RAIRO-Oper. Res. 59 (2025) 3675–3681
https://doi.org/10.1051/ro/2025148

RAIRO Operations Research
www.rairo-ro.org

CHARACTERIZING PATH-FACTOR UNIFORM GRAPHS WITH RESPECT TO THE DEGREE SUM OF NON-ADJACENT VERTICES

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Abstract. For a graph G and a set $\mathcal{H}$ of connected graphs, an $\mathcal{H}$-factor of G is a spanning subgraph of G with each component isomorphic to some member in $\mathcal{H}$. If each component of $\mathcal{H}$ is isomorphic to a path, then we call the $\mathcal{H}$-factor a path-factor. For each integer $k \geq 2$, a graph G is $P_{\geq k}$-factor uniform if for any two distinct edges e_1 and e_2, G admits a $P_{\geq k}$-factor including e_1 and excluding e_2. In this note, we determine two lower bounds on the degree sum of non-adjacent vertices to ensure that G is $P_{\geq k}$-factor uniform for $k = 2$ and $k = 3$. Furthermore, we construct some extremal graphs to show that the bounds are best possible. The results improve some known results slightly.

Mathematics Subject Classification. 05C38, 05C70.

Received November 27, 2024. Accepted October 23, 2025.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, we use $G - S$ to denote the graph obtained from G by deleting the vertices of S and edges with at least one endpoint in S. For $v \in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the *degree* of v and the *set of neighbors* of v in G, respectively. For $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$. For $S, U \subseteq V(G)$, we use $e_G(S, U)$ to denote the number of edges between S and U in G. If $S = \{s\}$, we also use $G - s$ and $e_G(s, U)$ to denote $G - \{s\}$ and $e_G(\{s\}, U)$ for short. For $e \in E(G)$, $G - e$ is the graph obtained from G by deleting e. We use $I(G)$ to denote the *set of isolated vertices* of G and write $i(G) = |I(G)|$. $\omega(G)$ is used to denote the *number of components* of G.

For two disjoint graphs G and H, the *union* of G and H, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For a given positive integer k and graph H, we also use kH to denote the union of k graphs isomorphic to H. The *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding the edges $\{uv | u \in V(G), v \in V(H)\}$. The *binding number* of G was introduced by Woodall [21] and defined as

$$bind(G) = \min\left\{\frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G)\right\}.$$

Keywords. Degree sum, edge-connectivity, path-factor uniform.

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Let $\alpha(G)$ be the *independence number* of G. If $\alpha(G) < k$, then $\sigma_k(G)$ is defined as $+\infty$. Otherwise, it is defined as follows

$$\sigma_k(G) = \min\left\{\sum_{x\in X} d_G(x) : X \text{ is an independent set of } k \text{ vertices}\right\}.$$

A subgraph H of G is called a *spanning subgraph* of G if $V(H) = V(G)$. For a graph G and a set $\mathcal{H}$ of connected graphs, an *$\mathcal{H}$-factor* of G is a spanning subgraph of G with each component isomorphic to some member in $\mathcal{H}$. If each component of $\mathcal{H}$ is isomorphic to a path, then we call the $\mathcal{H}$-factor a *path-factor*. For an integer $k \geq 2$, let P_k be the path of order k. A $\{P_2\}$-factor is also called a 1-factor. A $\{P_k, P_{k+1}, ...\}$-factor is also written as a $P_{\geq k}$-factor.

Tutte [18] in 1947 first gave a necessary and sufficient condition for graphs containing 1-factors. After that the path-factor problems have received a lot of attention, see for example [2–8, 10–16, 19, 20, 23–31]. There are also many interesting results on other factors, we refer the readers to [1, 17, 22].

A graph G is called *factor-critical* if $G - u$ contains a $\{P_2\}$-factor for any $u \in V(G)$. Kaneko [11] introduced the concept of sun as follows. For a graph H and a graph R, we call H a *corona* of R if H is obtained from R by adding a new vertex $w = w(v)$ together with a new edge vw for every $v \in V(R)$. A graph H is called a *sun* if $H \cong K_1$, $H \cong K_2$, or H is the corona of a factor-critical graph R with $|V(R)| \geq 3$. A sun with order one and order at least six is called a *trivial sun* and a *big sun*, respectively. For graphs G and K, if K is a component of G and K is a sun, then we call K a *sun component* of G. We denote by $sun(G)$ the *number of sun components* of G.

For $P_{\geq 3}$-factors, Kaneko [11] gave the following necessary and sufficient condition, which was also proved by Kano *et al.* [13].

Theorem 1.1 ([11,13]). *A graph G contains a $P_{\geq 3}$-factor if and only if $sun(G - S) \leq 2|S|$ for any $S \subseteq V(G)$.*

Zhang and Zhou [23] introduced the definition of $P_{\geq k}$-factor covered graphs. A graph G is *$P_{\geq k}$-factor covered* if G has a $P_{\geq k}$-factor containing e for any $e \in E(G)$. They also gave the following characterizations for graphs to be $P_{\geq 2}$-factor covered and $P_{\geq 3}$-factor covered.

Theorem 1.2 ([23]). *Let G be a connected graph. Then G is $P_{\geq 2}$-factor covered if and only if*

$$i(G - S) \leq 2|S| - \epsilon_1(S)$$

for any $S \subseteq V(G)$, where $\epsilon_1(S)$ is defined as follows:

$$\epsilon_1(S) = \begin{cases} 2, & \text{if } S \neq \emptyset \text{ and } S \text{ is not an independent set,} \\ 1, & \text{if } S \text{ is a nonempty independent set and } G - S \text{ admits a non-trivial component,} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.3 ([23]). *Let G be a connected graph. Then G is $P_{\geq 3}$-factor covered if and only if*

$$sun(G - S) \leq 2|S| - \epsilon_2(S)$$

for any $S \subseteq V(G)$, where $\epsilon_2(S)$ is defined as follows:

$$\epsilon_2(S) = \begin{cases} 2, & \text{if } S \neq \emptyset \text{ and } S \text{ is not an independent set,} \\ 1, & \text{if } S \text{ is a nonempty independent set and } G - S \text{ admits a non-sun component,} \\ 0, & \text{otherwise.} \end{cases}$$

Zhou and Sun [29] generalized the definition of $P_{\geq k}$-factor covered graphs to $P_{\geq k}$-factor uniform graphs. A graph G is *$P_{\geq k}$-factor uniform* if $G - f$ is $P_{\geq k}$-factor covered for any $f \in E(G)$. They also gave the following binding number condition for graphs to be $P_{\geq 3}$-factor uniform.

Theorem 1.4 ([29]). *Let G be a 2-edge-connected graph. If $bind(G) > \frac{9}{4}$, then G is $P_{\geq 3}$-factor uniform.*

Gao and Wang [9] improved the above binding number condition to $bind(G) > \frac{5}{3}$, which is tight. Dai [5] further considered 3-edge-connected graphs and gave the following two conditions for graphs to be $P_{\geq 2}$-factor uniform and $P_{\geq 3}$-factor uniform.

Theorem 1.5 ([5]). *Let G be a 3-edge-connected graph of order n. If $\sigma_2(G) > \frac{2n+5}{3}$, then G is $P_{\geq 2}$-factor uniform.*

Theorem 1.6 ([5]). *Let G be a 3-edge-connected graph of order n. If $\sigma_2(G) > \frac{2n+8}{3}$, then G is $P_{\geq 3}$-factor uniform.*

Let $t = 2, 3$. The maximum known $\sigma_2(G)$ of a 3-edge-connected graph G with order n that is not $P_{\geq t}$-factor uniform is $\frac{2n+t(t-1)}{3}$, which was given in [5]. In this note, we show that $\sigma_2(G) > \frac{2n+t(t-1)}{3}$ is sufficient for a 3-edge-connected graph G of order n to be $P_{\geq t}$-factor uniform.

2. Our main results

Theorem 2.1. *Let G be a 3-edge-connected graph of order n. If $\sigma_2(G) > \frac{2n+2}{3}$, then G is $P_{\geq 2}$-factor uniform.*

Theorem 2.2. *Let G be a 3-edge-connected graph of order n. If $\sigma_2(G) > \frac{2n+6}{3}$, then G is $P_{\geq 3}$-factor uniform.*

3. Proof of Theorem 2.1

Now we give the proof of Theorem 2.1. Suppose, to the contrary, that G is a 3-edge-connected graph of order n with $\sigma_2(G) > \frac{2n+2}{3}$, but G is not $P_{\geq 2}$-factor uniform. That is, there is an edge $e \in E(G)$ such that $G - e$ is not $P_{\geq 2}$-factor covered. Let $e = uv$ and $G' = G - e$. Then $|V(G')| = |V(G)| \geq 4$. By Theorem 1.2, there is a subset $S \subseteq V(G')$ such that

$$i(G' - S) \geq 2|S| - \epsilon_1(S) + 1. \tag{1}$$

By (1), we have the following statements.

Claim 3.1. $|S| \geq 2$.

Proof. Suppose, to the contrary, that $|S| \leq 1$. We divide the proof into two cases according to the value of $|S|$.

Case 1. $|S| = 0$.

According to the definition of $\epsilon_1(S)$, we have $\epsilon_1(S) = 0$ for $|S| = 0$. Then by (1), we have

$$i(G') = i(G' - S) \geq 1. \tag{2}$$

Since G is 3-edge-connected and $G' = G - e$, we have G' is connected. Together with (2), we have $i(G') = \omega(G') = 1$ and $G' \cong K_1$, which contradicts to $|V(G')| \geq 4$.

Case 2. $|S| = 1$.

By the definition of $\epsilon_1(S)$, we have $\epsilon_1(S) \leq 1$ for $|S| = 1$. Then by (1), we have $i(G' - S) \geq 2|S| - \epsilon_1(S) + 1 \geq 2|S| = 2$. Let $x \in I(G' - S)$. Then $d_{G'}(x) \leq |S| = 1$ and $d_G(x) \leq d_{G'}(x) + 1 \leq 2$. It means that G contains an edge cut of size at most 2, which contradicts to the 3-edge-connectivity of G.

This completes the proof of Claim 3.1. □

By Claim 3.1 and (1), we obtain that

$$i(G'-S) \geq 2|S| - \epsilon_1(S) + 1 \geq 2|S| - 1. \tag{3}$$

Since $G' = G - e$, we have

$$i(G-S) \geq i(G'-S) - 2. \tag{4}$$

Combining with Claim 3.1, (3) and (4), we have that

$$i(G-S) \geq i(G'-S) - 2 \geq 2|S| - 3 \geq 1. \tag{5}$$

Claim 3.2. $i(G-S) \geq 2$.

Proof. Suppose, to the contrary, that $i(G-S) = 1$. Then by (5), we have $i(G'-S) = 3$ and $|S| = 2$. Let $I(G-S) = \{y\}$. Then $y \notin \{u, v\}$, $N_G(y) \subseteq S$ and $d_G(y) \leq |S| = 2$. It means that G contains an edge cut of size at most 2, which contradicts to the 3-edge-connectivity of G. □

Let z_1, z_2 be two vertices of $I(G-S)$. Then $d_G(z_1) + d_G(z_2) \geq \sigma_2(G) > \frac{2n+2}{3}$ and $N_G(z_1) \cup N_G(z_2) \subseteq S$. So we have

$$|S| \geq \max\{d_G(z_1), d_G(z_2)\} \geq \frac{d_G(z_1) + d_G(z_2)}{2} > \frac{n+1}{3}. \tag{6}$$

Combining with (3) and (6), we have that

$$n = |V(G')| \geq |S| + i(G'-S) \geq 3|S| - 1 > 3 \cdot \frac{n+1}{3} - 1 = n,$$

a contradiction.

This completes the proof of Theorem 2.1. □

4. Proof of Theorem 2.2

Suppose, to the contrary, that G is a 3-edge-connected graph of order n with $\sigma_2(G) > \frac{2n+6}{3}$, but G is not $P_{\geq 3}$-factor uniform. That is, there is an edge $e \in E(G)$ such that $G - e$ is not $P_{\geq 3}$-factor covered. Let $e = uv$ and $G' = G - e$. Then by Theorem 1.3, there is a subset $S \subseteq V(G')$ such that

$$sun(G'-S) \geq 2|S| - \epsilon_2(S) + 1. \tag{7}$$

By (7), we have the following statements.

Claim 4.1. $|S| \geq 1$.

Proof. Suppose, to the contrary, that $|S| = 0$. According to the definition of $\epsilon_2(S)$, we have $\epsilon_2(S) = 0$. Then by (7), we have

$$sun(G') = sun(G'-S) \geq 1. \tag{8}$$

Since G is 3-edge-connected and $G' = G - e$, we have G' is connected. So

$$sun(G') \leq \omega(G') = 1. \tag{9}$$

Combining with (8) and (9), we have $sun(G') = \omega(G') = 1$, which means G' is a sun. Note that $|V(G')| = |V(G)| = n \geq 4$. So G' is a big sun, which implies that there is a vertex x of degree one in G'. So $d_G(x) \leq d_{G'}(x) + 1 \leq 2$. It means that G contains an edge cut of size at most 2, which contradicts to the 3-edge-connectivity of G. □

Claim 4.2. $|S| \geq 2$.

Proof. Suppose, to the contrary, that $|S| = 1$. By the definition of $\epsilon_2(S)$, we have that $\epsilon_2(S) \leq 1$. Then by (7), we have

$$sun(G' - S) \geq 2|S| - \epsilon_2(S) + 1 \geq 2|S| = 2.$$

If $G' - S$ has a big sun component X, then we can choose three vertices, say x_1, x_2, x_3, of degree one in X. So $d_{G'}(x_i) \leq d_X(x_i) + |S| = 2$. Note that $e = uv$, $G' = G - e$. Without loss of generality, we may assume that $x_1 \notin \{u, v\}$. Then $d_G(x_1) = d_{G'}(x_1) \leq 2$, which means that G contains an edge cut of size at most 2, contradicting to the 3-edge-connectivity of G.

If $G' - S$ has a trivial sun component, say $Y = \{y\}$, then $d_{G'}(y) \leq |S| = 1$ and $d_G(y) \leq d_{G'}(y) + 1 \leq 2$, a contradiction.

We now assume that all the sun components of $G' - S$ are isomorphic to K_2. Note that $sun(G' - S) \geq 2$ and $G' = G - e$. We can choose one vertex $z \notin \{u, v\}$ from some sun component of $G' - S$ such that $d_{G-S}(z) = d_{G'-S}(z) = 1$. Then $d_G(z) \leq d_{G-S}(z) + 1 = 2$, a contradiction. □

Combining with (7) and the definition of $\epsilon_2(S)$, we obtain that

$$sun(G' - S) \geq 2|S| - \epsilon_2(S) + 1 \geq 2|S| - 1. \tag{10}$$

Since $G' = G - e$, we have

$$sun(G - S) \geq sun(G' - S) - 2. \tag{11}$$

Combining with (10), (11) and Claim 4.2, we have that

$$sun(G - S) \geq sun(G' - S) - 2 \geq 2|S| - 3 \geq 1. \tag{12}$$

Claim 4.3. $sun(G - S) \geq 2$.

Proof. Suppose, to the contrary, that $sun(G - S) = 1$. Then by (12), we have $sun(G' - S) = 3$ and $|S| = 2$. Let Y_1, Y_2, Y_3 be the three sun components of $G' - S$ and Y_1 be the sun component of $G - S$. Then we can conclude that $uv \in E(V(Y_2), V(Y_3))$ and $Y_2 \cup Y_3 \cup uv$ is no longer a sun component in $G - S$, which means that at least one of Y_2, Y_3 is not an isolated vertex in $G' - S$. Without loss of generality, we may suppose that $|V(Y_2)| \geq 2$. Let y_1 be a vertex of Y_1 with $d_{Y_1}(y_1) \leq 1$ and y_2 be a vertex of Y_2 with $d_{Y_2}(y_2) = 1$ and $y_2 \notin \{u, v\}$. Then $e_G(y_i, S) \geq d_G(y_i) - 1$ for $i = 1, 2$ and $\{y_1, y_2\}$ forms an independent set of G. So $d_G(y_1) + d_G(y_2) \geq \sigma_2(G) > \frac{2n+6}{3}$. Then we have

$$|S| \geq \max\{e_G(y_1, S), e_G(y_2, S)\} \geq \frac{d_G(y_1) - 1 + d_G(y_2) - 1}{2} > \frac{n}{3}. \tag{13}$$

Note that $|S| = 2$. Together with (13), we have $n = 4$ or 5. If $n = 4$, then $|V(G' - S)| = 2$, which contradicts to $sun(G' - S) = 3$. If $n = 5$, then $|V(G' - S)| = 3$. Note that $sun(G' - S) = 3$. So Y_1, Y_2, Y_3 are all isolated vertices in $G' - S$, contradicts to our conclusion that at least one of Y_2, Y_3 is not an isolated vertex in $G' - S$. □

We divide the following proof into two cases.

Case 1. There are at least one trivial sun component in $G - S$.

Let W_1, W_2 be two sun components of $G - S$ with $V(W_1) = \{w_1\}$ and w_2 be a vertex of W_2 with $d_{W_2}(w_2) \leq 1$. Then $e_G(w_1, S) = d_G(w_1)$, $e_G(w_2, S) \geq d_G(w_2) - 1$ and $\{w_1, w_2\}$ forms an independent set of G. So

$$d_G(w_1) + d_G(w_2) \geq \sigma_2(G) > \frac{2n + 6}{3}. \tag{14}$$

Then we have

$$|S| \geq \max\{e_G(w_1, S), e_G(w_2, S)\} \geq \frac{d_G(w_1) + d_G(w_2) - 1}{2} > \frac{2n + 3}{6}. \tag{15}$$

Combining with (10) and (15), we have

$$n = |V(G')| \geq |S| + sun(G' - S) \geq 3|S| - 1 > \frac{2n+1}{2},$$

a contradiction.

Case 2. All sun components of $G - S$ are non-trivial.

Combining with Claims 4.2 and 4.3, we have that

$$n \geq |S| + 2 \cdot sun(G - S) \geq 6. \tag{16}$$

Let R_1, R_2 be two sun components of $G - S$ and $r_i \in V(R_i)$ with $d_{R_i}(r_i) = 1$ for $i = 1, 2$. Then $e_G(r_i, S) = d_G(r_i) - 1$ for $i = 1, 2$ and $\{r_1, r_2\}$ forms an independent set of G. So

$$d_G(r_1) + d_G(r_2) \geq \sigma_2(G) > \frac{2n+6}{3}. \tag{17}$$

Then we have

$$|S| \geq \max\{e_G(r_1, S), e_G(r_2, S)\} \geq \frac{d_G(r_1) - 1 + d_G(r_2) - 1}{2} > \frac{n}{3}. \tag{18}$$

Since all sun components of $G - S$ are non-trivial and $G' = G - e$, we have that at most two sun components of $G' - S$ are trivial. So there are at most two sun components of $G' - S$ containing one vertex, and all the other sun components of $G' - S$ contain at least two vertices. Combining with (10) and (18), we have

$$n = |V(G')| \geq |S| + 2 \cdot (sun(G' - S) - 2) + 2 \geq 5|S| - 4 > \frac{5n}{3} - 4,$$

which means $n < 6$ and contradicts to (16).

This completes the proof of Theorem 2.2. □

5. Concluding remarks

Remark 5.1. We now show that the degree sum condition in Theorem 2.1 is tight. Let $H_1 \cong K_3 \cup tK_1$, $H_2 \cong (3 + 2t)K_1 \cup K_2$ and $G \cong H_1 \vee H_2$. Then G is a 3-edge-connected graph of order $n = 8 + 3t$. $\sigma_2(G) = 6 + 2t = \frac{2n+2}{3}$. Let $e \in E(H_2)$, $G' = G - e$ and $S = V(H_1) \subseteq V(G')$. Then $|S| = 3 + t$, $\epsilon_1(S) = 2$ and $i(G' - S) = 5 + 2t > 4 + 2t = 2|S| - \epsilon_1(S)$. By Theorem 1.2, G' is not $P_{\geq 2}$-factor covered. So G is not $P_{\geq 2}$-factor uniform.

Remark 5.2. The graph given in [5] shows that the degree sum condition in Theorem 2.2 is tight. For the sake of completeness, we list the graph here. Let $H_1 \cong K_2, H_2 \cong 2K_2$ and $G \cong H_1 \vee H_2$. Then G is a 3-edge-connected graph of order $n = 6$. $\sigma_2(G) = 6 = \frac{2n+6}{3}$. Let $e \in E(H_2)$, $G' = G - e$ and $S = V(H_1) \subseteq V(G')$. Then $|S| = 2$, $\epsilon_2(S) = 2$ and $sun(G' - S) = 3 > 2 = 2|S| - \epsilon_2(S)$. By Theorem 1.3, G' is not $P_{\geq 3}$-factor covered. So G is not $P_{\geq 3}$-factor uniform.

Acknowledgments

We would like to thank the editors and the reviewers for providing many helpful comments and suggestions, which improve the presentation of the paper greatly.

Funding

This work is supported by National Natural Science Foundation of China (Nos. 12201408 and 12271362).

Conflicts of Interest

We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

No data was used for the research described in the article.

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