A3}^1 = 0.18, w_{A4}^1 = 0.21; w_{A1}^2 = 0.25, w_{A2}^2 = 0.25, w_{A3}^2 = 0.25, w_{A4}^2 = 0.25; w_{A1}^3 = 0.36, w_{A2}^3 = 0.28, w_{A3}^3 = 0.07, w_{A4}^3 = 0.28$ and $w_{A1}^4 = 0.26, w_{A2}^4 = 0.26, w_{A3}^4 = 0.26, w_{A4}^4 = 0.22$.
Based on equation (17), the priority weights are obtained as follows:
$\bar{w}_1^1 = [0.21, 0.23], \bar{w}_2^1 = [0.40, 0.40], \bar{w}_3^1 = [0.17, 0.18], \bar{w}_4^1 = [0.21, 0.21]; \bar{w}_1^2 = [0.25, 0.25], \bar{w}_2^2 = [0.21, 0.25], \bar{w}_3^2 = [0.25, 0.25], \bar{w}_4^2 = [0.25, 0.29]; \bar{w}_1^3 = [0.34, 0.36], \bar{w}_2^3 = [0.27, 0.28], \bar{w}_3^3 = [0.07, 0.11], \bar{w}_4^3 = [0.28, 0.29]$ and $\bar{w}_1^4 = [0.26, 0.31], \bar{w}_2^4 = [0.26, 0.27], \bar{w}_3^4 = [0.26, 0.29], \bar{w}_4^4 = [0.13, 0.22]$.

Table 1. The ranking results of the different methods.

Methods	Ranking values				Ranking results
	ϑ_1	ϑ_2	ϑ_3	ϑ_4	
Li *et al.* [23]'s method	3.39	2.11	0.50	1.99	$a_1 \succ a_2 \succ a_4 \succ a_3$
The proposed method	2.50	3.50	0.50	1.50	$a_2 \succ a_1 \succ a_4 \succ a_3$

Step 3: Obtain the collective priority weights.

Since the weight vector of DMs is $\lambda = (0.25, 0.25, 0.25, 0.25)$, the collective priority weights are derived with respective to equation (19) as: $\bar{w}_1 = [0.27, 0.29], \bar{w}_2 = [0.29, 0.30], \bar{w}_3 = [0.19, 0.21]$ and $\bar{w}_4 = [0.22, 0.25]$.

Step 4: Calculate the dominance relation.

According to equation (20), the dominance relation of interval-valued weight vector is determined as:

$$P = \begin{bmatrix} 0.50 & 0.00 & 1.00 & 1.00 \\ 1.00 & 0.50 & 1.00 & 1.00 \\ 0.00 & 0.00 & 0.50 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.50 \end{bmatrix}.$$

Step 5: Rank the alternatives.

Calculate the sum of the dominance relation by equation (21) as $\vartheta_1 = 2.50, \vartheta_2 = 3.50, \vartheta_3 = 0.50$ and $\vartheta_4 = 1.50$. Since $\vartheta_2 > \vartheta_1 > \vartheta_4 > \vartheta_3$, then the ranking order is $a_2 > a_1 > a_4 > a_3$. Thus, the information technology enterprises from a_2 is the best choice.

5.3. Comparative analysis and discussion

To validate the feasibility of the proposed method, a comparative study with other method based on the same illustrative example is conducted.

Li *et al.* [23] developed a new method for MCGDM problems with HFLPRs considering the BCI and ACI simultaneously. First, an optimization model to impute missing elements of an IHFLPRs is developed. And then, some local adjustment strategy-based feedback adjustment rules are proposed in consensus reaching process. Moreover, an iterative consensus reaching algorithm for MCGDM with IHFLPRs is also proposed. Finally, the ranking index is developed based on the uncertain 2-tuple linguistic averaging operator. To have a better comparison, the results obtained by Li *et al.* [23]'s method and the proposed method are summarized in Table 1.

As shown in Table 1, it can be seen that different ranking results are obtained from these two methods in the same illustrative example. The best choice derived from Li *et al.* [23]'s method is a_1, while obtained from the proposed method is a_2. These two methods both develop the processes to check the consistency of HFLPRs. For the unacceptable consistency one, the algorithms are designed to improve the consistency level. To check the consistency of the individual HFLPRs, Li *et al.* [23]'s method by constructing the completely additive consistent HFLPRs, and then obtained the consistency index by considering the BCI and the ACI. While the proposed method takes into accounted the WACI and AACI. Compared with the best case, the worst case is considered in most of the MCGDM environment. Based on this fact, the proposed method utilizes the WACI and AACI to check the consistency, which have a wider application background. What is more, in the process of calculating consistency indexes, the proposed method takes into accounted the PISs model. Since different DMs may have different understandings of the linguistic terms. In other words, words mean different things for different people. For that, the results derived from the proposed method seems more reasonable.

To verify the advantages of the proposed method, it will compare with several representative methods under the MCGDM environments with HFLPRs. Table 2 presents the performances of these methods regarding several indexes.

TABLE 2. Comparisons of different MCGDM methods with HFLPRs.

Methods	Whether consider the consistency measurement and improvement processes	Whether adding extra values	Whether consider the WCI and ACI simultaneous
The BCI, WCI and ACI-based methods	Yes	No	No
Normalization-based methods	Yes	Yes	No
Satisfaction degree-based methods	Yes	No	No
The proposed method	Yes	No	Yes

(1) BCI, WCI and ACI-based methods: These methods were developed to check and improve the consistency of the LPRs associated with HFLPRs. For instance, Li *et al.* [12] proposed the interval consistency index to estimate the consistency range of an HFLPR, and the interval consistency index consists of measuring the BCI and WCI of an HFLPR. Li *et al.* [23] developed an optimization model to impute missing elements for an IHFLPR, in which the BCI and ACI are utilized to measure the consistency level of an IHFLPR. Chen *et al.* [27] asserted the consistency checking by using the WCI. These methods do not need to add extra values into original HFLPRs, and can maintain the integrity of the original evaluation information. It is natural that the WCI and ACI of the LPRs associated with the HFLPR play an important role in analyzing the consistency of an HFLPR. However, current researches have not yet studied the measurement of these two indexes related to HFLPRs.
(2) Normalization-based methods: These methods developed consistency measurement and improvement processes based on the normalized HFLPRs (NHFLPRs). Zhang *et al.* [22] introduced the concept of multiplicative consistent HFLPRs based on NHFLPRs. In view of NHFLPRs, Wu *et al.* [45] developed a local feedback mechanism to improve the consistency of HFLPRs. Zhu *et al.* [21] showed several methods of consistency measures by α and β normalized methods. The normalized index needs to add other values into the original HFLPRs, and the consistency index may be different if we derive different NHFLPRs. As Li *et al.* [12] asserted, the normalization method measured several LPRs associated with an HFLPR. Owing to this partial measurement, the internal mechanism of the normalization method is not clear. Compared with normalization-based methods, the proposed method does not consider the normalized process. In view of these, the proposed method has an advantage in avoiding the distortion of information and the calculation process seems more reasonable.
(3) Satisfaction degree-based methods. These methods developed the priority weights deriving process without consistency measurement and improvement processes. Ren *et al.* [11] introduced a priority weights deriving method by introducing DMs' satisfaction degree to measure the differences between the IHFLPRs and its corresponding weight vector. These methods directly handle the original HFLPRs and obtain the weight vector under the consistent conditions, which reduces the complexities of the decision-making processes. However, the BCI, WCI and ACI of HFLPRs were disregarded. In view of these, the proposed method has advantage in analyzing the consistency of an HFLPR and avoiding the loss of information.

According to the comparative analysis, the method proposed in this study has the following advantages over other existing methods.

(1) The WACI and AACI are considered in the consistency measurement and improvement processes simultaneously. It not only overcomes the shortcoming of being difficult to achieve due to being too restricted, but also the loss of information due to being too loose.

(2) In the consistency improvement processes, two predefined thresholds for the WACI and AACI are taken into accounted. It is required all the LPRs associated with HFLPRs meet the threshold of WACI. This process considers all the evaluation information and avoids the loss of information.
(3) Since different DMs may have different understandings of the linguistic terms. In other words, words mean different things for different people. The proposed method taken into accounted the PISs in consistency measurement and improvement processes seems more reasonable.

6. Conclusion

To address the MCGDM problems with HFLPRs, this study develops an GDM method in view of WACI and AACI simultaneously. First, several optimization models are constructed for deriving the WACI and AACI of HFLPRs. And then, by considering the characteristic of WACI and AACI, the concept of acceptable additive consistent HFLPRs is developed. Second, several optimization models are constructed for improving the consistency of HFLPRs. Third, an algorithm is designed for deriving priority weights from acceptable consistent HFLPRs. Finally, the presented models are validated using a numerical example and extensive comparative analyses.

The present study provides several significant contributions for MCGDM problems with HFLPRs. They are summarized as follows: (1) it is natural that the worst and average consistency levels of the LPRs associated with an HFLPR play an important role in analyzing the consistency of HFLPRs. For that, the concepts of WACI and AACI are introduced. Meanwhile, several optimization models for deriving the WACI and AACI are proposed. (2) The concept of acceptable additive consistent HFLPRs is developed, which takes into accounted two predefined thresholds for the WACI and AACI. (3) Several programming models are constructed to improve the consistency of HFLPRs. This provides the direction for the DMs to revise their evaluation values, and the consistency improving process is time-saving and easy to been extended to incompletely evaluation information. In our future research, the proposed method is extended to other hesitant environments and applied to other practical MCGDM problems.

Acknowledgements

The authors thank the anonymous reviewers and the editor for their insightful and constructive comments and suggestions that have led to an improved version of this paper.

Funding

This work was supported by the National Natural Science Foundation of China (Nos. 72061026 and 72461027), the Natural Science Foundation of Guangxi (No. 2020GXNSFAA297239), and the Science and Technology Plan of Guangxi (gui ke AD20238006).

Conflict of Interest

All the authors declare that they have no conflict of interest.

Data availability statement

All data generated or analyzed during this study are included in this published article (and its supplementary information files).

Ethical approval

This article does not contain any studies with human participants performed by any of the authors.

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