RAIRO-Oper. Res. 59 (2025) 3913–3944
https://doi.org/10.1051/ro/2025152

RAIRO Operations Research
www.rairo-ro.org

THE INVESTMENT AND REINSURANCE GAME ON ASSET-LIABILITY MANAGEMENT WITH COMMON SHOCK DEPENDENCE UNDER CEV MODEL

Lu Li[1,*] and Zhijian Qiu[2]

Abstract. This paper investigates the non-zero-sum stochastic differential game problem of optimal asset-liability management (ALM) between insurers and reinsurers. Combining the investment-reinsurance problem with the ALM, this paper innovatively studies the investment-reinsurance ALM non-zero-sum game within the framework of the constant elasticity of variance (CEV) model under common shock dependence. We allow the insurance companies to purchase proportional reinsurance from reinsurance companies, and both companies can invest in a financial market composed of one risk-free asset and one risk asset whose price process follows the CEV model. By innovatively introducing a class of stochastic liability processes associated with the volatility of risky assets, we obtain the dynamic evolution of net wealth. Using stochastic control theory, we obtain an explicit expression of Nash equilibrium strategy for the problem and a closed-form expression for the corresponding equilibrium value function by maximizing the expected utility from the terminal net wealth. Finally, we illustrate the effect of relevant parameters on the optimal investment-reinsurance strategy by means of numerical examples.

Mathematics Subject Classification. 62P05, 91B30, 93E20.

Received December 17, 2024. Accepted November 16, 2025.

1. Introduction

In the field of actuarial science, reinsurance and investment strategies are currently very active areas of research. On one hand, reinsurance is a critical component of the insurance industry, providing a mechanism for insurance companies to transfer a portion of the risks. On the other hand, insurance companies can invest surpluses into financial markets to make profits and enhance their market competitiveness. This has generated significant interest in the research of optimal reinsurance and investment strategies. Liang and Yuen [18], Yuen [36], Zhang and Siu [39] and Li [17] discussed the problem for maximizing the expected utility. Promislow and Young [26], Schmidli [28], Cao and Zeng [37], Han *et al.* [14] studied studied the optimal investment and reinsurance strategies of an insurance company for minimizing the ruin probability of the risk process. Gulnaz

Keywords. Investment and reinsurance, stochastic differential game, asset-liability management, two-dimensional claim.

[1] School of Mathematics and Physics, Yibin University, Yibin, Sichuan, P.R. China.

[2] School of Mathematics, Southwestern University of Finance and Economics, Chengdu, Sichuan, P.R. China.

*Corresponding author: 1210202Z1004@smail.swufe.edu.cn

and Uğurlu [29] studied different portfolios by calculating their LPMs and concluded which portfolios are less risky than the others. Uğurlu [31] considered a continuous-time principal-agent model on a finite time horizon.

Asset-liability management is a crucial risk management tool for financial institutions like banks, insurers, and funds. It involves matching assets and liabilities while considering safety, liquidity, profitability, and diversification. Liabilities have far-reaching impacts, and prudent asset management can help institutions turn losses into profits. Therefore, the issue of asset-liability management has gradually attracted great attention in the field of financial engineering. Sharpe and Tint [30] are the first studies on the asset-liability management problem using portfolio theory under the mean–variance criterion. Since then, there has been a proliferation of research on asset-liability management in continuous-time scenarios. Chiu and Li [6] investigated the asset-liability management problem in which the risky asset price process and the liability evolution process are depicted by geometric Brownian motion, and obtained the analytic equations for the optimal investment strategy and the efficient frontier. Chen *et al.* [3] extended the above liability problem to the Markov switching mechanism. Xie *et al.* [34] studied the asset-liability management problem in the continuous-time case, where investor liabilities are portrayed by Brownian motion with drift and investor risky asset prices are described by geometric Brownian motion, and then Xie [20] continued to extend this problem to the Markov mechanism. Zeng and Li [38] investigated the asset-liability management problem under the jump-diffusion model and obtained closed-form solutions of the boundary and optimal investment strategy. Chiu and Wong [5] applied Lagrange duality theory and optimal feedback control theory to study the asset-liability management problem in terms of risky asset returns and insurance claims, and obtained the closed-form solution of the optimal investment strategy and the efficient frontier. Chang [2] applied a similar approach to consider a portfolio with endogenous liabilities, and obtained the efficient strategy and efficient frontier. Pan and Xiao [23] studied the optimal mean–variance asset-liability management problem with the stochastic interest rate and inflation risk. Wei and Wang [33] considered asset-liability management with stochastic coefficients and obtained the time-consistent optimal strategy. Zhang *et al.* [40] studied state-dependent risk-averse asset-liability management. Zhang *et al.* [41] considered asset-liability management portfolio selection problem with debt ratio, and obtained open-loop equilibrium strategy. Further, Peng and Chen [25] investigated the asset-liability management portfolios with inside information under the mean–variance criterion.

In addition, utility maximization is also a very important decision objective in the study of asset-liability management. Pan and Xiao [22] considered optimal asset-liability management with liquidity constraints and stochastic interest rates in an expected utility maximization framework, and obtained explicit expressions for the optimal solution under CRRA and CARA utility. Josa-Fombellida and Rincon-Zapatero [16] studied the management of DB pensions with liabilities with the objective of utility maximization. Hu and Wang [15] investigated optimal investment consumption strategies with liabilities under the value-at-risk constraint and mechanism switching models. Pan *et al.* [24] investigated optimal asset-liability management for investors with CARA preferences under the Heston stochastic volatility model. Chen *et al.* [4] considered the robust asset-liability portfolio problem with mechanism switching under the CRRA utility. Yuan and Mi [35] investigated robust optimal asset-liability management with ambiguity penalty. Uğurlu [32] studied the portfolio optimization problem of an investor seeking to maximize his terminal wealth.

Although many scholars have investigated the ALM problem under utility maximization, there are still three aspects worth discussing. Firstly, in the existing studies of asset-liability management, the subject of optimal decision-making is a single agent. However, in the real economy, the behavioral decisions of competitors are often taken into account when making decisions. Besides, the reinsurance strategy involves the interests of both the insurer and the reinsurer. When an insurer seeks reinsurance, it must assess the reinsurer's capacity to provide adequate coverage. Likewise, the reinsurer must evaluate the insurer's reinsurance needs. The strategic decisions of insurers and reinsurers constitute a typical strategic interaction. Therefore, similar to Espinosa and Touzi [11], Zhu *et al.* [43], and Dong *et al.* [10], we incorporate the game under relative performance into the investment-reinsurance asset-liability management problem. Different from them, our goal is to maximize the expected utility from the terminal net wealth, that is, wealth minus liabilities. Secondly, most papers assume that different insurance businesses are mutually independent. But in real life, insurance businesses often have a certain degree of dependence. A prime example is the COVID-19 pandemic (or other natural disasters like hurricanes or earthquakes), which can give rise to various types of insurance claims. Some research have proposed that these cumulative claims may be interrelated through a common shock, which can capture the impact of diverse insurance claims stemming from such natural catastrophes. Yuan *et al.* [36] studied the investment-reinsurance problem in the case that the claim quantity processes were associated with a common shock. Zhang *et al.* [42] considered optimal excess-of-loss reinsurance and investment problem with thinning dependent risks. Third, compared to Pan and Xiao [22], Pan *et al.* [24] and Chen *et al.* [4], they consider stochastic liability processes depicted by Brownian or geometric Brownian motion with drift, but in reality, liabilities generally fluctuate with asset prices. Based on this, we consider a class of liability processes associated with asset price fluctuations.

Inspired by the above works, this paper studies the non-zero-sum game between an insurer and a reinsurer under the asset-liability management problem. We assume that the insurance company can purchase reinsurance and invest its surplus in the financial market composed of risk-free assets and risky assets, where the risky assets are characterized by the CEV model. Besides, we set a conditional dependence risk model, in which common shocks depict the dependency between risks, that is, the insurance market is described by two-dimensional dependent claims. Reinsurers can also invest in this financial market. Under the above background setting, we consider the optimal investment and reinsurance ALM game problem with a common shock under the CEV model. Applying the stochastic control theory in the game-theoretic framework, we obtain the corresponding extended Hamilton–Jacobi–Bellman (HJB) equation. Furthermore, we derive both the explicit expression of optimal investment-reinsurance strategy and the corresponding equilibrium value function. Finally, through numerical simulations, we analyze the impact of model parameters on investment-reinsurance strategy, providing theoretical guidance for asset-liability management.

The main contributions of this paper are as follows: (1) A class of stochastic processes of liabilities associated with asset price fluctuations is considered. (2) Based on the risk sharing attribute established by the reinsurance contract between the insurance company and the reinsurance company, we construct a non-zero-sum game between the insurance company and the reinsurance company, which is different from the non-zero-sum game between insurance companies in the existing literature. (3) Compared with Pun [27], we study two insurance businesses with common shocks and analyze two reinsurance strategies influenced by some common factors, thus making the discussion more realistic and complicated. (4) Combining the investment reinsurance problem with the ALM, this paper innovatively studies the investment-reinsurance ALM non-zero-sum game within the framework of the CEV model under common shock dependence. (5) We study the asset-liability management game problem under relative performance, where the relative performance is based on net wealth after liabilities are removed. (6) We give a verification theorem for the asset-liability management game problem under the relative performance of net wealth.

The remainder of this paper is organized as follows: Section 2 constructs a mathematical model to illustrate the non-zero stochastic differential game problem for insurers and reinsurers with exogenous stochastic liability. In Section 3, by using stochastic control theory and solving the HJB equation, we obtain the Nash equilibrium solution of the problem and the corresponding equilibrium value function. In Section 4, we analyze the effect of the parameter depicting liability as well as the parameter characterizes relative performance on the Nash equilibrium strategy by providing a numerical example. Section 5 concludes this paper.

2. Model formulation

In this paper, we consider a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a right continuous and $\mathbb{P}$-complete filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ represents the information of the market available up to time t. $[0, T]$ is a fixed and finite time horizon. $\mathbb{P}$ is the reference measure. All stochastic processes introduced below are defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. In addition, we suppose that there are no transaction costs or taxes in the financial and insurance markets, and trading can continuously occur.

2.1. Surplus process

Suppose an insurer has an insurance portfolio that consists of two distinct insurance businesses, such as medical insurance and life insurance. The claims arrival in these businesses can be modeled as dependent Poisson processes with common shocks. Suppose that the random variables $\{L_i, i = 1, 2, \ldots,\}$ represent the claim size from the first-class claim and assume sequence $\{L_i, i \geq 1\}$ is independent with common distribution $F_L()$, $\mu_L = E(L_i)$, and $\mu'_L = E(L_i^2)$. $\{Y_i, i = 1, 2, \ldots,\}$ represent the claim size from the second-class claim and assume sequence $\{Y_i, i \geq 1\}$ is mutually independent with common distribution $F_Y()$, $\mu_Y = E(Y_i)$, and $\mu'_Y = E(Y_i^2)$. The claim settlement process shared between the two insurance businesses is as follows:

$$S_1(t) := \sum_{i=1}^{N_1(t)+N(t)} L_i, \quad S_2(t) := \sum_{i=1}^{N_2(t)+N(t)} Y_i,$$

where $\{N_1(t) + N(t)\}_{t>0}$ and $\{N_2(t) + N(t)\}_{t>0}$ represent the number of claims for the first and second categories of insurance business up to time t, respectively. In addition, $\{N_1(t)\}_{t\geq 0}$, $\{N_2(t)\}_{t\geq 0}$ and $\{N(t)\}_{t\geq 0}$ are independent Poisson processes with intensity $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda > 0$, respectively. Obviously, the dependence of the two types of business is controlled by the common impact caused by $\{N(t)\}_{t\geq 0}$. The larger λ means the higher degree of dependence of the two insurance businesses.

Similar to Liang [18] and Browne [1], for the convenience of calculation, we assume that the composite Poisson process $S_1(t)$ and $S_2(t)$ can be approximated by the following geometric Brownian motion:

$$\mathrm{d}S_1(t) = a_1 \ \mathrm{d}t - \sigma_1 \ \mathrm{d}B_1(t),$$

with $a_1 = (\lambda_1 + \lambda)E(L_i)$, and $\sigma_1^2 = (\lambda_1 + \lambda)E(L_i^2)$. In the same way,

$$\mathrm{d}S_2(t) = a_2 \ \mathrm{d}t - \sigma_2 \ \mathrm{d}B_2(t),$$

with $a_2 = (\lambda_2 + \lambda)E(Y_i)$, and $\sigma_2^2 = (\lambda_2 + \lambda)E(Y_i^2)$. Here $B_1(t)$ and $B_2(t)$ are standard Brownian motions, and the correlation coefficient of $B_1(t)$ and $B_2(t)$ is denoted as:

$$\rho = \frac{\lambda E(L_i)E(Y_i)}{\sqrt{(\lambda_1 + \lambda)E(L_i^2)(\lambda_2 + \lambda)E(Y_i^2)}} = \frac{\lambda \mu_L \mu_Y}{\sigma_1 \sigma_2}$$

where $E(B_1(t)B_2(t)) = \rho t$.

Therefore, the surplus process of the insurer up to time t is given by

$$R_1(t) = R_0 + ct - (a_1 + a_2)t + \sigma_1 B_1(t) + \sigma_2 B_2(t),$$

where R_0 is a deterministic initial reserve, and c represents the premium rate.

In addition, we assume that the premium rate of the insurance company is calculated based on the expected value principle:

$$c' = (1 + \theta_1)(\lambda_1 + \lambda)\mu_L + (1 + \theta_2)(\lambda_2 + \lambda)\mu_Y,$$

where θ_1 and θ_2 are the insurer's positive safety loads for the first and second class of claims. Besides, to be protected from potential large claims, we suppose insurers can govern their insurance risks by buying proportional reinsurance, but they can not get new business. Let $q_1(t)(\in (0, 1])$ and $q_2(t)(\in (0, 1])$ be the reinsurance retention levels for the first class claims and the second class claims at time t, respectively. When a claim occurs, the insurer pays $q_1(t)L_i$ (or $q_2(t)Y_i$) and the reinsurer pays $(1 - q_1(t))L_i$ (or $(1 - q_2(t))Y_i$). The reinsurance premium is given by

$$c'' = (1 - q_1(t))(1 + \eta_1)(\lambda_1 + \lambda)\mu_L + (1 - q_2(t))(1 + \eta_2)(\lambda_2 + \lambda)\mu_Y,$$

where η_1 and η_2 are positive safety loads for the first and second types of claims. Note that for the insurer, $q_1(t)(\in (0,1])$ and $q_2(t)(\in (0,1])$ correspond to a reinsurance coverage and $q_1(t) > 1$ (or $q_2(t) > 1$) would mean that the company can take an extra insurance business from other companies (*i.e.*, act as a reinsurer for other cedents). In general, suppose $\eta_i > \theta_i$, $i = 1, 2$ and η_1 and η_2 are positive safety loads for the reinsurer. After reinsurance, the insurer's premium becomes

$$\begin{aligned} c &= c' - c'' \\ &= [(1+\eta_1)q_1(t) + \theta_1 - \eta_1](\lambda_1 + \lambda)\mu_L + [(1+\eta_2)q_2(t) + \theta_2 - \eta_2](\lambda_2 + \lambda)\mu_Y. \end{aligned}$$

We introduce a new standard Brownian motion $\mathrm{d}W_0(t)$ such that we can express the combination of $\mathrm{d}B_1(t)$ and $\mathrm{d}B_2(t)$ using $\mathrm{d}W_0(t)$. First, recall that for two correlated Brownian motions $B_1(t)$ and $B_2(t)$ with correlation ρ, we can write $\mathrm{d}B_2(t) = \rho\,\mathrm{d}B_1(t) + \sqrt{1-\rho^2}\,\mathrm{d}B_3(t)$, where $B_3(t)$ is another independent standard Brownian motion. However, in our case, we aim to combine $\sigma_1 q_1(t)\,\mathrm{d}B_1(t) + \sigma_2 q_2(t)\,\mathrm{d}B_2(t)$ into a single term involving $\mathrm{d}W_0(t)$. The variance of the combined noise term $\sigma_1 q_1(t)\,\mathrm{d}B_1(t) + \sigma_2 q_2(t)\,\mathrm{d}B_2(t)$ is:

$$E\Big[\big(\sigma_1 q_1(t)\,\mathrm{d}B_1(t) + \sigma_2 q_2(t)\,\mathrm{d}B_2(t)\big)^2\Big] = \sigma_1^2 q_1(t)^2\,\mathrm{d}t + \sigma_2^2 q_2^2(t)\,\mathrm{d}t + 2\sigma_1\sigma_2 q_1(t) q_2(t)\rho\,\mathrm{d}t.$$

We define $\mathrm{d}W_0(t)$ such that:

$$\sigma_1 q_1(t)\,\mathrm{d}B_1(t) + \sigma_2 q_2(t)\,\mathrm{d}B_2(t) = \sqrt{\sigma_1^2 q_1(t)^2 + \sigma_2^2 q_2^2(t) + 2\sigma_1\sigma_2 q_1(t)q_2(t)\rho}\,\mathrm{d}W_0(t),$$

where $\mathrm{d}W_0(t)$ is a standard Brownian motion. Substituting the combined noise term into the reserve process, we get:

$$\begin{aligned} \mathrm{d}R_1(t) &= c\,\mathrm{d}t - q_1(t)\,\mathrm{d}S_1(t) - q_2(t)\,\mathrm{d}S_2(t) \\ &= [(\theta_1-\eta_1)a_1 + (\theta_2-\eta_2)a_2 + a_1\eta_1 q_1(t) + a_2\eta_2 q_2(t)]\,\mathrm{d}t \\ &\quad + \sqrt{\sigma_1^2 q_1^2(t) + \sigma_2^2 q_2^2(t) + 2q_1(t)q_2(t)\rho}\,\mathrm{d}W_0(t). \end{aligned}$$

Using the given expression for $\rho = \frac{\lambda\mu_L\mu_Y}{\sigma_1\sigma_2}$, we can rewrite the square root term: $\sqrt{\sigma_1^2 q_1^2(t) + \sigma_2^2 q_2^2(t) + 2q_1(t)q_2(t)\rho} = \sqrt{\sigma_1^2 q_1^2(t) + \sigma_2^2 q_2^2(t) + 2q_1(t)q_2(t)\lambda\mu_L\mu_Y}$.

Then, the reserve process $\{R_1(t)\}_{t\geq 0}$ of the insurer evolves as

$$\begin{aligned} \mathrm{d}R_1(t) &= [(\theta_1-\eta_1)a_1 + (\theta_2-\eta_2)a_2 + a_1\eta_1 q_1(t) + a_2\eta_2 q_2(t)]\,\mathrm{d}t \\ &\quad + \sqrt{\sigma_1^2 q_1^2(t) + \sigma_2^2 q_2^2(t) + 2q_1(t)q_2(t)\lambda\mu_L\mu_Y}\,\mathrm{d}W_0(t), \end{aligned} \tag{1}$$

where $W_0(t)$ is a standard Brownian motion.

After selling the reinsurance contract, the surplus process of the reinsurer is

$$\begin{aligned} \mathrm{d}R_2(t) &= c''\,\mathrm{d}t - (1-q_1(t))\,\mathrm{d}S_1(t) - (1-q_2(t))\,\mathrm{d}S_2(t) \\ &= [(1-q_1(t))\eta_1 a_1 + (1-q_2(t))\eta_2 a_2]\,\mathrm{d}t \\ &\quad + \sqrt{\sigma_1^2(1-q_1(t))^2 + \sigma_2^2(1-q_2(t))^2 + 2(1-q_1(t))(1-q_2(t))\lambda\mu_L\mu_Y}\,\mathrm{d}W_0(t), \end{aligned} \tag{2}$$

2.2. Financial market

We consider a financial market that consists of one risk-free asset and one risk asset whose price process follows the CEV model.

The price process of the risk-free asset is given by the following ordinary differential equation (ODE):

$$\begin{cases} \mathrm{d}S_0(t) = rS_0(t)\,\mathrm{d}t, \quad t \in [0,T], \\ S_0(0) = 1, \end{cases} \tag{3}$$

in which $r(>0)$ represents the risk-free interest rate.

Referring to Cox and Ross [8], Gu *et al.* [13] and Luo *et al.* [19], the price process of the risky asset follows a constant elasticity of variance (CEV) model

$$\begin{cases} \mathrm{d}S(t) = S(t)\big[\alpha\,\mathrm{d}t + \sigma S^{\beta}(t)\,\mathrm{d}W_1(t)\big], & t \in [0,T], \\ S(0) = s_0, \end{cases} \tag{4}$$

where $\alpha(>r)$ is the appreciation rate, σ is the volatility coefficient and β is the elasticity parameter. The term $\sigma S^{\beta}(t)$ denotes the instantaneous volatility. If $\beta = 0$, the CEV model simplifies to a geometric Brownian motion (GBM) model. If $\beta > 0$, the instantaneous volatility $\sigma S^{\beta}(t)$ increases as the risky asset price increases. If $\beta < 0$, the instantaneous volatility $\sigma S^{\beta}(t)$ increases as the stock price decreases, potentially resulting in a distribution with a fatter left tail. Therefore, in order to keep the risky asset price process positive, we let $\beta \geq 0$. $W_1(t)$ is a standard one-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Brownian motion which is independent of $W_0(t)$. We suppose r, α and σ are given constants.

2.3. Wealth process

In order to use capital efficiently, we assume that both insurers and reinsurers can invest assets in the financial market. Suppose that there are no transaction costs or taxes for investment and reinsurance; financial assets can be traded continuously; short-selling of the risky asset is not allowed, and the investment is not enough to affect the prices of the risky asset.

Assume that $X_1(t)$ and $X_2(t)$ denote the wealth of the insurer and reinsurer at time $t \in [0,T]$, respectively; $\pi_1(t)$ and $\pi_2(t)$ represent the investment amount of insurer and reinsurer in the risky asset, respectively; then $X_1(t)-\pi_1(t)$ and $X_2(t)-\pi_2(t)$ represent their respective investments in the risk-free asset. Define the investment and reinsurance strategy adopted by the insurer and the reinsurer as

$$u_1(t) = (q_1(t), q_2(t), \pi_1(t)), \quad u_2(t) = (\pi_2(t)).$$

After a simple calculation, under the strategies $u_1(t) = (q_1(t), q_2(t), \pi_1(t))$ and $u_2(t) = (\pi_2(t))$, the wealth processes of the insurer and the reinsurer can be described by the following stochastic differential equations:

$$\begin{aligned} \mathrm{d}X_1^{u_1}(t) &= \frac{X_1^{u_1}(t) - \pi_1(t)}{S_0(t)}\,\mathrm{d}S_0(t) + \frac{\pi_1(t)}{S(t)}\,\mathrm{d}S(t) + \mathrm{d}R_1(t) \\ &= [rX_1^{u_1}(t) + (\alpha - r)\pi_1(t) + (\theta_1 - \eta_1)a_1 + (\theta_2 - \eta_2)a_2 + a_1\eta_1 q_1(t) + a_2\eta_2 q_2(t)]\,\mathrm{d}t \\ &\quad + \pi_1(t)\sigma S^{\beta}(t)\,\mathrm{d}W_1(t) + \sqrt{\sigma_1^2 q_1^2(t) + \sigma_2^2 q_2^2(t) + 2q_1(t)q_2(t)\lambda\mu_L\mu_Y}\,\mathrm{d}W_0(t) \\ \mathrm{d}X_2^{u_2}(t) &= \frac{X_2^{u_2}(t) - \pi_2(t)}{S_0(t)}\,\mathrm{d}S_0(t) + \frac{\pi_2(t)}{S(t)}\,\mathrm{d}S(t) + \mathrm{d}R_2(t) \\ &= [rX_2^{u_2}(t) + (\alpha - r)\pi_2(t) + (1 - q_1(t))\eta_1 a_1 + (1 - q_2(t))\eta_2 a_2]\,\mathrm{d}t \\ &\quad + \pi_2(t)\sigma S^{\beta}(t)\,\mathrm{d}W_1(t) + \sqrt{\sigma_1^2(1 - q_1(t))^2 + \sigma_2^2(1 - q_2(t))^2 + 2(1 - q_1(t))(1 - q_2(t))\lambda\mu_L\mu_Y}\,\mathrm{d}W_0(t). \end{aligned}$$

To simplify the formula, we suppose

$$p_1(t) = \sqrt{\sigma_1^2 q_1^2(t) + \sigma_2^2 q_2^2(t) + 2q_1(t)q_2(t)\lambda\mu_L\mu_Y},$$

and

$$p_2(t) = \sqrt{\sigma_1^2(1 - q_1(t))^2 + \sigma_2^2(1 - q_2(t))^2 + 2(1 - q_1(t))(1 - q_2(t))\lambda\mu_L\mu_Y}.$$

Then the wealth processes of the insurer and the reinsurer can be expressed as

$$\mathrm{d}X_1^{u_1}(t) = [rX_1^{u_1}(t) + (\alpha - r)\pi_1(t) + (\theta_1 - \eta_1)a_1 + (\theta_2 - \eta_2)a_2 + a_1\eta_1 q_1(t) + a_2\eta_2 q_2(t)]\,\mathrm{d}t$$

$$
\begin{aligned}
&+ \pi_1(t)\sigma S^\beta(t)\,\mathrm{d}W_1(t) + p_1(t)\,\mathrm{d}W_0(t) && (5)\\
\mathrm{d}X_2^{u_2}(t) = {}& [rX_2^{u_2}(t) + (\alpha - r)\pi_2(t) + (1-q_1(t))\eta_1 a_1 + (1-q_2(t))\eta_2 a_2]\,\mathrm{d}t \\
&+ \pi_2(t)\sigma S^\beta(t)\,\mathrm{d}W_1(t) + p_2(t)\,\mathrm{d}W_0(t). && (6)
\end{aligned}
$$

To make the problem more realistic, setting liability is also a common issue for both the insurer and the reinsurer. In the financial market, debt is an important problem that investors will inevitably face. Different from other literature, we assume that the insurer and the reinsurer face an uncontrollable exogenous liability related to stochastic fluctuations. Let $L_1(t)$ and $L_2(t)$ be the liability of the insurer and the reinsurer at time t, and the evolution dynamics of these liabilities ($L_1(t)$ and $L_2(t)$) are as follows respectively:

$$
\mathrm{d}L_1(t) = \alpha_1\,\mathrm{d}t + \sigma_{11}S^\beta(t)\,\mathrm{d}W_1(t) + \sigma_{12}\,\mathrm{d}W_2(t), \quad L_1(0) = l_{10} > 0, \tag{7}
$$

and

$$
\mathrm{d}L_2(t) = \alpha_2\,\mathrm{d}t + \sigma_{21}S^\beta(t)\,\mathrm{d}W_1(t) + \sigma_{22}\,\mathrm{d}W_2(t), \quad L_2(0) = l_{20} > 0, \tag{8}
$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ denote the debt ratio, $\sigma_{11} > 0$ and $\sigma_{21} > 0$ represent the volatilities of the uncontrollable exogenous liabilities, and $\sigma_{12} > 0$ and $\sigma_{22} > 0$ represent the volatilities of the liabilities. $W_2(t)$ is a one-dimensional standard Brownian motion, which is independent of $W_1(t)$ and $W_0(t)$.

Considering the liability problem, investors focus on the difference between the asset value and the liability value. For convenience, we denote the net wealth of the insurer and the reinsurer as follows:

$$
Y_1(t) = X_1(t) - L_1(t), \quad Y_2(t) = X_2(t) - L_2(t).
$$

Thus, the net wealth processes of the insurer and the reinsurer are governed by the following stochastic differential equations (SDEs), respectively:

$$
\begin{aligned}
\mathrm{d}Y_1^{u_1}(t) &= \mathrm{d}X_1^{u_1}(t) - \mathrm{d}L_1(t) \\
&= [r(Y_1^{u_1}(t) + L_1(t)) + (\alpha - r)\pi_1(t) + (\theta_1 - \eta_1)a_1 + (\theta_2 - \eta_2)a_2 + a_1\eta_1 q_1(t) + a_2\eta_2 q_2(t) - \alpha_1]\,\mathrm{d}t \\
&\quad + (\sigma\pi_1(t) - \sigma_{11})S^\beta(t)\,\mathrm{d}W_1(t) - \sigma_{12}\,\mathrm{d}W_2(t) + p_1(t)\,\mathrm{d}W_0(t), && (9)\\
\mathrm{d}Y_2^{u_2}(t) &= \mathrm{d}X_2^{u_2}(t) - \mathrm{d}L_2(t) \\
&= [r(Y_2^{u_2}(t) + L_2(t)) + (\alpha - r)\pi_2(t) + (1-q_1(t))\eta_1 a_1 + (1-q_2(t))\eta_2 a_2 - \alpha_2]\,\mathrm{d}t \\
&\quad + (\sigma\pi_2(t) - \sigma_{21})S^\beta(t)\,\mathrm{d}W_1(t) - \sigma_{22}\,\mathrm{d}W_2(t) + p_2(t)\,\mathrm{d}W_0(t). && (10)
\end{aligned}
$$

Definition 1 (Admissible Control-measure Strategy)**.** For any fixed $t \in [0,T]$, the investment-reinsurance strategy $u(t) = u_1(t) \times u_2(t) = (q_1(t), q_2(t), \pi_1(t)) \times (\pi_2(t))$ is said to be admissible if it satisfies:

(i) $\{u_1(t)\}_{t\in[0,T]}$ and $\{u_2(t)\}_{t\in[0,T]}$ are $\{\mathcal{F}_t\}_{t\in[0,T]}$-progressively measurable processes, such that $1 \geq q_1(t) \geq 0, 1 \geq q_2(t) \geq 0$ for any $t \in [0,T]$;

(ii) for all $t \in [0,T]$ and $\forall v \in [t,T]$, $\pi_1(t) \geq 0$, $\pi_2(t) \geq 0$ and

$$
E\left[\int_t^T \left(\pi_1(v)S^\beta(v)\right)^2 \mathrm{d}v\right] < +\infty, \quad E\left[\int_t^T \left(\pi_2(v)S^\beta(v)\right)^2 \mathrm{d}v\right] < +\infty,
$$

$$
E_{t,y_1,l_1,s}\left[\int_0^T \left[(q_1(t))^2 + (q_2(t))^2 + (\pi_1(t))^2\right] \mathrm{d}t\right] < +\infty;
$$

$$
E_{t,y_2,l_2,s}\left[\int_0^T \left[(\pi_2(t))^2\right] \mathrm{d}t\right] < +\infty;
$$

(iii) for $\forall(t, y_1, l_1, s) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, $\forall v \in [t, T]$, the state equation (9) associated with $u_1(t)$ has a unique strong solution $Y_1^{u_1}(v)$, which satisfies

$$\left\{E_{t,y_1,l_1,s}\left[\sup|Y_1^{u_1}(v)|^2\right]\right\}^{\frac{1}{2}} < +\infty,$$

where $E_{t,y_1,l_1,s}[\cdot]$ is the conditional expectation given $Y_1^{u_1}(t) = y_1$, $L_1(t) = l_1$ and $S(t) = s$;

(iv) for $\forall(t, y_2, l_2, s) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, $\forall v \in [t, T]$, the state equation (10) associated with $u_2(t)$ has a unique strong solution $Y_2^{u_2}(v)$, which satisfies

$$\left\{E_{t,y_2,l_2,s}\left[\sup|Y_2^{u_2}(v)|^2\right]\right\}^{\frac{1}{2}} < +\infty,$$

where $E_{t,y_2,l_2,s}[\cdot]$ is the conditional expectation given $Y_2^{u_2}(t) = y_2$, $L_2(t) = l_2$ and $S(t) = s$.

Let $U = U_1 \times U_2$ be the sets of all admissible strategies, where U_1 and U_2 denote the set of all admissible strategies of the insurer and the reinsurer, respectively.

2.4. Non-zero-sum game models setup

Game theory is an important discipline of operations research, which has been widely used in the field of insurance in recent years. Through a literature review of the investment-reinsurance problem, it can be found that in a competitive market, insurers and reinsurers will havea psychological tendency to compare with each other, which can be depicted by relative performance. Therefore, due to the mind of rivalry, the insurer and the reinsurer are concerned not only with their own terminal wealth but also with the differences between them. Hence in this section, we will discuss the equilibrium investment-reinsurance strategy by constructing a non-zero-sum stochastic differential game between insurer and reinsurer.

Following Espinosa and Touzi [11], we define the relative performance $\hat{Y}_1^{u_1}(t)$ for the insurer and $\hat{Y}_2^{u_2}(t)$ for the reinsurer as follows:

$$\widehat{Y}_1^{u_1}(t) = (1-\kappa_1)Y_1^{u_1}(t) + \kappa_1(Y_1^{u_1}(t) - Y_2^{u_2}(t)), \tag{11}$$
$$\widehat{Y}_2^{u_2}(t) = (1-\kappa_2)Y_2^{u_2}(t) + \kappa_2(Y_2^{u_2}(t) - Y_1^{u_1}(t)), \tag{12}$$

where $\kappa_i \in [0, 1], i = 1, 2$ represents the intensity of the insurer's and reinsurer's relative concern between each other, the larger κ_i is, the higher the degree of dependence between the two companies. Together with equations (11) and (12), we derive

$$\begin{aligned}
\mathrm{d}\widehat{Y}_1^{u_1}(t) &= \mathrm{d}Y_1^{u_1}(t) - \kappa_1\,\mathrm{d}Y_2^{u_2}(t)\\
&= \Big[r\Big(\widehat{Y}_1^{u_1}(t) + L_1(t) - \kappa_1 L_2(t)\Big) + (\alpha - r)\pi_1(t) + (\theta_1 - \eta_1)a_1\\
&\quad + (\theta_2 - \eta_2)a_2 + a_1\eta_1 q_1(t) + a_2\eta_2 q_2(t) - \alpha_1 + \kappa_1\alpha_2\\
&\quad - (\alpha - r)\kappa_1\pi_2(t) - (1 - q_1(t))\kappa_1\eta_1 a_1 - (1 - q_2(t))\kappa_1\eta_2 a_2\Big]\,\mathrm{d}t\\
&\quad - (\sigma_{12} - \kappa_1\sigma_{22})\,\mathrm{d}W_2(t) + (p_1(t) - \kappa_1 p_2(t))\,\mathrm{d}W_0(t)\\
&\quad + (\sigma\pi_1(t) - \sigma_{11} - \kappa_1\sigma\pi_2(t) + \kappa_1\sigma_{21})S^{\beta}(t)\,\mathrm{d}W_1(t),
\end{aligned} \tag{13}$$
$$\begin{aligned}
\mathrm{d}\widehat{Y}_2^{u_2}(t) &= \mathrm{d}Y_2^{u_2}(t) - \kappa_2\,\mathrm{d}Y_1^{u_1}(t)\\
&= \Big[r\Big(\widehat{Y}_2^{u_2}(t) + L_2(t) - \kappa_2 L_1(t)\Big) + (\alpha - r)\pi_2(t) + (1 - q_1(t))\eta_1 a_1\\
&\quad + (1 - q_2(t))\eta_2 a_2 - \alpha_2 - \kappa_2(\alpha - r)\pi_1(t) - \kappa_2(\theta_1 - \eta_1)a_1\\
&\quad - \kappa_2(\theta_2 - \eta_2)a_2 - \kappa_2 a_1\eta_1 q_1(t) - \kappa_2 a_2\eta_2 q_2(t) + \kappa_2\alpha_1\Big]\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}&-(\sigma_{22}-\kappa_2\sigma_{12})\,\mathrm{d}W_2(t)+(p_2(t)-\kappa_2 p_1(t))\,\mathrm{d}W_0(t)\\&+(\sigma\pi_2(t)-\sigma_{21}-\kappa_2\sigma\pi_1(t)+\kappa_2\sigma_{11})S^{\beta}(t)\,\mathrm{d}W_1(t).\end{aligned}\tag{14}$$

This study explores the non-zero-sum game problem with ALM based on the expected utility criterion. Under the criterion of expected utility, investors assess the risks and returns of various strategies by using utility functions. As a stable combination of strategies, the Nash equilibrium can balance risks and returns to a certain degree, thus assisting investors in reducing their overall risks.

We assume that the preferences of the insurer and the reinsurer are described by the following exponential utility functions:

$$\mathcal{U}_1(\hat{y}_1)=-\frac{1}{\gamma_1}e^{-\gamma_1\hat{y}_1},\quad \mathcal{U}_2(\hat{y}_2)=-\frac{1}{\gamma_2}e^{-\gamma_2\hat{y}_2},$$

where $\gamma_1>0$ and $\gamma_2>0$ represent the constant absolute risk aversion coefficients of insurers and reinsurers respectively.

Investors aim to maximize the expected utility of relative net wealth at the terminal moment. Referring to Cyert and Degroot [9] and Colombo and Labrecciosa [7], we define the objective functions of the insurer and the reinsurer as follows:

$$J_1^{u_1,u_2}(t,y_1,y_2,l_1,s):=E_{\{t,y_1,y_2,l_1,s\}}[\mathcal{U}_1(Y_1^{u_1}(T)-\kappa_1Y_2^{u_2}(T))],\tag{15}$$

and

$$J_2^{u_1,u_2}(t,y_1,y_2,l_2,s):=E_{\{t,y_1,y_2,l_2,s\}}[\mathcal{U}_2(Y_2^{u_2}(T)-\kappa_2Y_1^{u_1}(T))],\tag{16}$$

where $E_{\{t,y_1,y_2,l_1,s\}}[\cdot]$ and $E_{\{t,y_1,y_2,l_2,s\}}[\cdot]$ represent the conditional expectation given by

$$(Y_1^{u_1}(t),Y_2^{u_2}(t),L_1(t),S(t))=(y_1,y_2,l_1,s),$$

and

$$(Y_1^{u_1}(t),Y_2^{u_2}(t),L_2(t),S(t))=(y_1,y_2,l_2,s),$$

Definition 2 (Nash Equilibrium)**.** For $\forall(u_1,u_2)\in(U_1,U_2)$, the admissible investment-reinsurance ALM strategy is said to be a Nash equilibrium if it satisfies:

$$\begin{cases}J_1^{u_1,u_2^*}(t,y_1,y_2,l_1,s)\le J_1^{u_1^*,u_2^*}(t,y_1,y_2,l_1,s),\\ J_2^{u_1^*,u_2}(t,y_1,y_2,l_2,s)\le J_2^{u_1^*,u_2^*}(t,y_1,y_2,l_2,s).\end{cases}\tag{17}$$

Problem 3. The objective of the non-zero-sum stochastic differential game between the insurer and the reinsurer under the expected utility criterion is to find a Nash equilibrium strategy such that:

$$\begin{cases}\max\limits_{u_1\in U_1}E_{\{t,\hat{y}_1,l_1,s\}}\left[\mathcal{U}_1\left(\widehat{Y}_1^{u_1}(T)\right)|\widehat{Y}_1^{u_1}(t)=\hat{y}_1,L_1(t)=l_1,S(t)=s\right],\\ \max\limits_{u_2\in U_2}E_{\{t,\hat{y}_2,l_2,s\}}\left[\mathcal{U}_2\left(\widehat{Y}_2^{u_2}(T)\right)|\widehat{Y}_2^{u_2}(t)=\hat{y}_2,L_2(t)=l_2,S(t)=s\right].\end{cases}$$

For any given that $\widehat{Y}_1^{u_1}(t)=\hat{y}_1,\widehat{Y}_2^{u_2}(t)=\hat{y}_2,L_1(t)=l_1,L_2(t)=l_2$ and $S(t)=s$, by applying stochastic control theory, the value functions of the insurer and the reinsurer which are given by (15) and (16) respectively, can be rewritten as follows:

$$\begin{aligned}J_1(t,\hat{y}_1,l_1,s)&=J_1^{u_1^*,u_2^*}(t,\hat{y}_1,l_1,s)\\&=\sup_{u_1\in U_1}E_{\{t,\hat{y}_1,l_1,s\}}\left[\mathcal{U}_1\left(\widehat{Y}_1^{u_1}(T)\right)|\widehat{Y}_1^{u_1}(t)=\hat{y}_1,L_1(t)=l_1,S(t)=s\right],\\ J_2(t,\hat{y}_2,l_2,s)&=J_2^{u_1^*,u_2^*}(t,\hat{y}_2,l_2,s)\\&=\sup_{u_2\in U_2}E_{\{t,\hat{y}_2,l_2,s\}}\left[\mathcal{U}_2\left(\widehat{Y}_2^{u_2}(T)\right)|\widehat{Y}_2^{u_2}(t)=\hat{y}_2,L_2(t)=l_2,S(t)=s\right].\end{aligned}$$

We refer the admissible strategies u_1^* and u_2^* to as the competitively optimal reinsurance and investment strategies.

3. Solution to the Non-zero-sum game

In this section, we derive the explicit solutions to the optimal investment-reinsurance strategy and the corresponding equilibrium value function.

3.1. Assumption

To present the extended HJB equations conveniently, we define two variational operators.

Denote that $C^{1,2,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ is the space of any function $\phi(t,\hat{y},l,s)$ such that $\phi(t,\hat{y},l,s)$ and its derivatives $\phi_t(t,\hat{y},l,s)$, $\phi_{\hat{y}}(t,\hat{y},l,s)$, $\phi_l(t,\hat{y},l,s)$, $\phi_s(t,\hat{y},l,s)$, $\phi_{\hat{y}\hat{y}}(t,\hat{y},l,s)$, $\phi_{ll}(t,\hat{y},l,s)$, $\phi_{ss}(t,\hat{y},l,s)$, $\phi_{\hat{y}l}(t,\hat{y},l,s)$, $\phi_{\hat{y}s}(t,\hat{y},l,s)$, $\phi_{ls}(t,\hat{y},l,s)$ are continuous on $[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

For the value function $J_1(t,\hat{y}_1,l_1,s)\in C^{1,2,2,2}$, the variational operator is defined as follows:

$$
\begin{aligned}
\mathcal{L}^{u_1}J_1(t,\hat{y}_1,l_1,s) = {} & \frac{\partial J_1}{\partial t} + \big[r(\hat{y}_1+l_1-\kappa_1 l_2)+(\alpha-r)(\pi_1(t)-\kappa_1\pi_2(t))-\alpha_1+\kappa_1\alpha_2 \\
& +(\theta_1-\eta_1)a_1+(\theta_2-\eta_2)a_2+a_1\eta_1 q_1(t)+a_2\eta_2 q_2(t)-(1-q_1(t))\kappa_1\eta_1 a_1 \\
& -(1-q_2(t))\kappa_1\eta_2 a_2\big]\frac{\partial J_1}{\partial \hat{y}_1}+\alpha s\frac{\partial J_1}{\partial s}+\alpha_1\frac{\partial J_1}{\partial l_1}+\frac{1}{2}\big[(\sigma_{12}-\kappa_1\sigma_{22})^2+\xi_1(t) \\
& +(\sigma\pi_1(t)-\sigma_{11}-\kappa_1\sigma\pi_2(t)+\kappa_1\sigma_{21})^2 s^{2\beta}\big]\frac{\partial^2 J_1}{\partial \hat{y}_1^2}+\frac{1}{2}\big[\sigma_{11}^2 s^{2\beta}+\sigma_{12}^2\big]\frac{\partial^2 J_1}{\partial l_1^2} \\
& +\big[(\sigma\pi_1(t)-\sigma_{11}-\kappa_1\sigma\pi_2(t)+\kappa_1\sigma_{21})\sigma s^{2\beta+1}\big]\frac{\partial^2 J_1}{\partial \hat{y}_1\partial s} \\
& +\big[(\sigma\pi_1(t)-\sigma_{11}-\kappa_1\sigma\pi_2(t)+\kappa_1\sigma_{21})\sigma_{11}s^{2\beta}-(\sigma_{12}-\kappa_1\sigma_{22})\sigma_{12}\big] \\
& \frac{\partial^2 J_1}{\partial \hat{y}_1\partial l_1}+\frac{1}{2}\sigma^2 s^{2\beta+2}\frac{\partial^2 J_1}{\partial s^2}+\sigma\sigma_{11}s^{2\beta+1}\frac{\partial^2 J_1}{\partial l_1\partial s}
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
\xi_1(t) &= (p_1(t)-\kappa_1 p_2(t))^2 \\
&= \sum_{i=1}^{2}[(1+\kappa_1)q_i(t)-\kappa_1]^2\sigma_i^2+2\lambda\mu_L\mu_Y\prod_{i=1}^{2}[(1+\kappa_1)q_i(t)-\kappa_1].
\end{aligned}
$$

For the value function $J_2(t,\hat{y}_2,l_2,s)\in C^{1,2,2,2}$, the variational operator is defined as follows:

$$
\begin{aligned}
\mathcal{L}^{u_2}J_2(t,\hat{y}_2,l_2,s) = {} & \frac{\partial J_2}{\partial t} + \Big[r(\hat{y}_2+l_2-\kappa_2 l_1)+(\alpha-r)(\pi_2(t)-\kappa_2\pi_1(t))-\alpha_2+\kappa_2\alpha_1 \\
& -\kappa_2(\theta_1-\eta_1)a_1-\kappa_2(\theta_2-\eta_2)a_2-\kappa_2 a_1\eta_1 q_1(t)-\kappa_2 a_2\eta_2 q_2(t) \\
& +(1-q_1(t))\eta_1 a_1+(1-q_2(t))\eta_2 a_2\Big]\frac{\partial J_2}{\partial \hat{y}_2}+\alpha s\frac{\partial J_2}{\partial s}+\alpha_2\frac{\partial J_2}{\partial l_2} \\
& +\frac{1}{2}\Big[(\sigma\pi_2(t)-\sigma_{21}-\kappa_2\sigma\pi_1(t)+\kappa_2\sigma_{11})^2 s^{2\beta}+(\sigma_{22}-\kappa_2\sigma_{12})^2+\xi_2(t)\Big]\frac{\partial^2 J_2}{\partial \hat{y}_2^2} \\
& +\frac{1}{2}\sigma^2 s^{2\beta+2}\frac{\partial^2 J_2}{\partial s^2}+\frac{1}{2}\big[\sigma_{21}^2 s^{2\beta}+\sigma_{22}^2\big]\frac{\partial^2 J_2}{\partial l_2^2} \\
& +\big[(\sigma\pi_2(t)-\sigma_{21}-\kappa_2\sigma\pi_1(t)+\kappa_2\sigma_{11})\sigma s^{2\beta+1}\big]\frac{\partial^2 J_2}{\partial \hat{y}_2\partial s}
\end{aligned}
$$

$$+ \Big[(\sigma\pi_2(t) - \sigma_{21} - \kappa_2\sigma\pi_1(t) + \kappa_2\sigma_{11})\sigma_{21}s^{2\beta} \\ - (\sigma_{22} - \kappa_2\sigma_{12})\sigma_{22}\Big]\frac{\partial^2 J_2}{\partial\hat{y}_2\partial l_2} + \sigma\sigma_{21}s^{2\beta+1}\frac{\partial^2 J_2}{\partial l_2\partial s} \tag{19}$$

where

$$\xi_2(t) = (p_2(t) - \kappa_2 p_1(t))^2 \\ = \sum_{i=1}^{2}[(-\kappa_2 - 1)q_i(t) + 1]^2\sigma_i^2 + 2\lambda\mu_L\mu_Y\prod_{i=1}^{2}[(-1-\kappa_2)q_i(t) + 1].$$

3.2. Verification theorem

According to Fleming and Soner [12], if the value function $J_i(t,\hat{y}_i,l_i,s) \in \mathcal{C}^{1,2,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$, then $J_i(t,\hat{y}_i,l_i,s)$ satisfies the following extended Hamilton–Jacobi–Bellman (HJB) equation:

$$\sup_{u_i\in U_i}\mathcal{L}^{u_i}J_i(t,\hat{y}_i,l_i,s) = 0, \quad J_i(T,\hat{y}_i,l_i,s) = \mathcal{U}_i(\hat{y}_i), \quad i\in\{1,2\}. \tag{20}$$

The following theorem provides verification for the extended HJB equation associated with the Problem 3.

Theorem 4 (Verification Theorem)**.** *Suppose that $V_i(t,\hat{y}_i,l_i,s) \in \mathcal{C}^{1,2,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ is a classical solution to equation (20), i.e., $V_i(t,\hat{y}_i,l_i,s), i\in\{1,2\}$ satisfies the following HJB equation*

$$\sup_{u_i\in U_i}\{\mathcal{L}^{u_i}V_i(t,\hat{y}_i,l_i,s)\} = 0, \quad V_i(T,\hat{y}_i,l_i,s) = \mathcal{U}_i(\hat{y}_i), \tag{21}$$

then we have $J_i(t,\hat{y}_i,l_i,s) \le V_i(t,\hat{y}_i,l_i,s)$ for an arbitrary admissible strategy $u_i\in U_i$. Furthermore assume that there exists an optimal strategy $u_i^*\in U_i$ such that

$$u_i^* \in \arg\sup\{\mathcal{L}^{u_i}V_i(t,\hat{y}_i,l_i,s)\},$$

then $V_i(T,\hat{y}_i,l_i,s) = J_i(T,\hat{y}_i,l_i,s)$ if $u_i = u_i^*$. Hence u_1^* and u_2^* are indeed the optimal strategies for the insurer and the reinsurer.

Proof. See Appendix B. □

3.3. Nash equilibrium and value functions

In the following, we derive the explicit solutions to the HJB equation (20), the expressions of the Nash equilibrium strategies, and the corresponding equilibrium value functions for the problem have been obtained. According to Theorem 4, the equation (20) can be rewritten as

$$\sup_{u_1\in U_1}\Big\{\frac{\partial J_1}{\partial t} + \Big[r(\hat{y}_1 + l_1 - \kappa_1 l_2) + (\alpha - r)(\pi_1(t) - \kappa_1\pi_2^*(t)) - \alpha_1 + \kappa_1\alpha_2 \\ + (\theta_1 - \eta_1)a_1 + (\theta_2 - \eta_2)a_2 + a_1\eta_1 q_1(t) + a_2\eta_2 q_2(t) \\ - (1 - q_1(t))\kappa_1\eta_1 a_1 - (1 - q_2(t))\kappa_1\eta_2 a_2\Big]\frac{\partial J_1}{\partial\hat{y}_1} + \alpha s\frac{\partial J_1}{\partial s} + \alpha_1\frac{\partial J_1}{\partial l_1} \\ + \frac{1}{2}\Big[(\sigma\pi_1(t) - \sigma_{11} - \kappa_1\sigma\pi_2^*(t) + \kappa_1\sigma_{21})^2 s^{2\beta} + (\sigma_{12} - \kappa_1\sigma_{22})^2 + \xi_1(t)\Big]\frac{\partial^2 J_1}{\partial\hat{y}_1^2} \\ + \frac{1}{2}\sigma^2 s^{2\beta+2}\frac{\partial^2 J_1}{\partial s^2} + \frac{1}{2}\big[\sigma_{11}^2 s^{2\beta} + \sigma_{12}^2\big]\frac{\partial^2 J_1}{\partial l_1^2}$$

$$
\begin{aligned}
&+ \left[(\sigma\pi_1(t) - \sigma_{11} - \kappa_1\sigma\pi_2^*(t) + \kappa_1\sigma_{21})\sigma s^{2\beta+1}\right] \frac{\partial^2 J_1}{\partial \hat{y}_1 \partial s} \\
&+ \Big[(\sigma\pi_1(t) - \sigma_{11} - \kappa_1\sigma\pi_2^*(t) + \kappa_1\sigma_{21})\sigma_{11} s^{2\beta} \\
&- (\sigma_{12} - \kappa_1\sigma_{22})\sigma_{12}\Big] \frac{\partial^2 J_1}{\partial \hat{y}_1 \partial l_1} + \sigma\sigma_{11} s^{2\beta+1} \frac{\partial^2 J_1}{\partial l_1 \partial s} \Bigg\} = 0,
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
\sup_{u_1 \in U_1} \Bigg\{ &\frac{\partial J_2}{\partial t} + \Big[r(\hat{y}_2 + l_2 - \kappa_2 l_1) + (\alpha - r)(\pi_2(t) - \kappa_2\pi_1^*(t)) - \alpha_2 + \kappa_2\alpha_1 \\
&- \kappa_2(\theta_1 - \eta_1)a_1 - \kappa_2(\theta_2 - \eta_2)a_2 - \kappa_2 a_1\eta_1 q_1^*(t) - \kappa_2 a_2 \eta_2 q_2^*(t) \\
&+ (1 - q_1^*(t))\eta_1 a_1 + (1 - q_2^*(t))\eta_2 a_2\Big] \frac{\partial J_2}{\partial \hat{y}_2} + \alpha s \frac{\partial J_2}{\partial s} + \alpha_2 \frac{\partial J_2}{\partial l_2} \\
&+ \frac{1}{2}\Big[(\sigma\pi_2(t) - \sigma_{21} - \kappa_2\sigma\pi_1^*(t) + \kappa_2\sigma_{11})^2 s^{2\beta} + (\sigma_{22} - \kappa_2\sigma_{12})^2 \\
&+ \xi_2^*(t)\Big] \frac{\partial^2 J_2}{\partial \hat{y}_2^2} + \frac{1}{2}\sigma^2 s^{2\beta+2} \frac{\partial^2 J_2}{\partial s^2} + \frac{1}{2}\left[\sigma_{21}^2 s^{2\beta} + \sigma_{22}^2\right] \frac{\partial^2 J_2}{\partial l_2^2} \\
&+ \left[(\sigma\pi_2(t) - \sigma_{21} - \kappa_2\sigma\pi_1^*(t) + \kappa_2\sigma_{11})\sigma s^{2\beta+1}\right] \frac{\partial^2 J_2}{\partial \hat{y}_2 \partial s} \\
&+ \Big[(\sigma\pi_2(t) - \sigma_{21} - \kappa_2\sigma\pi_1^*(t) + \kappa_2\sigma_{11})\sigma_{21} s^{2\beta} \\
&- (\sigma_{22} - \kappa_2\sigma_{12})\sigma_{22}\Big] \frac{\partial^2 J_2}{\partial \hat{y}_2 \partial l_2} + \sigma\sigma_{21} s^{2\beta+1} \frac{\partial^2 J_2}{\partial l_2 \partial s} \Bigg\} = 0,
\end{aligned} \tag{23}
$$

where

$$
\xi_2^*(t) = \sum_{i=1}^{2} [(-\kappa_2 - 1)q_i^*(t) + 1]^2 \sigma_i^2 + 2\lambda\mu_L\mu_Y \prod_{i=1}^{2} [(-1 - \kappa_2)q_i^*(t) + 1].
$$

To solve equation (20), by applying the first-order condition and differentiating the expression inside of the bracket of the above equation (22) w.r.t. $\pi_1(t)$, $q_1(t)$ and $q_2(t)$ respectively, and differentiating equation (23) w.r.t. $\pi_2(t)$, we obtain the following equations for $u_1^*(t) = (\pi_1^*(t), q_1^*(t), q_2^*(t))$ and $u_2^*(t) = (\pi_2^*(t))$:

$$
\begin{aligned}
&(\alpha - r)\frac{\partial J_1}{\partial \hat{y}_1} + (\sigma\pi_1(t) - \sigma_{11} - \kappa_1\sigma\pi_2^*(t) + \kappa_1\sigma_{21})\sigma s^{2\beta} \frac{\partial^2 J_1}{\partial \hat{y}_1^2} \\
&\qquad + \sigma^2 s^{2\beta+1} \frac{\partial^2 J_1}{\partial \hat{y}_1 \partial s} + \sigma\sigma_{11} s^{2\beta} \frac{\partial^2 J_1}{\partial \hat{y}_1 \partial l_1} = 0, \\
&a_1\eta_1 \frac{\partial J_1}{\partial \hat{y}_1} + \left[2((1 + \kappa_1)q_1(t) - \kappa_1)\sigma_1^2 + 2\lambda\mu_L\mu_Y((1 + \kappa_1)q_2(t) - \kappa_1)\right] \frac{\partial^2 J_1}{\partial \hat{y}_1^2} = 0, \\
&a_2\eta_2 \frac{\partial J_1}{\partial \hat{y}_1} + \left[2((1 + \kappa_1)q_2(t) - \kappa_1)\sigma_2^2 + 2\lambda\mu_L\mu_Y((1 + \kappa_1)q_1(t) - \kappa_1)\right] \frac{\partial^2 J_1}{\partial \hat{y}_1^2} = 0, \\
&(\alpha - r)\frac{\partial J_2}{\partial \hat{y}_2} + (\sigma\pi_2(t) - \sigma_{21} - \kappa_2\sigma\pi_1^*(t) + \kappa_2\sigma_{11})\sigma s^{2\beta} \frac{\partial^2 J_2}{\partial \hat{y}_2^2} \\
&\qquad + \sigma^2 s^{2\beta+1} \frac{\partial^2 J_2}{\partial \hat{y}_2 \partial s} + \sigma\sigma_{21} s^{2\beta} \frac{\partial^2 J_2}{\partial \hat{y}_2 \partial l_2} = 0.
\end{aligned}
$$

Then we can derive

$$\pi_1^*(t) = \frac{-(\alpha-r)\frac{\partial J_1}{\partial \hat{y}_1} - \sigma^2 s^{2\beta+1}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial s} - \sigma\sigma_{11}s^{2\beta}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial l_1}}{\sigma^2 s^{2\beta}\frac{\partial^2 J_1}{\partial \hat{y}_1^2}} + \frac{\sigma_{11}-\kappa_1\sigma_{21}}{\sigma} + \kappa_1\pi_2^*(t), \tag{24}$$

$$q_1^*(t) = \frac{-a_1\eta_1\frac{\partial J_1}{\partial \hat{y}_1} - 2\lambda\mu_L\mu_Y((1+\kappa_1)q_2(t)-\kappa_1)\frac{\partial^2 J_1}{\partial \hat{y}_1^2}}{2\sigma_1^2(1+\kappa_1)} + \frac{\kappa_1}{1+\kappa_1}, \tag{25}$$

$$q_2^*(t) = \frac{-a_2\eta_2\frac{\partial J_2}{\partial \hat{y}_2} - 2\lambda\mu_L\mu_Y((1+\kappa_1)q_1(t)-\kappa_1)\frac{\partial^2 J_2}{\partial \hat{y}_2^2}}{2\sigma_2^2(1+\kappa_1)} + \frac{\kappa_1}{1+\kappa_1}, \tag{26}$$

$$\pi_2^*(t) = \frac{-(\alpha-r)\frac{\partial J_2}{\partial \hat{y}_2} - \sigma^2 s^{2\beta+1}\frac{\partial^2 J_2}{\partial \hat{y}_2 \partial s} - \sigma\sigma_{21}s^{2\beta}\frac{\partial^2 J_2}{\partial \hat{y}_2 \partial l_2}}{\sigma^2 s^{2\beta}\frac{\partial^2 J_2}{\partial \hat{y}_2^2}} + \frac{\sigma_{21}-\kappa_2\sigma_{11}}{\sigma} + \kappa_2\pi_1^*(t). \tag{27}$$

Inserting (24)–(26) into (22), we have

$$\begin{aligned}
&\frac{\partial J_1}{\partial t} + \Bigg[r(\hat{y}_1 + l_1 - \kappa_1 l_2) - \alpha_1 + \kappa_1\alpha_2 + (\theta_1-\eta_1)a_1 + (\theta_2-\eta_2)a_2 \\
&\quad + (\alpha-r)\left(\frac{-(\alpha-r)\frac{\partial J_1}{\partial \hat{y}_1} - \sigma^2 s^{2\beta+1}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial s} - \sigma\sigma_{11}s^{2\beta}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial l_1}}{\sigma^2 s^{2\beta}\frac{\partial^2 J_1}{\partial \hat{y}_1^2}} + \frac{\sigma_{11}-\kappa_1\sigma_{21}}{\sigma}\right) \\
&\quad + a_1\eta_1 q_1^*(t) + a_2\eta_2 q_2^*(t) - (1-q_1^*(t))\kappa_1\eta_1 a_1 - (1-q_2^*(t))\kappa_1\eta_2 a_2\Bigg]\frac{\partial J_1}{\partial \hat{y}_1} \\
&\quad + \alpha s\frac{\partial J_1}{\partial s} + \alpha_1\frac{\partial J_1}{\partial l_1} + \frac{1}{2}\Big[(\sigma_{12}-\kappa_1\sigma_{22})^2 + \xi_1^*(t)\Big]\frac{\partial^2 J_1}{\partial \hat{y}_1^2} + \sigma\sigma_{11}s^{2\beta+1}\frac{\partial^2 J_1}{\partial l_1 \partial s} \\
&\quad + \left[\frac{-(\alpha-r)\frac{\partial J_1}{\partial \hat{y}_1} - \sigma^2 s^{2\beta+1}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial s} - \sigma\sigma_{11}s^{2\beta}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial l_1}}{\frac{\partial^2 J_1}{\partial \hat{y}_1^2}}\right]s\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial s} + \frac{1}{2}\sigma^2 s^{2\beta+2}\frac{\partial^2 J_1}{\partial s^2} \\
&\quad + \left[\frac{-(\alpha-r)\frac{\partial J_1}{\partial \hat{y}_1} - \sigma^2 s^{2\beta+1}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial s} - \sigma\sigma_{11}s^{2\beta}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial l_1}}{\sigma\frac{\partial^2 J_1}{\partial \hat{y}_1^2}}\sigma_{11} - (\sigma_{12}-\kappa_1\sigma_{22})\sigma_{12}\right]\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial l_1} \\
&\quad + \frac{1}{2}\left[\sigma_{11}^2 s^{2\beta} + \sigma_{12}^2\right]\frac{\partial^2 J_1}{\partial l_1^2} + \frac{\left[-(\alpha-r)\frac{\partial J_1}{\partial \hat{y}_1} - \sigma^2 s^{2\beta+1}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial s} - \sigma\sigma_{11}s^{2\beta}\frac{\partial^2 J_1}{\partial \hat{y}_1 \partial l_1}\right]^2}{2\sigma^2 s^{2\beta}\frac{\partial^2 J_1}{\partial \hat{y}_1^2}} = 0.
\end{aligned} \tag{28}$$

Substituting equation (27) into equation (23), we derive

$$\begin{aligned}
&\frac{\partial J_2}{\partial t} + \Bigg[r(\hat{y}_2 + l_2 - \kappa_2 l_1) - \alpha_2 + \kappa_2\alpha_1 - \kappa_2(\theta_1-\eta_1)a_1 - \kappa_2(\theta_2-\eta_2)a_2 \\
&\quad + (\alpha-r)\left(\frac{-(\alpha-r)\frac{\partial J_2}{\partial \hat{y}_2} - \sigma^2 s^{2\beta+1}\frac{\partial^2 J_2}{\partial \hat{y}_2 \partial s} - \sigma\sigma_{21}s^{2\beta}\frac{\partial^2 J_2}{\partial \hat{y}_2 \partial l_2}}{\sigma^2 s^{2\beta}\frac{\partial^2 J_2}{\partial \hat{y}_2^2}} + \frac{\sigma_{21}-\kappa_2\sigma_{11}}{\sigma}\right) \\
&\quad - \kappa_2 a_1\eta_1 q_1^*(t) - \kappa_2 a_2\eta_2 q_2^*(t) + (1-q_1^*(t))\eta_1 a_1 + (1-q_2^*(t))\eta_2 a_2\Bigg]\frac{\partial J_2}{\partial \hat{y}_2} \\
&\quad + \alpha s\frac{\partial J_2}{\partial s} + \alpha_2\frac{\partial J_2}{\partial l_2} + \frac{1}{2}\Big[(\sigma_{22}-\kappa_2\sigma_{12})^2 + \xi_2^*(t)\Big]\frac{\partial^2 J_2}{\partial \hat{y}_2^2} + \sigma\sigma_{21}s^{2\beta+1}\frac{\partial^2 J_2}{\partial l_2 \partial s}
\end{aligned}$$

$$+\frac{-(\alpha-r)\frac{\partial J_2}{\partial \hat{y}_2}-\sigma^2 s^{2\beta+1}\frac{\partial^2 J_2}{\partial \hat{y}_2\partial s}-\sigma\sigma_{21}s^{2\beta}\frac{\partial^2 J_2}{\partial \hat{y}_2\partial l_2}}{\frac{\partial^2 J_2}{\partial \hat{y}_2^2}}s\frac{\partial^2 J_2}{\partial \hat{y}_2\partial s}+\frac{1}{2}\sigma^2 s^{2\beta+2}\frac{\partial^2 J_2}{\partial s^2}$$

$$+\left[\frac{-(\alpha-r)\frac{\partial J_2}{\partial \hat{y}_2}-\sigma^2 s^{2\beta+1}\frac{\partial^2 J_2}{\partial \hat{y}_2\partial s}-\sigma\sigma_{21}s^{2\beta}\frac{\partial^2 J_2}{\partial \hat{y}_2\partial l_2}}{\sigma\frac{\partial^2 J_2}{\partial \hat{y}_2^2}}\sigma_{21}-(\sigma_{22}-\kappa_2\sigma_{12})\sigma_{22}\right]\frac{\partial^2 J_2}{\partial \hat{y}_2\partial l_2}$$

$$+\frac{1}{2}\left[\sigma_{21}^2 s^{2\beta}+\sigma_{22}^2\right]\frac{\partial^2 J_2}{\partial l_2^2}+\frac{\left[-(\alpha-r)\frac{\partial J_2}{\partial \hat{y}_2}-\sigma^2 s^{2\beta+1}\frac{\partial^2 J_2}{\partial \hat{y}_2\partial s}-\sigma\sigma_{21}s^{2\beta}\frac{\partial^2 J_2}{\partial \hat{y}_2\partial l_2}\right]^2}{2\sigma^2 s^{2\beta}\frac{\partial^2 J_2}{\partial \hat{y}_2^2}}=0, \tag{29}$$

where

$$q_1^*(t)=\frac{-a_1\eta_1\sigma_2^2\frac{\partial J_1}{\partial \hat{y}_1}+a_2\eta_2\lambda\mu_L\mu_Y\frac{\partial J_2}{\partial \hat{y}_2}\frac{\partial^2 J_1}{\partial \hat{y}_1^2}-2\lambda^2\mu_L^2\mu_Y^2\kappa_1\frac{\partial^2 J_1}{\partial \hat{y}_1^2}\frac{\partial^2 J_2}{\partial \hat{y}_2^2}+2\kappa_1\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2(1+\kappa_1)-2\lambda^2\mu_L^2\mu_Y^2(1+\kappa_1)\frac{\partial^2 J_1}{\partial \hat{y}_1^2}\frac{\partial^2 J_2}{\partial \hat{y}_2^2}}, \tag{30}$$

$$q_2^*(t)=\frac{-a_2\eta_2\sigma_2^2\frac{\partial J_2}{\partial \hat{y}_2}+a_1\eta_1\lambda\mu_L\mu_Y\frac{\partial J_1}{\partial \hat{y}_1}\frac{\partial^2 J_2}{\partial \hat{y}_2^2}-2\lambda^2\mu_L^2\mu_Y^2\kappa_1\frac{\partial^2 J_1}{\partial \hat{y}_1^2}\frac{\partial^2 J_2}{\partial \hat{y}_2^2}+2\kappa_1\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2(1+\kappa_1)-2\lambda^2\mu_L^2\mu_Y^2(1+\kappa_1)\frac{\partial^2 J_1}{\partial \hat{y}_1^2}\frac{\partial^2 J_2}{\partial \hat{y}_2^2}}. \tag{31}$$

In order to solve (20), by the exponential utility function depicted by the subsection (2.4), we assume the value functions $J_1(t,\hat{y}_1,l_1,s)$ and $J_2(t,\hat{y}_2,l_2,s)$ of insurers and reinsurers have the following forms:

$$J_1(t,\hat{y}_1,l_1,s)=-\frac{1}{\gamma_1}\exp\Big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1+l_1+f_1(t,l_1)]+g_1(t,s)\Big\}, \tag{32}$$

and

$$J_2(t,\hat{y}_2,l_2,s)=-\frac{1}{\gamma_2}\exp\Big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2+l_2+f_2(t,l_2)]+g_2(t,s)\Big\}, \tag{33}$$

where $f_1(T,l_1)=-l_1$, $f_2(T,l_2)=-l_2$, $g_1(T,s)=0$ and $g_2(T,s)=0$, then we have the corresponding partial derivatives:

$$\begin{aligned}
\frac{\partial J_i}{\partial t}&=\left[\gamma_i r e^{r(T-t)}(\hat{y}_i+l_i+f_i)-\gamma_i e^{r(T-t)}\frac{\partial f_i}{\partial t}+\frac{\partial g_i}{\partial t}\right]J_i, \quad \frac{\partial J_i}{\partial \hat{y}_j}=-\gamma_i e^{r(T-t)}J_i,\\
\frac{\partial J_i}{\partial l_j}&=-\gamma_i e^{r(T-t)}\left(1+\frac{\partial f_j}{\partial l_j}\right)J_i, \quad \frac{\partial J_i}{\partial s}=\frac{\partial g_j}{\partial s}J_i, \quad \frac{\partial^2 J_i}{\partial \hat{y}_j^2}=\gamma_i^2 e^{2r(T-t)}J_i,\\
\frac{\partial^2 J_i}{\partial l_j^2}&=\left[\gamma_i^2 e^{2r(T-t)}\left(1+\frac{\partial f_j}{\partial l_j}\right)^2-\gamma_i e^{r(T-t)}\frac{\partial^2 f_j}{\partial l_j^2}\right]J_i,\\
\frac{\partial^2 J_i}{\partial \hat{y}_j\partial l_j}&=\gamma_i^2 e^{2r(T-t)}\left(1+\frac{\partial f_j}{\partial l_j}\right)J_i, \quad \frac{\partial^2 J_i}{\partial s^2}=\left[\frac{\partial^2 g_j}{\partial s^2}+\left(\frac{\partial g_j}{\partial s}\right)^2\right]J_i,\\
\frac{\partial^2 J_i}{\partial l_j\partial s}&=-\gamma_i e^{r(T-t)}\left(1+\frac{\partial f_j}{\partial l_j}\right)\frac{\partial g_j}{\partial s}J_i, \quad \frac{\partial^2 J_i}{\partial \hat{y}_j\partial s}=-\gamma_i e^{r(T-t)}\frac{\partial g_j}{\partial s}J_i.
\end{aligned}$$

Substituting the above partial derivatives into (28) and (29) yields

$$\gamma_1 e^{r(T-t)}\bigg[rf_1-\frac{\partial f_1}{\partial t}-\left(\alpha_1-\frac{(\alpha-r)\sigma_{11}}{\sigma}-\kappa_1\sigma_{12}\sigma_{22}\gamma_1 e^{r(T-t)}\right)\frac{\partial f_1}{\partial l_1}$$

$$+\frac{\sigma_{12}^2}{2}\gamma_1 e^{r(T-t)}\left(\frac{\partial f_1}{\partial l_1}\right)^2-\frac{\sigma_{11}^2 s^{2\beta}+\sigma_{12}^2}{2}\frac{\partial^2 f_1}{\partial l_1^2}+\frac{\kappa_1^2\sigma_{22}^2\gamma_1}{2}e^{r(T-t)}$$

$$
\begin{aligned}
&+ r\kappa_1 l_2 + \frac{(\alpha-r)\kappa_1\sigma_{21}}{\sigma} - \kappa_1\alpha_2\Bigg] + \frac{\partial g_1}{\partial t} + rs\frac{\partial g_1}{\partial s} \\
&+ \frac{\sigma^2 s^{2\beta+2}}{2}\frac{\partial^2 g_1}{\partial s^2} - \frac{(\alpha-r)^2}{2\sigma^2 s^{2\beta}} + \inf_{q_1,q_2}\{H_1(q_1,q_2,t)\} = 0,
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
&\gamma_2 e^{r(T-t)}\Bigg[rf_2 - \frac{\partial f_2}{\partial t} - \left(\alpha_2 - \frac{(\alpha-r)\sigma_{21}}{\sigma} - \kappa_2\sigma_{22}\sigma_{12}\gamma_2 e^{r(T-t)}\right)\frac{\partial f_2}{\partial l_2} \\
&+ \frac{\sigma_{22}^2}{2}\gamma_2 e^{r(T-t)}\left(\frac{\partial f_2}{\partial l_2}\right)^2 - \frac{\sigma_{21}^2 s^{2\beta}+\sigma_{22}^2}{2}\frac{\partial^2 f_2}{\partial l_2^2} + \frac{\kappa_2^2\sigma_{12}^2\gamma_2}{2}e^{r(T-t)} \\
&+ r\kappa_2 l_1 + \frac{(\alpha-r)\kappa_2\sigma_{11}}{\sigma} - \kappa_2\alpha_1\Bigg] + \frac{\partial g_2}{\partial t} + rs\frac{\partial g_2}{\partial s} \\
&+ \frac{\sigma^2 s^{2\beta+2}}{2}\frac{\partial^2 g_2}{\partial s^2} - \frac{(\alpha-r)^2}{2\sigma^2 s^{2\beta}} + H_2(q_1^*,q_2^*,t) = 0,
\end{aligned}
\tag{35}
$$

where

$$
\begin{aligned}
H_1(q_1,q_2,t) = {}& \gamma_1 e^{r(T-t)}\big[-(\theta_1-\eta_1)-(\theta_2-\eta_2)-a_1\eta_1 q_1 - a_2\eta_2 q_2 \\
&+ (1-q_1)\kappa_1\eta_1 a_1 + (1-q_2)\kappa_1\eta_2 a_2\big] + \frac{1}{2}\Big\{[(1+\kappa_1)q_1(t)-\kappa_1]^2\sigma_1^2 \\
&+ [(1+\kappa_1)q_2(t)-\kappa_1]^2\sigma_2^2 + 2\lambda\mu_L\mu_Y[(1+\kappa_1)q_1(t)-\kappa_1] \\
&[(1+\kappa_1)q_2(t)-\kappa_1]\Big\}\gamma_1^2 e^{2r(T-t)},
\end{aligned}
\tag{36}
$$

and

$$
\begin{aligned}
H_2(q_1^*,q_2^*,t) = {}& \gamma_2 e^{r(T-t)}\big[\kappa_2(\theta_1-\eta_1)a_1 + \kappa_2(\theta_2-\eta_2)a_2 + \kappa_2 a_1\eta_1 q_1^* + \kappa_2 a_2\eta_2 q_2^* \\
&- (1-q_1^*)\eta_1 a_1 - (1-q_2^*)\eta_2 a_2\big] + \frac{1}{2}\Big\{[(-\kappa_2-1)q_1^*+1]^2\sigma_1^2 \\
&+ [(-\kappa_2-1)q_2^*+1]^2\sigma_2^2 + 2\lambda\mu_L\mu_Y[(-\kappa_2-1)q_1^*+1] \\
&[(-\kappa_2-1)q_2^*+1]\Big\}\gamma_2^2 e^{2r(T-t)}.
\end{aligned}
\tag{37}
$$

By differentiating the above equation (36) w.r.t $q_1(t)$ and $q_1(t)$ respectively, we can derive

$$
\begin{aligned}
\frac{\partial H_1}{\partial q_1} &= (1+\kappa_1)\gamma_1 e^{r(T-t)}\Big[-a_1\eta_1 + \big\{[(1+\kappa_1)q_1-\kappa_1]\sigma_1^2 + \lambda\mu_L\mu_Y[(1+\kappa_1)q_2-\kappa_1]\big\}\gamma_1^2 e^{2r(T-t)}\Big], \\
\frac{\partial H_1}{\partial q_2} &= (1+\kappa_1)\gamma_1 e^{r(T-t)}\Big[-a_2\eta_2 + \big\{[(1+\kappa_1)q_2-\kappa_1]\sigma_2^2 + \lambda\mu_L\mu_Y[(1+\kappa_1)q_1-\kappa_1]\big\}\gamma_1^2 e^{2r(T-t)}\Big], \\
\frac{\partial^2 H_1}{\partial q_1^2} &= \sigma_1^2(1+\kappa_1)^2\gamma_1^2 e^{2r(T-t)}, \\
\frac{\partial^2 H_1}{\partial q_2^2} &= \sigma_2^2(1+\kappa_1)^2\gamma_1^2 e^{2r(T-t)}, \\
\frac{\partial^2 H_1}{\partial q_1\partial q_2} &= (1+\kappa_1)^2\lambda\mu_L\mu_Y\gamma_1^2 e^{2r(T-t)}.
\end{aligned}
$$

Then we have following Hessian matrix:

$$
\begin{bmatrix} \frac{\partial^2 H_1}{\partial q_1^2} & \frac{\partial^2 H_1}{\partial q_1\partial q_2} \\ \frac{\partial^2 H_1}{\partial q_2\partial q_1} & \frac{\partial^2 H_1}{\partial q_2^2} \end{bmatrix} = (1+\kappa_1)^2\gamma_1^2 e^{2r(T-t)}\begin{bmatrix} \sigma_1^2 & \lambda\mu_L\mu_Y \\ \lambda\mu_L\mu_Y & \sigma_2^2 \end{bmatrix}.
$$

By Yuen *et al.* [36]

$$\begin{bmatrix} \sigma_1^2 & \lambda\mu_L\mu_Y \\ \lambda\mu_L\mu_Y & \sigma_2^2 \end{bmatrix}$$

is a positive definite matrix. By the first-order condition, we derive

$$\hat{q}_1(t) = \frac{A_1}{(1+\kappa_1)\gamma_1 e^{r(T-t)} A_3} + \frac{\kappa_1}{1+\kappa_1}, \quad \hat{q}_2(t) = \frac{A_2}{(1+\kappa_1)\gamma_1 e^{r(T-t)} A_3} + \frac{\kappa_1}{1+\kappa_1}, \tag{38}$$

where $A_1 = \sigma_2^2 a_1\eta_1 - \lambda\mu_L\mu_Y a_2\eta_2, A_2 = \sigma_1^2 a_2\eta_2 - \lambda\mu_L\mu_Y a_1\eta_1, A_3 = \sigma_1^2\sigma_2^2 - \lambda^2\mu_L^2\mu_Y^2$.

Note that if $q_i < \hat{q}_i$ then $\frac{\partial H_1}{\partial q_i} < 0$ and if $q_i > \hat{q}_i$ then $\frac{\partial H_1}{\partial q_i} > 0$, beside $\frac{\partial^2 H_1}{\partial q_i^2} > 0$. Therefore $H_1(q_1, q_2, t)$ has the a minimum point at $(\hat{q}_1, \hat{q}_2)$.

Remark 5. Note that for the insurer and the reinsurer, $q_1(t) \in [0,1]$ and $q_2(t) \in [0,1]$ correspond to a reinsurance cover. If $0 < \hat{q}_i(t) < 1$, then the equilibrium reinsurance strategy $q_i^*(t) = \hat{q}_i(t)(t)$; if $\hat{q}_i(t)(t) < 0$, then $q_i^*(t) = 0$; if $\hat{q}_i(t)(t) > 1$, then $q_i^*(t) = 1$; this case means that the company can take an extra insurance business from other companies (*i.e.*, act as a reinsurer for other cedents).

Next, we discuss the equilibrium investment-reinsurance strategy $u_i^*(t)$ under different cases.

Substituting (38) into equation (34) and separating the variables of equations (34) and (35), we can derive the following equations:

$$\begin{cases} rf_1 - \frac{\partial f_1}{\partial t} - \left(\alpha_1 - \frac{(\alpha-r)\sigma_{11}}{\sigma} - \kappa_1\sigma_{12}\sigma_{22}\gamma_1 e^{r(T-t)}\right)\frac{\partial f_1}{\partial l_1} \\ + \frac{\sigma_{12}^2}{2}\gamma_1 e^{r(T-t)}\left(\frac{\partial f_1}{\partial l_1}\right)^2 - \frac{\sigma_{11}^2 s^{2\beta}+\sigma_{12}^2}{2}\frac{\partial^2 f_1}{\partial l_1^2} + \frac{\kappa_1^2\sigma_{22}^2\gamma_1}{2}e^{r(T-t)} \\ + r\kappa_1 l_2 + \frac{(\alpha-r)\kappa_1\sigma_{21}}{\sigma} - \kappa_1\alpha_2 = 0, f_1(T, l_1) = -l_1, \\ \frac{\partial g_1}{\partial t} + rs\frac{\partial g_1}{\partial s} + \frac{\sigma^2 s^{2\beta+2}}{2}\frac{\partial^2 g_1}{\partial s^2} - \frac{(\alpha-r)^2}{2\sigma^2 s^{2\beta}} + H_1(q_1^*, q_2^*, t) = 0, g_1(T,s) = 0, \end{cases} \tag{39}$$

and

$$\begin{cases} rf_2 - \frac{\partial f_2}{\partial t} - \left(\alpha_2 - \frac{(\alpha-r)\sigma_{21}}{\sigma} - \kappa_2\sigma_{22}\sigma_{12}\gamma_2 e^{r(T-t)}\right)\frac{\partial f_2}{\partial l_2} \\ + \frac{\sigma_{22}^2}{2}\gamma_2 e^{r(T-t)}\left(\frac{\partial f_2}{\partial l_2}\right)^2 - \frac{\sigma_{21}^2 s^{2\beta}+\sigma_{22}^2}{2}\frac{\partial^2 f_2}{\partial l_2^2} + \frac{\kappa_2^2\sigma_{12}^2\gamma_2}{2}e^{r(T-t)} \\ + r\kappa_2 l_1 + \frac{(\alpha-r)\kappa_2\sigma_{11}}{\sigma} - \kappa_2\alpha_1 = 0, f_2(T, l_2) = -l_2, \\ \frac{\partial g_2}{\partial t} + rs\frac{\partial g_2}{\partial s} + \frac{\sigma^2 s^{2\beta+2}}{2}\frac{\partial^2 g_2}{\partial s^2} - \frac{(\alpha-r)^2}{2\sigma^2 s^{2\beta}} + H_2(q_1^*, q_2^*, t) = 0, g_2(T,s) = 0. \end{cases} \tag{40}$$

Since the structure of the value functions $J_1(t, \hat{y}_1, l_1, s)$ and $J_2(t, \hat{y}_1, l_1, s)$ is given, as well as the boundary conditions of $f_i(t)$ and $g_i(t)$ are given, it is natural to assume

$$f_i(t, l_i) = F_i(t)l_i + G_i(t), \tag{41}$$

$$g_i(t,s) = P_i(t)s^{-2\beta} + Q_i(t). \tag{42}$$

Substituting equations (41) and (42) into the terminal value problems (39) and (40), combining the boundary conditions $f_i(T, l_i) = -l_i$ and $g_i(T, s) = 0$, we derive the following equations:

$$\begin{cases} rF_1(t) - \frac{\mathrm{d}F_1(t)}{\mathrm{d}t} = 0, & F_1(T) = -1, \\ rG_1(t) - \frac{\mathrm{d}G_1(t)}{\mathrm{d}t} + \left[\frac{(\alpha-r)\sigma_{11}}{\sigma} + \kappa_1\sigma_{12}\sigma_{22}\gamma_1 e^{r(T-t)} - \alpha_1\right]F_1(t) & \\ + \frac{\sigma_{12}^2}{2}\gamma_1 e^{r(T-t)}(F_1(t))^2 + \frac{\kappa_1^2\sigma_{22}^2}{2}\gamma_1 e^{r(T-t)} + r\kappa_1 l_2 + \frac{(\alpha-r)\kappa_1\sigma_{21}}{\sigma} & \\ -\kappa_1\sigma_{21} = 0, & G_1(T) = 0 \end{cases} \tag{43}$$

$$\begin{cases} \frac{\mathrm{d}P_1(t)}{\mathrm{d}t} = \big(2r\beta - \sigma^2\beta(2\beta+1)\big)P_1(t) + \frac{(\alpha-r)^2}{2\sigma^2}, & P_1(T) = 0, \\ \frac{\mathrm{d}Q_1(t)}{\mathrm{d}t} = -H_1(q_1^*, q_2^*, t), & Q_1(T) = 0 \end{cases} \tag{44}$$

and

$$\begin{cases} rF_2(t) - \frac{\mathrm{d}F_2(t)}{\mathrm{d}t} = 0, & F_2(T) = -1, \\ rG_2(t) - \frac{\mathrm{d}G_2(t)}{\mathrm{d}t} + \left[\frac{(\alpha-r)\sigma_{21}}{\sigma} + \kappa_2\sigma_{22}\sigma_{12}\gamma_2 e^{r(T-t)} - \alpha_2\right]F_2(t) & \\ + \frac{\sigma_{22}^2}{2}\gamma_2 e^{r(T-t)}(F_2(t))^2 + \frac{\kappa_2^2\sigma_{12}^2}{2}\gamma_2 e^{r(T-t)} + r\kappa_2 l_1 + \frac{(\alpha-r)\kappa_2\sigma_{11}}{\sigma} & \\ -\kappa_2\sigma_{11} = 0, & G_2(T) = 0. \end{cases} \tag{45}$$

$$\begin{cases} \frac{\mathrm{d}P_2(t)}{\mathrm{d}t} = \big(2r\beta - \sigma^2\beta(2\beta+1)\big)P_2(t) + \frac{(\alpha-r)^2}{2\sigma^2}, & P_2(T) = 0, \\ \frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = -H_2(q_1^*, q_2^*, t), & Q_2(T) = 0. \end{cases} \tag{46}$$

According to the boundary conditions and the solution of the corresponding differential equations, we obtain:

$$\begin{aligned}
F_1(t) &= F_2(t) = -e^{-r(T-t)}, \\
G_1(t) &= \left\{\left[\frac{(\alpha-r)\sigma_{11}}{\sigma} - \alpha_1 - \frac{\sigma_{12}^2\gamma_1}{2}\right](T-t) + \frac{\kappa_1^2\sigma_{22}^2\gamma_1}{4r}\Big(1 - e^{2r(T-t)}\Big)\right. \\
&\quad \left. - \frac{1}{r}\left[\kappa_1\sigma_{12}\sigma_{22}\gamma_1 - r\kappa_1 l_2 - \frac{(\alpha-r)\kappa_1\sigma_{21}}{\sigma} + \kappa_1\sigma_{22}\right]\Big(1 - e^{r(T-t)}\Big)\right\}e^{-r(T-t)}, \\
G_2(t) &= \left\{\left[\frac{(\alpha-r)\sigma_{21}}{\sigma} - \alpha_2 - \frac{\sigma_{22}^2\gamma_2}{2}\right](T-t) + \frac{\kappa_2^2\sigma_{12}^2\gamma_2}{4r}\Big(1 - e^{2r(T-t)}\Big)\right. \\
&\quad \left. - \frac{1}{r}\left[\kappa_2\sigma_{22}\sigma_{12}\gamma_2 - r\kappa_2 l_1 - \frac{(\alpha-r)\kappa_2\sigma_{11}}{\sigma} + \kappa_2\sigma_{12}\right]\Big(1 - e^{r(T-t)}\Big)\right\}e^{-r(T-t)}, \\
P_1(t) &= P_2(t) = \frac{(\alpha-r)^2\left[e^{-(2r\beta-\sigma^2\beta(2\beta+1))(T-t)} - 1\right]}{4r\beta\sigma^2 - 2\sigma^4\beta(2\beta+1)}, \\
Q_1(t) &= \int_t^T H_1(q_1^*(s), q_2^*(s), s)\,\mathrm{d}s, \\
Q_2(t) &= \int_t^T H_2(q_1^*(s), q_2^*(s), s)\,\mathrm{d}s.
\end{aligned}$$

Substituting the above results into equations (24) and (27), we derive

$$\pi_1^*(t) = \frac{\big(\alpha - r - 2\sigma^2\beta P_1(t)\big)(\gamma_2+\gamma_1\kappa_1) + (\kappa_1\sigma_{21}+\sigma_{11})\sigma s^{2\beta}\gamma_1\gamma_2\big(1-e^{r(T-t)}\big)}{(1-\kappa_1\kappa_2)\sigma s^{2\beta}\gamma_1\gamma_2 e^{r(T-t)}} + \frac{\sigma_{11}}{\sigma}, \tag{47}$$

$$\pi_2^*(t) = \frac{\big(\alpha - r - 2\sigma^2\beta P_2(t)\big)(\gamma_1+\gamma_2\kappa_2) + (\kappa_2\sigma_{11}+\sigma_{21})\sigma s^{2\beta}\gamma_1\gamma_2\big(1-e^{r(T-t)}\big)}{(1-\kappa_1\kappa_2)\sigma s^{2\beta}\gamma_1\gamma_2 e^{r(T-t)}} + \frac{\sigma_{21}}{\sigma}, \tag{48}$$

and the corresponding equilibrium value function is

$$J_i(t,\hat{y}_i,l_i,s) = -\frac{1}{\gamma_i}\exp\Big\{-\gamma_i e^{r(T-t)}[\hat{y}_i + l_i + f_i(t,l_i)] + g_i(t,s)\Big\}, \quad i\in\{1,2\},$$

where $f_i(t,l_i)$ and $g_i(t,s)$ are given by (41) and (42), respectively. Note that the investment strategy and the reinsurance strategy are independent.

To guarantee that the insurance retention $q_1(t)$ and $q_2(t)$ are non-negative, we give the following lemma.

Lemma 6. *The parameters $\lambda,\lambda_1,\lambda_2,\mu_L,\mu_Y,\mu_L'$ and μ_Y' are given in Section 2 above and satisfy the following inequalities:*

$$\sigma_1^2\sigma_2^2 > \lambda^2\mu_L^2\mu_Y^2, \frac{\lambda\mu_L\mu_Y}{\sigma_2^2}\frac{a_2}{a_1} < 1 < \frac{\sigma_1^2}{\lambda\mu_L\mu_Y}\frac{a_2}{a_1}.$$

Proof. See Appendix A. □

Considering that $q_1(t)$ and $q_2(t)$ are non-negative, according to Lemma 6, the following cases need to be discussed.

Case 1. For $\eta_1 < \frac{\lambda\mu_L\mu_Y}{(\lambda+\lambda_2)\mu_Y'}\frac{(\lambda+\lambda_2)\mu_Y}{(\lambda+\lambda_1)\mu_L}\eta_2$, we have $m_1 \le 0$ and $m_2 > 0$;

Case 2. For $\frac{\lambda\mu_L\mu_Y}{(\lambda+\lambda_2)\mu_Y'}\frac{(\lambda+\lambda_2)\mu_Y}{(\lambda+\lambda_1)\mu_L}\eta_2 < \eta_1 < \frac{(\lambda+\lambda_1)\mu_L'}{\lambda\mu_L\mu_Y}\frac{(\lambda+\lambda_2)\mu_Y}{(\lambda+\lambda_1)\mu_L}\eta_2$, we have $m_1 > 0$ and $m_2 > 0$;

Case 3. For $\eta_1 \ge \frac{(\lambda+\lambda_1)\mu_L'}{\lambda\mu_L\mu_Y}\frac{(\lambda+\lambda_2)\mu_Y}{(\lambda+\lambda_1)\mu_L}\eta_2$, we have $m_1 > 0$ and $m_2 \le 0$.

Here $m_1 = \frac{A_1}{A_3} = \frac{a_1\eta_1\sigma_2^2 - a_2\eta_2\lambda\mu_L\mu_Y}{\sigma_1^2\sigma_2^2-\lambda^2\mu_L^2\mu_Y^2}$ and $m_2 = \frac{A_2}{A_3} = \frac{a_2\eta_2\sigma_1^2 - a_1\eta_1\lambda\mu_L\mu_Y}{\sigma_1^2\sigma_2^2-\lambda^2\mu_L^2\mu_Y^2}$.
Furthermore, we suppose that

$$t_1 = \begin{cases} T, & A_1 \le (1+\kappa_1)\gamma_1 A_3, \\ T - \frac{1}{r}\ln\frac{A_1}{(1+\kappa_1)\gamma_1 A_3}, & (1+\kappa_1)\gamma_1 A_3 < A_1 < (1+\kappa_1)\gamma_1 A_3 e^{rT}, \\ 0, & A_1 \ge (1+\kappa_1)\gamma_1 A_3 e^{rT}, \end{cases} \tag{49}$$

$$t_2 = \begin{cases} T, & A_2 \le (1+\kappa_1)\gamma_1 A_3, \\ T - \frac{1}{r}\ln\frac{A_2}{(1+\kappa_1)\gamma_1 A_3}, & (1+\kappa_1)\gamma_1 A_3 < A_2 < (1+\kappa_1)\gamma_1 A_3 e^{rT}, \\ 0, & A_2 \ge (1+\kappa_1)\gamma_1 A_3 e^{rT}. \end{cases} \tag{50}$$

Next, we discuss the optimal reinsurance strategy and $Q_i(t)$ in different cases.

Case 1. If $\eta_1 < \frac{\lambda\mu_L\mu_Y}{(\lambda+\lambda_2)\mu_Y'}\frac{(\lambda+\lambda_2)\mu_Y}{(\lambda+\lambda_1)\mu_L}\eta_2$, we have $A_1 \le 0$ and $A_2 > 0$.

(1) When $0 \le t < t_2$, the equilibrium reinsurance strategy and $Q_i(t)$ are $(q_1^*(t), q_2^*(t)) = (0, \hat{q}_2(t))$, $Q_i^1(t) = \int_t^T H_i(0,\hat{q}_2(s),s)\,\mathrm{d}s$.

(2) Let $t_{21} = T - \frac{1}{r}\ln\frac{a_2\eta_2}{\gamma_1[(1+\kappa_1)\sigma_2^2 - \kappa_1\lambda\mu_L\mu_Y - \kappa_1\sigma_2^2]}$, then for $t_2 \le t < t_{21}$, the optimal reinsurance strategies and $Q_i(t)$ are given by

$$q_1^*(t) = 0,$$

$$\begin{aligned} q_2^{1*}(t) &= \frac{a_2\eta_2 + \kappa_1\lambda\mu_L\mu_Y\gamma_1 e^{r(T-t)} + \kappa_1\sigma_2^2\gamma_1 e^{r(T-t)}}{\gamma_1 e^{r(T-t)}(1+\kappa_1)\sigma_2^2}, \\ Q_i^2(t) &= \int_t^T H_i\big(0, q_2^{1*}(s), s\big)\,\mathrm{d}s. \end{aligned} \tag{51}$$

(3) When $t_{21} \leq t < T$, the equilibrium reinsurance strategy and $Q_i(t)$ are $(q_1^*(t), q_2^*(t)) = (0,1)$, $Q_i^3(t) = \int_t^T H_i(0,1,s)\,\mathrm{d}s$.

Case 2. If $\frac{\lambda\mu_L\mu_Y}{(\lambda+\lambda_2)\mu_Y'}\frac{(\lambda+\lambda_2)\mu_Y}{(\lambda+\lambda_1)\mu_L}\eta_2 < \eta_1 < \frac{(\lambda+\lambda_1)\mu_L'}{\lambda\mu_L\mu_Y}\frac{(\lambda+\lambda_2)\mu_Y}{(\lambda+\lambda_1)\mu_L}\eta_2$, we have $A_1 > 0$ and $A_2 > 0$, and we can note that $A_1, A_2, A_3 > 0$, thus $\hat{q}_i(t) > 0$.

Case 2.1. $A_1 \leq A_2$, then $t_1 \geq t_2$.

(1) When $0 \leq t < t_2$, the optimal reinsurance proportion are given by (38), therefore we can derive

$$Q_i^4(t) = \int_t^T H_i(\hat{q}_1(s), \hat{q}_2(s), s)\,\mathrm{d}s.$$

(2) When $t \geq t_2$, we have $q_2^*(t) = 1$, according to (34) and (36), in order to solve the equation (34), we need to solve $\inf_{q_1,q_2}\{H_1(q_1,q_2,t)\}$. Then, substituting $q_2^*(t) = 1$ into $\inf_{q_1,q_2}\{H_1(q_1,q_2,t)\}$ and by the first-order conditions we get

$$q_1^{1*}(t) = \frac{a_1\eta_1 - \lambda\mu_L\mu_Y\gamma_1 e^{r(T-t)} + \kappa_1\sigma_1^2\gamma_1 e^{r(T-t)}}{\gamma_1 e^{r(T-t)}(1+\kappa_1)\sigma_1^2}. \tag{52}$$

Let $t_{11} = T - \frac{1}{r}\ln\frac{a_1\eta_1}{\gamma_1[(1+\kappa_1)\sigma_1^2 + \lambda\mu_L\mu_Y - \kappa_1\sigma_1^2]}$, then for $t_2 \leq t < t_{11}$, the optimal reinsurance strategies and $Q_i(t)$ are given by

$$\begin{aligned} q_1^{1*}(t) &= \frac{a_1\eta_1 - \lambda\mu_L\mu_Y\gamma_1 e^{r(T-t)} + \kappa_1\sigma_1^2\gamma_1 e^{r(T-t)}}{\gamma_1 e^{r(T-t)}(1+\kappa_1)\sigma_1^2}, \\ q_2^*(t) &= 1, \\ Q_i^5(t) &= \int_t^T H_i\big(q_1^{1*}(s), 1, s\big)\,\mathrm{d}s. \end{aligned} \tag{53}$$

(3) When $t_1' \leq t < T$, the equilibrium reinsurance strategy and $Q_i(t)$ are $(q_1^*(t), q_2^*(t)) = (1,1)$ and $Q_i^6(t) = \int_t^T H_i(1,1,s)\,\mathrm{d}s$.

Case 2.2. $A_1 > A_2$, then $t_1 < t_2$.

(1) When $0 \leq t < t_1$, the optimal reinsurance proportion are given by (38), and $Q_i(t)$ is the same as $Q_i^4(t)$.

(2) When $t \geq t_1$, we have $q_1^*(t) = 1$. Substituting $q_1^*(t) = 1$ into $\inf_{q_1,q_2}\{H_1(q_1,q_2,t)\}$ and by the first-order conditions we get

$$q_2^{2*}(t) = \frac{a_2\eta_2 - \lambda\mu_L\mu_Y\gamma_1 e^{r(T-t)} + \kappa_1\sigma_2^2\gamma_1 e^{r(T-t)}}{\gamma_1 e^{r(T-t)}(1+\kappa_1)\sigma_2^2}. \tag{54}$$

Let $t_{22} = T - \frac{1}{r}\ln\frac{a_2\eta_2}{\gamma_1[(1+\kappa_1)\sigma_2^2 + \lambda\mu_L\mu_Y - \kappa_1\sigma_2^2]}$, then for $t_1 \leq t < t_{22}$, the optimal reinsurance strategies and $Q_i(t)$ are given by

$$\begin{aligned} q_1^*(t) &= 1, \\ q_2^{2*}(t) &= \frac{a_2\eta_2 - \lambda\mu_L\mu_Y\gamma_1 e^{r(T-t)} + \kappa_1\sigma_2^2\gamma_1 e^{r(T-t)}}{\gamma_1 e^{r(T-t)}(1+\kappa_1)\sigma_2^2}, \\ Q_i^7(t) &= \int_t^T H_i\big(1, q_2^{1*}(s), s\big)\,\mathrm{d}s. \end{aligned} \tag{55}$$

(3) For $t_{22} \le t < T$, it is easy to see that $q_1^*(t) = 1, q_2^*(t) = 1$, and $Q_i(t)$ is the same as $Q_i^6(t)$.

Case 3. If $\eta_1 \ge \frac{(\lambda+\lambda_1)\mu_L'}{\lambda\mu_L\mu_Y}\frac{(\lambda+\lambda_2)\mu_Y}{(\lambda+\lambda_1)\mu_L}\eta_2$, we have $A_1 > 0$ and $A_2 \le 0$.

(1) When $0 \le t < t_1$, the equilibrium reinsurance strategy and $Q_i(t)$ are $(q_1^*(t), q_2^*(t)) = (\hat{q}_1(t), 0)$, $Q_i^8(t) = \int_t^T H_i(0, \hat{q}_2(s), s)\,\mathrm{d}s$.

(2) Let $t_{12} = T - \frac{1}{r}\ln\frac{a_1\eta_1}{\gamma_1[(1+\kappa_1)\sigma_1^2 - \kappa_1\lambda\mu_L\mu_Y - \kappa_1\sigma_1^2]}$, then for $t_1 \le t < t_{12}$, the optimal reinsurance strategies and $Q_i(t)$ are given by

$$
\begin{aligned}
q_1^{2*}(t) &= \frac{a_1\eta_1 + \kappa_1\lambda\mu_L\mu_Y\gamma_1 e^{r(T-t)} + \kappa_1\sigma_1^2\gamma_1 e^{r(T-t)}}{\gamma_1 e^{r(T-t)}(1+\kappa_1)\sigma_1^2}, \\
q_2^*(t) &= 0, \\
Q_i^9(t) &= \int_t^T H_i\big(q_1^{2*}(s), 0, s\big)\,\mathrm{d}s.
\end{aligned} \tag{56}
$$

(3) When $t_{12} \le t < T$, the equilibrium reinsurance strategy and $Q_i(t)$ are $(q_1^*(t), q_2^*(t)) = (1, 0)$, $Q_i^{10}(t) = \int_t^T H_i(1, 0, s)\,\mathrm{d}s$.

Remark 7. If $A_1 < 0$ and $A_2 < 0$, then the equilibrium reinsurance strategy and $Q_i(t)$ are $(q_1^*(t), q_2^*(t)) = (0, 0)$, $Q_i^{11}(t) = \int_t^T H_i(0, 0, s)\,\mathrm{d}s$.

According to the above results, we can obtain the following theorem.

Theorem 8. *For the non-zero-sum game Problem 3 with the exponential utility function, the candidate equilibrium investment strategies of the insurer and the reinsurer are given by*

$$
\begin{aligned}
\pi_1^*(t) &= \frac{\big(\alpha - r - 2\sigma^2\beta P_1(t)\big)(\gamma_2 + \gamma_1\kappa_1) + (\kappa_1\sigma_{21} + \sigma_{11})\sigma s^{2\beta}\gamma_1\gamma_2\big(1 - e^{r(T-t)}\big)}{(1-\kappa_1\kappa_2)\sigma s^{2\beta}\gamma_1\gamma_2 e^{r(T-t)}} + \frac{\sigma_{11}}{\sigma}, \\
\pi_2^*(t) &= \frac{\big(\alpha - r - 2\sigma^2\beta P_2(t)\big)(\gamma_1 + \gamma_2\kappa_2) + (\kappa_2\sigma_{11} + \sigma_{21})\sigma s^{2\beta}\gamma_1\gamma_2\big(1 - e^{r(T-t)}\big)}{(1-\kappa_1\kappa_2)\sigma s^{2\beta}\gamma_1\gamma_2 e^{r(T-t)}} + \frac{\sigma_{21}}{\sigma},
\end{aligned}
$$

where $P_1(t) = P_2(t) = \frac{(\alpha-r)^2[e^{-(2r\beta-\sigma^2\beta(2\beta+1))(T-t)} - 1]}{4r\beta\sigma^2 - 2\sigma^4\beta(2\beta+1)}$.

The equilibrium reinsurance strategies and the corresponding equilibrium value functions are given under different cases:

Case 1: $A_1 \le 0, A_2 > 0$**.**

The optimal reinsurance strategies are

$$
(q_1^*(t), q_2^*(t)) = \begin{cases} (0, \hat{q}_2(t)), & 0 \le t < t_2, \\ \big(0, q_2^{1*}(t)\big), & t_2 \le t < t_{21}, \\ (0, 1), & t_{21} \le t < T. \end{cases}
$$

The equilibrium value functions $J_1(t, \hat{y}_1, l_1, s)$ and $J_2(t, \hat{y}_2, l_2, s)$ of the insurer and the reinsurer are given by

$$
J_1(t, \hat{y}_1, l_1, s) = \begin{cases} -\frac{1}{\gamma_1}\exp\big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1 + l_1 + F_1(t)l_1 + G_1(t)] + P_1(t)s^{-2\beta} + Q_1^1(t)\big\}, & 0 \le t < t_2, \\ -\frac{1}{\gamma_1}\exp\big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1 + l_1 + F_1(t)l_1 + G_1(t)] + P_1(t)s^{-2\beta} + Q_1^2(t)\big\}, & t_2 \le t < t_{21}, \\ -\frac{1}{\gamma_1}\exp\big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1 + l_1 + F_1(t)l_1 + G_1(t)] + P_1(t)s^{-2\beta} + Q_1^3(t)\big\}, & t_{21} \le t < T, \end{cases}
$$

and

$$
J_2(t, \hat{y}_2, l_2, s) = \begin{cases} -\frac{1}{\gamma_2}\exp\big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2 + l_2 + F_2(t)l_2 + G_2(t)] + P_2(t)s^{-2\beta} + Q_2^1(t)\big\}, & 0 \le t < t_2, \\ -\frac{1}{\gamma_2}\exp\big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2 + l_2 + F_2(t)l_2 + G_2(t)] + P_2(t)s^{-2\beta} + Q_2^2(t)\big\}, & t_2 \le t < t_{21}, \\ -\frac{1}{\gamma_2}\exp\big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2 + l_2 + F_2(t)l_2 + G_2(t)] + P_2(t)s^{-2\beta} + Q_2^3(t)\big\}, & t_{21} \le t < T. \end{cases}
$$

Case 2: $A_2 > A_1 > 0$**.**

The optimal reinsurance strategies are

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (\hat{q}_1(t), \hat{q}_2(t)), & 0 \le t < t_2, \\ \big(q_1^{1*}(t), 0\big), & t_2 \le t < t_{11}, \\ (1, 1), & t_{11} \le t < T. \end{cases}$$

The equilibrium value functions $J_1(t, \hat{y}_1, l_1, s)$ and $J_2(t, \hat{y}_2, l_2, s)$ of the insurer and the reinsurer are given by

$$J_1(t, \hat{y}_1, l_1, s) = \begin{cases} -\frac{1}{\gamma_1}\exp\big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1 + l_1 + F_1(t)l_1 + G_1(t)] + P_1(t)s^{-2\beta} + Q_1^4(t)\big\}, & 0 \le t < t_2, \\ -\frac{1}{\gamma_1}\exp\big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1 + l_1 + F_1(t)l_1 + G_1(t)] + P_1(t)s^{-2\beta} + Q_1^5(t)\big\}, & t_2 \le t < t_{11}, \\ -\frac{1}{\gamma_1}\exp\big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1 + l_1 + F_1(t)l_1 + G_1(t)] + P_1(t)s^{-2\beta} + Q_1^6(t)\big\}, & \mathrm{d}t_{11} \le t < T, \end{cases}$$

and

$$J_2(t, \hat{y}_2, l_2, s) = \begin{cases} -\frac{1}{\gamma_2}\exp\big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2 + l_2 + F_2(t)l_2 + G_2(t)] + P_2(t)s^{-2\beta} + Q_2^4(t)\big\}, & 0 \le t < t_2, \\ -\frac{1}{\gamma_2}\exp\big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2 + l_2 + F_2(t)l_2 + G_2(t)] + P_2(t)s^{-2\beta} + Q_2^5(t)\big\}, & t_2 \le t < t_{11}, \\ -\frac{1}{\gamma_2}\exp\big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2 + l_2 + F_2(t)l_2 + G_2(t)] + P_2(t)s^{-2\beta} + Q_2^6(t)\big\}, & t_{11} \le t < T. \end{cases}$$

Case 3: $A_1 > A_2 > 0$**.**

The optimal reinsurance strategies are

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (\hat{q}_1(t), \hat{q}_2(t)), & 0 \le t < t_1, \\ \big(1, q_2^{2*}(t)\big), & t_1 \le t < t_{22}, \\ (1, 1), & t_{22} \le t < T. \end{cases}$$

The equilibrium value functions $J_1(t, \hat{y}_1, l_1, s)$ and $J_2(t, \hat{y}_2, l_2, s)$ of the insurer and the reinsurer are given by

$$J_1(t, \hat{y}_1, l_1, s) = \begin{cases} -\frac{1}{\gamma_1}\exp\big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1 + l_1 + F_1(t)l_1 + G_1(t)] + P_1(t)s^{-2\beta} + Q_1^4(t)\big\}, & 0 \le t < t_1, \\ -\frac{1}{\gamma_1}\exp\big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1 + l_1 + F_1(t)l_1 + G_1(t)] + P_1(t)s^{-2\beta} + Q_1^7(t)\big\}, & t_1 \le t < t_{22}, \\ -\frac{1}{\gamma_1}\exp\big\{-\gamma_1 e^{r(T-t)}[\hat{y}_1 + l_1 + F_1(t)l_1 + G_1(t)] + P_1(t)s^{-2\beta} + Q_1^6(t)\big\}, & t_{22} \le t < T, \end{cases}$$

and

$$J_2(t, \hat{y}_2, l_2, s) = \begin{cases} -\frac{1}{\gamma_2}\exp\big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2 + l_2 + F_2(t)l_2 + G_2(t)] + P_2(t)s^{-2\beta} + Q_2^4(t)\big\}, & 0 \le t < t_1, \\ -\frac{1}{\gamma_2}\exp\big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2 + l_2 + F_2(t)l_2 + G_2(t)] + P_2(t)s^{-2\beta} + Q_2^7(t)\big\}, & t_1 \le t < t_{22}, \\ -\frac{1}{\gamma_2}\exp\big\{-\gamma_2 e^{r(T-t)}[\hat{y}_2 + l_2 + F_2(t)l_2 + G_2(t)] + P_2(t)s^{-2\beta} + Q_2^6(t)\big\}, & t_{22} \le t < T. \end{cases}$$

Case 4: $A_1 > 0, A_2 \le 0$**.**

The optimal reinsurance strategies are

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (\hat{q}_1(t), 0), & 0 \le t < t_1, \\ \big(q_1^{2*}(t), 0\big), & t_1 \le t < t_{12}, \\ (1, 0), & t_{12} \le t < T. \end{cases}$$

The equilibrium value functions $J_1(t,\hat{y}_1,l_1,s)$ and $J_2(t,\hat{y}_2,l_2,s)$ of the insurer and the reinsurer are given by

$$J_1(t,\hat{y}_1,l_1,s)=\begin{cases}-\frac{1}{\gamma_1}\exp\big\{-\gamma_1e^{r(T-t)}[\hat{y}_1+l_1+F_1(t)l_1+G_1(t)]+P_1(t)s^{-2\beta}+Q_1^8(t)\big\}, & 0\le t<t_1,\\ -\frac{1}{\gamma_1}\exp\big\{-\gamma_1e^{r(T-t)}[\hat{y}_1+l_1+F_1(t)l_1+G_1(t)]+P_1(t)s^{-2\beta}+Q_1^9(t)\big\}, & t_1\le t<t_{12},\\ -\frac{1}{\gamma_1}\exp\big\{-\gamma_1e^{r(T-t)}[\hat{y}_1+l_1+F_1(t)l_1+G_1(t)]+P_1(t)s^{-2\beta}+Q_1^{10}(t)\big\}, & t_{12}\le t<T,\end{cases}$$

and

$$J_2(t,\hat{y}_2,l_2,s)=\begin{cases}-\frac{1}{\gamma_2}\exp\big\{-\gamma_2e^{r(T-t)}[\hat{y}_2+l_2+F_2(t)l_2+G_2(t)]+P_2(t)s^{-2\beta}+Q_2^8(t)\big\}, & 0\le t<t_1,\\ -\frac{1}{\gamma_2}\exp\big\{-\gamma_2e^{r(T-t)}[\hat{y}_2+l_2+F_2(t)l_2+G_2(t)]+P_2(t)s^{-2\beta}+Q_2^9(t)\big\}, & t_1\le t<t_{12},\\ -\frac{1}{\gamma_2}\exp\big\{-\gamma_2e^{r(T-t)}[\hat{y}_2+l_2+F_2(t)l_2+G_2(t)]+P_2(t)s^{-2\beta}+Q_2^{10}(t)\big\}, & t_{12}\le t<T.\end{cases}$$

Case 5: $A_1\le 0, A_2\le 0$**.**

The optimal reinsurance strategy is $(q_1^*(t),q_2^*(t))=(0,0)$, and the value functions $J_1(t,\hat{y}_1,l_1,s)$ and $J_2(t,\hat{y}_2,l_2,s)$ of the insurer and the reinsurer are

$$J_1(t,\hat{y}_1,l_1,s)=-\frac{1}{\gamma_1}\exp\Big\{-\gamma_1e^{r(T-t)}[\hat{y}_1+l_1+F_1(t)l_1+G_1(t)]+P_1(t)s^{-2\beta}+Q_1^{11}(t)\Big\},$$

and

$$J_2(t,\hat{y}_2,l_2,s)=-\frac{1}{\gamma_2}\exp\Big\{-\gamma_2e^{r(T-t)}[\hat{y}_2+l_2+F_2(t)l_2+G_2(t)]+P_2(t)s^{-2\beta}+Q_2^{11}(t)\Big\},$$

where

$$\begin{cases}f_1(t,l_1)=F_1(t)l_1+G_1(t), \quad g_1(t,s)=P_1(t)s^{-2\beta}+Q_1(t),\\ f_2(t,l_2)=F_2(t)l_1+G_2(t), \quad g_2(t,s)=P_2(t)s^{-2\beta}+Q_2(t),\\ F_1(t)=F_2(t)=-e^{-r(T-t)},\\ G_1(t)=\Big\{\Big[\frac{(\alpha-r)\sigma_{11}}{\sigma}-\alpha_1-\frac{\sigma_{12}^2\gamma_1}{2}\Big](T-t)+\frac{\kappa_1^2\sigma_{22}^2\gamma_1}{4r}\big(1-e^{2r(T-t)}\big)\\ \qquad -\frac{1}{r}\Big[\kappa_1\sigma_{12}\sigma_{22}\gamma_1-r\kappa_1l_2-\frac{(\alpha-r)\kappa_1\sigma_{21}}{\sigma}+\kappa_1\sigma_{22}\Big]\big(1-e^{r(T-t)}\big)\Big\}e^{-r(T-t)},\\ G_2(t)=\Big\{\Big[\frac{(\alpha-r)\sigma_{21}}{\sigma}-\alpha_2-\frac{\sigma_{22}^2\gamma_2}{2}\Big](T-t)+\frac{\kappa_2^2\sigma_{12}^2\gamma_2}{4r}\big(1-e^{2r(T-t)}\big)\\ \qquad -\frac{1}{r}\Big[\kappa_2\sigma_{22}\sigma_{12}\gamma_2-r\kappa_2l_1-\frac{(\alpha-r)\kappa_2\sigma_{11}}{\sigma}+\kappa_2\sigma_{12}\Big]\big(1-e^{r(T-t)}\big)\Big\}e^{-r(T-t)},\\ P_1(t)=P_2(t)=\frac{(\alpha-r)^2\left[e^{-(2r\beta-\sigma^2\beta(2\beta+1))(T-t)}-1\right]}{4r\beta\sigma^2-2\sigma^4\beta(2\beta+1)},\\ Q_1(t)=\int_t^T H_1(q_1^*(s),q_2^*(s),s)\,\mathrm{d}s,\\ Q_2(t)=\int_t^T H_2(q_1^*(s),q_2^*(s),s)\,\mathrm{d}s,\\ t_{11}=T-\frac{1}{r}\ln\frac{a_1\eta_1}{\gamma_1\left[(1+\kappa_1)\sigma_1^2+\lambda\mu_L\mu_Y-\kappa_1\sigma_1^2\right]},\\ t_{12}=T-\frac{1}{r}\ln\frac{a_1\eta_1}{\gamma_1\left[(1+\kappa_1)\sigma_1^2-\kappa_1\lambda\mu_L\mu_Y-\kappa_1\sigma_1^2\right]},\\ t_{21}=T-\frac{1}{r}\ln\frac{a_2\eta_2}{\gamma_1\left[(1+\kappa_1)\sigma_2^2-\kappa_1\lambda\mu_L\mu_Y-\kappa_1\sigma_2^2\right]},\\ t_{22}=T-\frac{1}{r}\ln\frac{a_2\eta_2}{\gamma_1\left[(1+\kappa_1)\sigma_2^2+\lambda\mu_L\mu_Y-\kappa_1\sigma_2^2\right]}.\end{cases}$$

Proposition 9. *The optimal reinsurance strategies $\hat{q}_1(t)$ and $\hat{q}_2(t)$ in (38) increase as κ_1 increases.*

Proof. We can derive the first-order condition with respect to κ_1 as follows:

$$\frac{\mathrm{d}\hat{q}_1}{\mathrm{d}\kappa_1}=-\frac{A_1}{(1+\kappa_1)^2\gamma_1e^{r(T-t)}A_3}+\frac{1}{(1+\kappa_1)^2},$$

$$\frac{\mathrm{d}\hat{q}_2}{\mathrm{d}\kappa_1} = -\frac{A_2}{(1+\kappa_1)^2\gamma_1 e^{r(T-t)}A_3} + \frac{1}{(1+\kappa_1)^2}.$$

It's clear that they are positive, so the optimal reinsurance strategies are increasing with κ_1 □

From Proposition 9, we know that the equilibrium retained proportion $\hat{q}_1(t)$ and $\hat{q}_2(t)$ are increasing with the growth of κ_1. It is intuitive that a larger κ_1 means that the insurer becomes more competitive and more concerned with relative wealth towards the reinsurer. That is to say competition makes the insurer more risk-seeking. An explanation may be that may be that the insurer with greater competition sensitivity parameter κ_1 is more concerned about its expected terminal surplus relative to that of the reinsurer, and hence the insurer tends to pay less reinsurance premium through retaining more risks, and then more capital could be accumulated at terminal time.

Remark 10. From Theorem 8, we have the following conclusions:

(i) The optimal investment strategy of the insurer can be divided into two parts, one for hedging the market risk in the investment such as $\frac{(\alpha-r-2\sigma^2\beta P_1(t))(\gamma_2+\gamma_1\kappa_1)}{(1-\kappa_1\kappa_2)\sigma s^{2\beta}\gamma_1\gamma_2 e^{r(T-t)}}$, and the other for hedging the debt risk in the investment such as $\frac{-(\kappa_1\sigma_{21}+\sigma_{11})\sigma s^{2\beta}\gamma_1\gamma_2(1-e^{r(T-t)})}{(1-\kappa_1\kappa_2)\sigma s^{2\beta}\gamma_1\gamma_2 e^{r(T-t)}} + \frac{\sigma_{11}}{\sigma}$. So, the debt has a significant impact on investment decisions. The reinsurer has similar results.

(ii) The equilibrium investment-reinsurance strategies of insurers and reinsurers are related to the sensitivity coefficient $\kappa_i, i \in \{1,2\}$, which is the hedge components coming from the relative concerns, and this means the insurer would imitate the reinsurer's investment strategy. Specifically, if the insurer does not consider the wealth gap between itself and the reinsurer, and similarly, the reinsurer's amount invested in the risky asset is negatively related to the risk-averse coefficient. Besides, from Proposition 9, it shows that the optimal reinsurance strategy is quadratic in relation to κ_1.

(iii) When debt and game are not considered and let $\beta = 0$, the optimal investment strategy of the insurer and the reinsurer are as follows:

$$\pi_1^*(t) = \frac{\alpha - r}{\sigma^2\gamma_1 e^{r(T-t)}},$$
$$\pi_2^*(t) = \frac{\alpha - r}{\sigma^2\gamma_2 e^{r(T-t)}}.$$

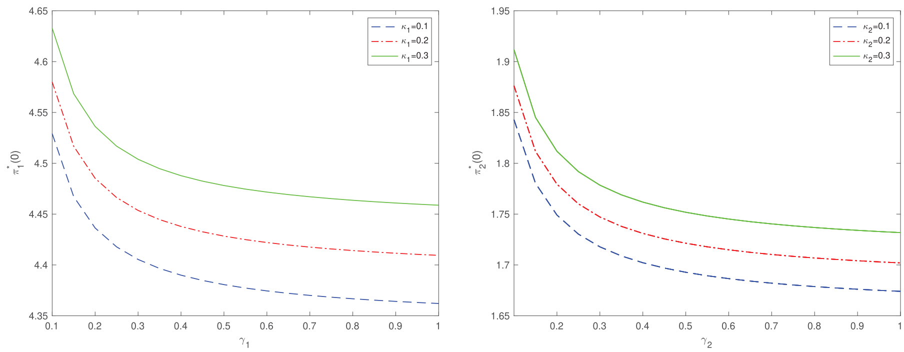
This is consistent with the real economic phenomenon, because the insurer makes its investment strategy only based on its own situation without considering the wealth gap. Meanwhile, the reinsurer's investment strategy only contains the standard part because it only considers its own wealth. This is exactly the optimal investment strategy for CARA utility in Merton [21].

4. Sensitivity analysis

In this section, we provide some numerical examples focusing on the impact of the liability parameter and the relative net wealth concern coefficient on the equilibrium investment and reinsurance strategy. Since we use numerical methods to determine the optimal investment and reinsurance decisions, the parameter values need to be specified. According to the model setting in Section 2, we assume that the claim size random variables L_i and Y_i are exponentially distributed with parameters v_1 and v_2, respectively, then $\mu_L = 1/v_1$, $\mu_Y = 1/v_2$, $\mu'_L = 2/v_1^2$ and $\mu'_Y = 2/v_2^2$. Throughout numerical analysis, unless otherwise stated, the basic parameters are given as Table 1.

TABLE 1. Model parameters.

Financial market								
t	T	r	α	β	σ	s	α_1	α_2
0	10	0.5	0.08	1	1	1	0.07	0.04
σ_{11}	σ_{12}	σ_{21}	σ_{22}	κ_1	κ_2	γ_1	γ_2	
3	0.14	1	0.15	0.3	0.2	0.3	0.1	
Insurance market								
v_1	v_2	a_1	a_2	σ_1^2	σ_1^2	μ_L	μ_Y	μ_L'
4	5	1	1	0.5	0.4	0.25	0.2	0.125
μ_Y'	λ	λ_1	λ_2	θ_1	θ_2	η_1	η_2	
0.08	1	4	3	0.2	0.2	0.4	0.4	



(a) (b)

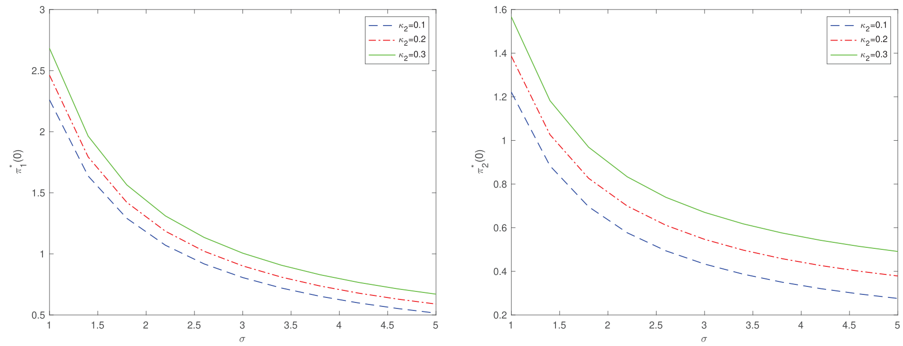
FIGURE 1. The effects of γ_i on $\pi_i^*(0)$.

4.1. Equilibrium investment strategy

We now investigate the sensitivity of the equilibrium strategies $\pi_1^*(0)$ and $\pi_2^*(0)$ with respect to the constant absolute risk aversion coefficient γ_i and the intensity of the relative concern κ_i by varying the parameters within reasonable intervals. From Figure 1, it can be observed that as γ_i increases, the value of $\pi_i^*(0)$ decreases. This is because a larger value of γ_i implies that the insurer and the reinsurer become more risk-aversive and thus choose to invest less in risky assets.

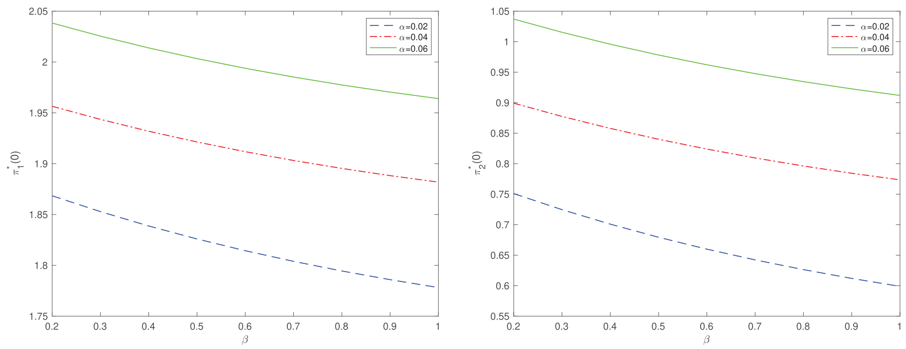
Figure 2 shows the effect of σ on the equilibrium investment strategy. It is easy to find that $\pi_i^*(0)$ decreases as the risky asset volatility σ increases. In fact, the larger the value of σ is, the larger the instantaneous volatility is, which means the more risk of investment, and thus the insurer and the reinsurer would invest less in risky assets.

As shown in Figure 3, we analyze the effect of the expected return of risky asset α and the elasticity parameter β on the equilibrium investment strategy $\pi_i^*(0)$. Note that $\pi_i^*(0)$ increases with α and decreases with β, a larger α implies an increase in the return of risky assets, insurers and reinsurers will obtain more returns from investment. Therefore, they would increase the amounts invested in risky assets. In addition, we can find that $\pi_i^*(0)$ has a negative relationship with β. This is consistent with intuition. As β increases, s^β will be greater, which leads to a larger expected decrease in volatility and an increased probability of a large adverse movement in the risky asset's price, which indicates that higher investment risk reduces the attractiveness of the risky asset. Thus, they will invest less in risky assets to reduce risks.



(a) (b)

FIGURE 2. The effects of σ on $\pi_i^*(0)$.



(a) (b)

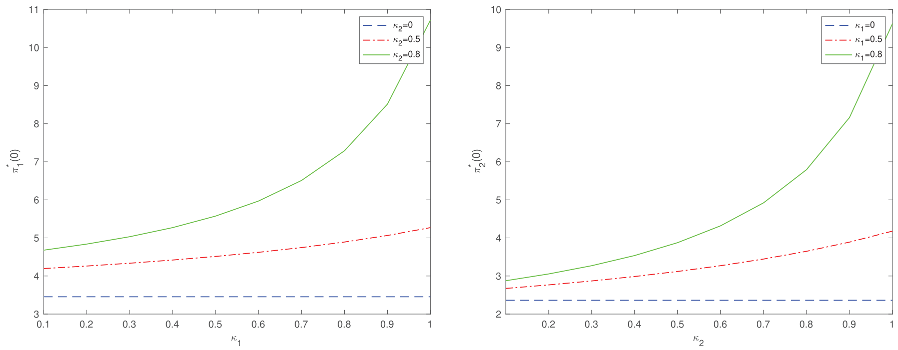
FIGURE 3. The effects of α and β on $\pi_i^*(0)$.

To illustrate the effect of competition, Figure 4 depicts the effects of relative performance parameters $\kappa_i, i \in \{1, 2\}$ on the optimal investment strategy $\pi_i^*(0)$. From the subfigures, we observe that $\pi_i^*(0)$ increases as κ_i increases, a larger κ_i produces a riskier investment strategy in response to the competition. That is, the more concerned about outperforming the competitor, the larger amount the insurer or the reinsurer would like to invest. In addition, from the Figure 4a for the same level of κ_1, we can find that $\pi_1^*(0)$ has a positive relationship with κ_2: a larger κ_2 implies that the insurer would face a higher competition intensity, and this also leads the insurer to hold more shares. The reinsurer yields similar results to the insurer. Figures 1 and 2 also capture the same effects of relative performance parameters on the optimal investment strategy.

Figure 5 provides graphical illustrations of the effect of the volatility of the uncontrollable exogenous liability on the equilibrium strategy $\pi_i^*(0)$. As can be seen in the Figure 5a, the value of parameter σ_{11} determines the magnitude of debt volatility of the insurer, holding all other parameters constant. A larger σ_{11} will cause an increase in debt. As a result, the insurer will tend to increase its investment in the risky asset to reduce the debt. In the same way, from the Figure 5b, we can observe that the reinsurer has a similar result.

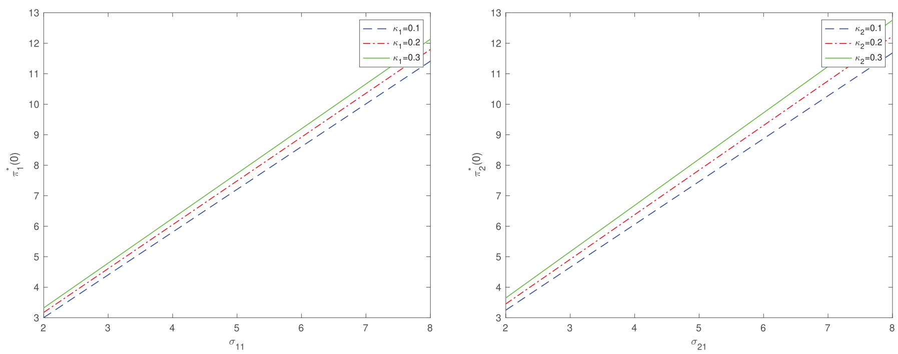
4.2. Equilibrium reinsurance strategy

According to the model parameters we have set above, we can derive that $A_1 < A_2$ and $A_2 < (1+\kappa_1)\gamma_1 A_3$ always hold; therefore, we let $t_2 = T$ and the equilibrium reinsurance strategy $(q_1^*(t), q_2^*(t)) = (\hat{q}_1{}^*(t), \hat{q}_2{}^*(t))$.



(a) (b)

FIGURE 4. The effects of κ_i on $\pi_i^*(0)$.



(a) (b)

FIGURE 5. The effects of σ_{i1} on $\pi_i^*(0)$.

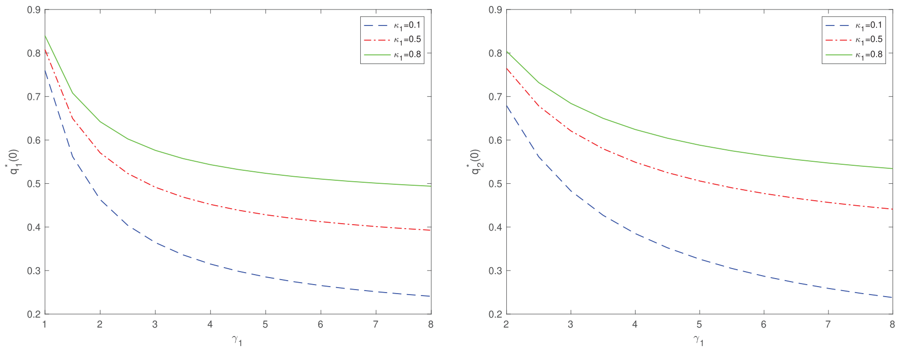
From Figure 6, we can observe that the equilibrium reinsurance strategy $q_i^*(t)$ is inversely proportional to the risk-aversion coefficient γ_1. A larger γ_1 causes the insurer to be more risk-averse; thus, the insurer will tend to decrease its reinsurance retention level to reduce the claim risk. The phenomenon is consistent with the actual situation and the mathematical conclusions.

The impact of the safety load η_i on the equilibrium reinsurance strategy $q_i^*(t)$ is described in Figure 7. As shown in the figure, an increase in the safety load η_i leads to an increase in the reinsurance retention level. This is because a larger value of η_i implies that the insurer has a higher capacity to take on risk and makes the insurer have more confidence to increase its retention ratio.

Figure 8 demonstrates how the equilibrium reinsurance strategy $q_i^*(t)$ varies with respect to σ_i. From these subfigures, we find that $q_i^*(0)$ is a decreasing function of σ_i when $\kappa_1 > 0$. Note that a larger σ_i indicates greater uncertainty and larger potential losses; thus, the company may decrease its retention ratio to diversify risks by purchasing more reinsurance contracts.

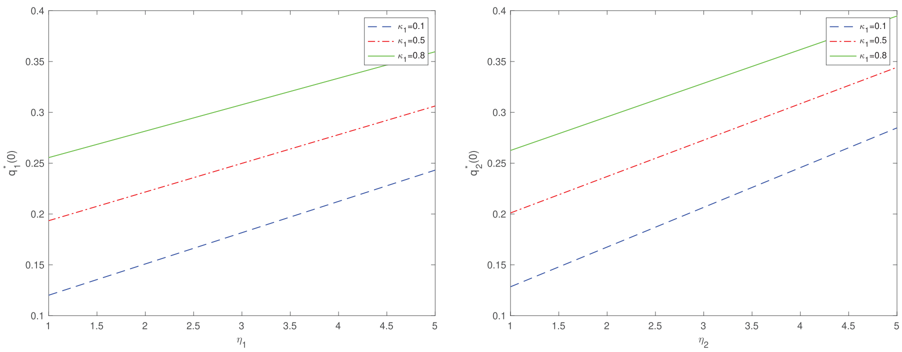
In addition, Figures 1–8 reveal the equilibrium reinsurance strategy as increasing functions of the relative performance parameter κ_1. A larger κ_1 means that the insurance companies are more sensitive to competition; thus the insurer intends to purchase fewer reinsurance contracts to win in a higher degree of competition. Therefore, the insurer may bear more risk due to more claims resulting from purchasing fewer reinsurance contracts, while obtaining a higher terminal wealth and thus a higher probability of beating its competitor.

Figure 9 illustrates the impacts of the parameters λ, λ_1 and λ_2 on the equilibrium reinsurance strategy $q_i^*(0)$, respectively. From Figure 9, we have the following findings. Firstly, the optimal reinsurance strategy decreases



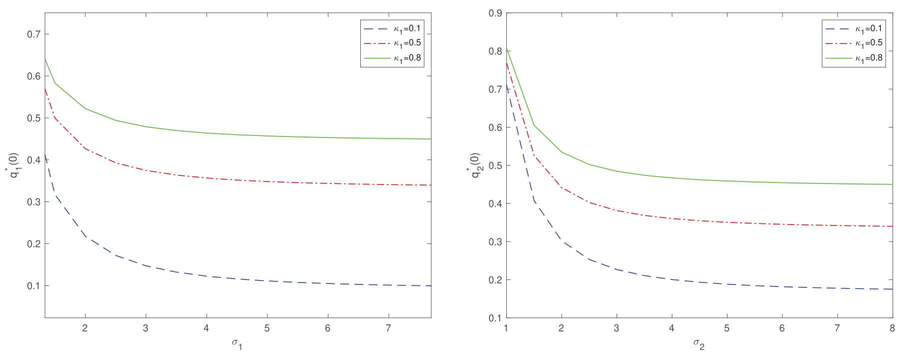
(a) (b)

FIGURE 6. The effects of γ_1 on $q_i^*(0)$.



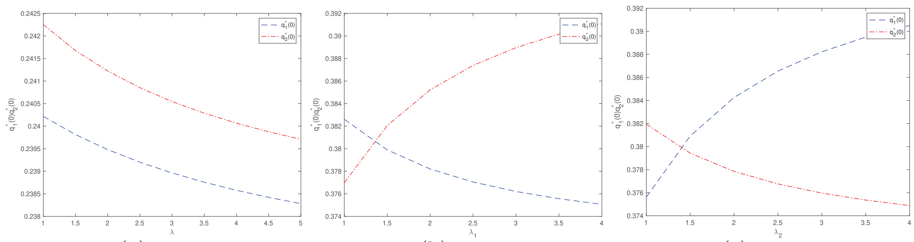
(a) (b)

FIGURE 7. The effects of η_i on $q_i^*(0)$.



(a) (b)

FIGURE 8. The effects of σ_i on $q_i^*(0)$.



(a) (b) (c)

FIGURE 9. The effects of λ, λ_1 and λ_2 on $q_i^*(0)$.

as the overall risk parameter λ increases, it is evident that a larger λ leads to greater average claim amounts for both lines of business; thus, the insurer tends to reduce the retention levels $q_1^*(0)$ and $q_2^*(0)$ to maintain the expected risk level. Besides, a larger λ means that the higher degree of dependence of the two insurance businesses, so the insurer may face greater potential risks, then the insurer will diversify risks by purchasing more reinsurance contracts. Secondly, from the Figure 9b, we can derive that $q_1^*(0)$ increases with λ_1, while $q_2^*(0)$ decreases. This is because higher average claim amounts for the first line lead the insurer to purchase more reinsurance for that line, while reducing reinsurance for the second line which has a relatively lower claim risk. Conversely, the Figure 9c indicates that as the risk parameter λ_2 for the second line increases, the optimal retention $q_1^*(0)$ for the first line increases, while $q_2^*(0)$ for the second line decreases. This reflects that the insurer will tend to decrease the retention of the second line while increasing that of the first line.

5. Conclusion

This paper considers Non-zero-sum stochastic differential games of investment and reinsurance for asset-liability management with common shock dependence under CEV model between insurers and reinsurers. The financial market consists of one risk-free asset and one risk asset whose price process follows the CEV model. We assume that the insurer can purchase proportional reinsurance from the reinsurer, and both companies can invest in the financial market. The reinsurance premium is calculated by the expected value principle. The goal of the insurer and the reinsurer is to maximize their CARA utility from its terminal wealth with relative performance concerns. We introduce a class of stochastic liability processes associated with the volatility of risky assets, on the basis of which the dynamic evolution of net wealth is obtained. By applying stochastic control theory and a game theoretic framework, we find the generalized HJB system under the classical Cramér-Lundberg insurance model, and derive the explicit expressions of the equilibrium investment strategy with the effect of liabilities and the optimal reinsurance strategy under the exponential utility. We find that the volatility of the uncontrollable exogenous liability distorts the equilibrium investment strategy of insurers, where a larger volatility leads the insurer to increase its investment in the risky asset to mitigate the impact of debt. Finally, we provide the numerical sensitivity analysis to illustrate the effects of model parameters on equilibrium investment strategy and reinsurance strategy.

Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

[1] S. Browne, Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. *Math. Oper. Res.* **20** (1995) 937–958.

[2] H. Chang, Dynamic mean–variance portfolio selection with liability and stochastic interest rate. *Econ. Modell.* **51** (2015) 172–182.

[3] P. Chen, H. Yang and G. Yin, Markowitz's mean–variance asset-liability management with regime switching: a continuous-time model. *Insurance Math. Econ.* **43** (2008) 456–465.

[4] X. Chen, F. Huang and X. Li, Robust asset-liability management under CRRA utility criterion with regime switching: a continuous-time model. *Stoch. Models* **38** (2022) 167–189.

[5] M.C. Chin and H.Y. Wong, Mean–variance asset-liability management with asset correlation risk and insurance liabilities. *Insurance Math. Econ.* **59** (2014) 300–310.

[6] M.C. Chiu and D. Li, Asset and liability management under a continuous-time mean–variance optimization framework. *Insurance Math. Econ.* **39** (2006) 330–355.

[7] L. Colombo and P. Labrecciosa, Consumer surplus-enhancing cooperation in a natural resource oligopoly. *J. Environ. Econ. Manage.* **92** (2018) 185–193.

[8] J.C. Cox and S.A. Ross, The valuation of options for alternative stochastic processes. *J. Finan. Econ.* **3** (1976) 145–166.

[9] R.M. Cyert and M.H. Degroot, An analysis of cooperation and learning in a duopoly context. *Am. Econ. Rev.* **63** (1973) 24–37.

[10] X. Dong, X. Rong and H. Zhao, Non-zero-sum reinsurance and investment game with correlation between insurance market and financial market under CEV model. *J. Ind. Manage. Optim.* **19** 2023.

[11] G.-E. Espinosa and N. Touzi, Optimal investment under relative performance concerns. *Math. Finan.* **25** (2015) 221–257.

[12] W.H. Fleming and H.M. Soner, Controlled Markov Processes and Viscosity Solutions. Springer (2006).

[13] M. Gu, Y. Yang, S. Li and J. Zhang, Constant elasticity of variance model for proportional reinsurance and investment strategies. *Insurance Math. Econ.* **46** (2010) 580–587.

[14] X. Han, Z. Liang and K.C. Yuen, Minimizing the probability of absolute ruin under the mean–variance premium principle. *Optim. Control App. Methods* **42** (2021) 786–806.

[15] F. Hu and R. Wang, Optimal investment-consumption strategy with liability and regime switching model under value-at-risk constraint. *Appl. Math. Comput.* **313** (2017) 103–118.

[16] R. Josa-Fombellida and J.P. Rincón-Zapatero, Optimal investment decisions with a liability: the case of defined benefit pension plans. *Insurance Math. Econ.* **39** (2006) 81–98.

[17] D. Li, Y. Zeng and H. Yang, Robust optimal excess-of-loss reinsurance and investment strategy for an insurer in a model with jumps. *Scand. Actuarial J.* **2018** (2018) 145–171.

[18] Z. Liang and K.C. Yuen, Optimal dynamic reinsurance with dependent risks: variance premium principle. *Scand. Actuarial J.* **2016** (2016) 18–36.

[19] X. Luo and Q. Zhou, Stochastic differential games on robust optimal asset-liability management with delay under the CEV model. *J. Ind. Manage. Optim.* **21** (2025) 1771–1796.

[20] S. Me, Continuous-time mean–variance portfolio selection with liability and regime switching. *Insurance Math. Econ.* **45** (2009) 148–155.

[21] R.C. Merton, Optimum consumption and portfolio rules in a continuous-time model. *J. Econ. Theory* **3** (1971) 373–413.

[22] J. Pan and Q. Xiao, Optimal asset-liability management with liquidity constraints and stochastic interest rates in the expected utility framework. *J. Comput. Appl. Math.* **317** (2017) 371–387.

[23] J. Pan and Q. Xiao, Optimal mean–variance asset-liability management with stochastic interest rates and inflation risks. *Math. Methods Oper. Res.* **85** (2017) 491–519.

[24] J. Pan, S. Hu and X. Zhou, Optimal investment strategy for asset-liability management under the Heston model. *Optimization* **68** (2019) 895–920.

[25] X. Peng and F. Chen, Mean–variance asset-liability management with inside information. *Commun. Stat.-Theory Methods* **51** (2022) 2281–2302.

[26] S.D. Promislow and V.R. Young, Minimizing the probability of ruin when claims follow Brownian motion with drift. *North Am. Actuarial J.* **9** (2005) 110–128.

[27] C.S. Pun, Robust time-inconsistent stochastic control problems. *Automatica* **94** (2018) 249–257.

[28] H. Schmidli, On optimal investment and subexponential claims. *Insurance Math. Econ.* **36** (2005) 25–35.

[29] G. Shaidolda and K. Uğurlu, Lower partial moments for skew elliptical distributions. *J. Ind. Manage. Optim.* **21** (2025) 4503–4535.

[30] W.F. Sharpe and L.G. Tint, Liabilities-a new approach. *J. Portfolio Manage.* **16** (1990) 5–10.
[31] K. Uğurlu, Dynamic optimal contract under parameter uncertainty with risk-averse agent and principal. *Turkish J. Math.* **42** (2018) 977–992.
[32] K. Uğurlu, Terminal wealth maximization under drift uncertainty. *Optimization* **74** (2025) 1743–1761.
[33] J. Wei and T. Wang, Time-consistent mean–variance asset-liability management with random coefficients. *Insurance Math. Econ.* **77** (2017) 84–96.
[34] S. Xie, Z. Li and S. Wang, Continuous-time portfolio selection with liability: mean–variance model and stochastic LQ approach. *Insurance Math. Econ.* **42** (2008) 943–953.
[35] Y. Yuan and H. Mi, Robust optimal asset-liability management with penalization on ambiguity. *J. Ind. Manage. Optim.* **18** (2022).
[36] K.C. Yuen, Z. Liang and M. Zhou, Optimal proportional reinsurance with common shock dependence. *Insurance Math. Econ.* **64** (2015) 1–13.
[37] C. Yusong and Z. Xianquan, Optimal proportional reinsurance and investment with minimum probability of ruin. *Appl. Math. Comput.* **218** (2012) 5433–5438.
[38] Y. Zeng and Z. Li, Asset-liability management under benchmark and mean–variance criteria in a jump diffusion market. *J. Syst. Sci. Complexity* **24** (2011) 317–327.
[39] X. Zhang and T.K. Siu, On optimal proportional reinsurance and investment in a Markovian regime-switching economy. *Acta Math. Sin. English Ser.* **28** (2012) 67–82.
[40] Y. Zhang, Y. Wu, S. Li and B. Wiwatanapataphee, Mean–variance asset liability management with state-dependent risk aversion. *North Am. Actuarial J.* **21** (2017) 87–106.
[41] J. Zhang, P. Chen, Z. Jin and S. Li, Open-loop equilibrium strategy for mean–variance asset-liability management portfolio selection problem with debt ratio. *J. Comput. Appl. Math.* **380** (2020) 112951.
[42] Y. Zhang, P. Zhao and B. Kou, Optimal excess-of-loss reinsurance and investment problem with thinning dependent risks under Heston model. *J. Comput. Appl. Math.* **382** (2021) 113082.
[43] H. Zhu, M. Cao and Y. Zhu, Non-zero-sum reinsurance and investment game between two mean–variance insurers under the CEV model. *Optimization* **70** (2021) 2579–2606.

Appendix A. Proof of Lemma 6

Using Cauchy–Schwarz inequality, we have

$$(\lambda_1+\lambda)E\left[L_i^2\right](\lambda_2+\lambda)E\left[Y_i^2\right] \geq \left(\sqrt{(\lambda_1+\lambda)(\lambda_2+\lambda)}E[L_i]E[Y_i]\right)^2 > (\lambda E[L_i]E[Y_i])^2.$$

By $E[L_i^2] > (E[L_i])^2, E[Y_i^2] > (E[Y_i])^2$ and

$$a_1 = (\lambda_1+\lambda)E[L_i], \sigma_1^2 = (\lambda_1+\lambda)E\left[L_i^2\right]$$
$$a_2 = (\lambda_2+\lambda)E[Y_i], \sigma_2^2 = (\lambda_2+\lambda)E\left[Y_i^2\right]$$

we derive

$$\sigma_1^2\sigma_2^2 > \lambda^2\mu_L^2\mu_Y^2.$$

Moreover, we also have

$$\begin{aligned}\frac{\lambda\mu_L\mu_Y}{\sigma_2^2}\frac{a_2}{a_1} &= \frac{\lambda\mu_L\mu_Y}{(\lambda_2+\lambda)\mu_Y'}\frac{(\lambda_2+\lambda)\mu_Y}{(\lambda_1+\lambda)\mu_L} = \frac{\lambda\mu_Y^2}{(\lambda_1+\lambda)\mu_Y'}\\ \frac{\sigma_1^2}{\lambda\mu_L\mu_Y}\frac{a_2}{a_1} &= \frac{(\lambda_1+\lambda)\mu_L'}{\lambda\mu_L\mu_Y}\frac{(\lambda_2+\lambda)\mu_Y}{(\lambda_1+\lambda)\mu_L} = \frac{(\lambda_2+\lambda)\mu_L'}{\lambda\mu_L^2}.\end{aligned}$$

Noting that $\mu_L' > \mu_L^2, \mu_Y' > \mu_Y^2$, then we have $\frac{\lambda\mu_Y^2}{(\lambda_1+\lambda)\mu_Y'} < 1$ and $\frac{(\lambda_2+\lambda)\mu_L'}{\lambda\mu_L^2} > 1$. Thus, the lemma holds.

Appendix B. Proof of Theorem 4

Let $\Theta = [0,+\infty)\times[0,+\infty)\times[0,+\infty)$, we take a sequence of bounded open sets Θ_i meeting $\Theta_i \subset \Theta_{i+1} \subset \cdots \subset \Theta$, $i = 1, 2, \cdots$, and $\Theta = \cup_{i=1}^{\infty}\Theta_i$. For $(\hat{y}_j, l_j, s) \in \Theta_i$, suppose that the exit time of $(\hat{Y}_j(t), L_j(t), S(t))$ from Θ_i is represented by τ_i. If $i \to \infty$, then $\tau_i \wedge T \to T$.

(i) For the investor j (the insurer or the reinsurer), take an arbitrary admissible strategy $u_j(t)$. Applying Ito formula for $V_j(t, \hat{y}_j, l_j, s)$ on $[t, T]$, we derive:

$$\begin{aligned} V_j\Big(T, \hat{Y}_j(T), L_j(T), S(T)\Big) &= V_j(t, \hat{y}_j, l_j, s) + \int_t^T \mathcal{L}^{u_j} V_j\Big(v, \hat{Y}_j(v), L_j(v), S(v)\Big)\,\mathrm{d}v\\ &+ \int_t^T [(\sigma\pi_j(v) - \sigma_{j1}) - \kappa_j(\sigma\pi_k(v) - \sigma_{k1})]S^{\beta}(v)\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_1(u)\\ &- \int_t^T (\sigma_{j2} - \kappa_j\sigma_{k2})\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_2(v) + \int_t^T (p_j(v) - \kappa_j p_k(v))\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_0(v)\\ &+ \int_t^T \sigma_{j1}S^{\beta}(v)\frac{\partial V_j}{\partial L_j}\,\mathrm{d}W_1(v) + \int_t^T \sigma_{j2}\frac{\partial V_j}{\partial L_j}\,\mathrm{d}W_2(v) + \int_t^T \sigma S^{\beta+1}(v)\frac{\partial V_j}{\partial s}\,\mathrm{d}W_1(v). \end{aligned} \tag{B.1}$$

From $\sup_{u_j\in U_j}\{\mathcal{L}^{u_j}V_j(t, \hat{y}_j, l_j, s)\} = 0$, it is to see that $\mathcal{L}^{u_j}V_j(t, \hat{y}_j, l_j, s) \le 0$.
Furthermore, we have

$$\begin{aligned} V_j\Big(T, \hat{Y}_j(T), L_j(T), S(T)\Big) &\le V_j(t, \hat{y}_j, l_j, s)\\ &+ \int_t^T [(\sigma\pi_j(v) - \sigma_{j1}) - \kappa_j(\sigma\pi_k(v) - \sigma_{k1})]S^{\beta}(v)\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_1(u)\\ &- \int_t^T (\sigma_{j2} - \kappa_j\sigma_{k2})\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_2(v) + \int_t^T (p_j(v) - \kappa_j p_k(v))\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_0(v)\\ &+ \int_t^T \sigma_{j1}S^{\beta}(v)\frac{\partial V_j}{\partial L_j}\,\mathrm{d}W_1(v) + \int_t^T \sigma_{j2}\frac{\partial V_j}{\partial L_j}\,\mathrm{d}W_2(v) + \int_t^T \sigma S^{\beta+1}(v)\frac{\partial V_j}{\partial s}\,\mathrm{d}W_1(v). \end{aligned} \tag{B.2}$$

In equation (B.2), the last six terms on the right-hand side of the inequality are the square-integrable martingale with expectation equal to 0. Thus, we obtain

$$E\Big[V_j(T, \hat{Y}_j(T), L_j(T), S(T))\Big|\hat{Y}_j(t) = \hat{y}_j, L_j(t) = l_j, S(t) = s\Big] \le V_j(t, \hat{y}_j, l_j, s).$$

Furthermore, we have

$$\sup_{u_j\in U_j}\Big\{E\Big[V_j\Big(T, \hat{Y}_j(T), L_j(T), S(T)\Big)\Big|\hat{Y}_j(t) = \hat{y}_j, L_j(t) = l_j, S(t) = s\Big]\Big\} \le V_j(t, \hat{y}_j, l_j, s),$$

that is to say $J_j(t, \hat{y}_j, l_j, s) \le V_j(t, \hat{y}_j, l_j, s)$.

(ii) $E[V_j(\tau_i\wedge T, \hat{Y}_j(\tau_i\wedge T), L_j(\tau_i\wedge T), S(\tau_i\wedge T))] < \infty$ for a specific strategy u_j^*.

Using Ito formula for $V_j(t,\hat{y}_j,l_j,s)$ on $[0,\tau_i\wedge T]$, we obtain

$$
\begin{aligned}
V_j\Big(\tau_i\wedge T,\hat{Y}_j(\tau_i\wedge T),L_j(\tau_i\wedge T),S(\tau_i\wedge T)\Big) &= V_j(0,\hat{y}_{j0},l_{j0},s_0)\\
&+\int_0^{\tau_i\wedge T}\mathcal{L}^{u_j}V_j\Big(v,\hat{Y}_j(v),L_j(v),S(v)\Big)\,\mathrm{d}v+\int_0^{\tau_i\wedge T}(p_j(v)-\kappa_j p_k(v))\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_0(v)\\
&+\int_0^{\tau_i\wedge T}[(\sigma\pi_j(v)-\sigma_{j1})-\kappa_j(\sigma\pi_k(v)-\sigma_{k1})]S^{\beta}(v)\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_1(u)\\
&-\int_0^{\tau_i\wedge T}(\sigma_{j2}-\kappa_j\sigma_{k2})\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_2(v)+\int_0^{\tau_i\wedge T}\sigma_{j1}S^{\beta}(v)\frac{\partial V_j}{\partial L_j}\,\mathrm{d}W_1(v)\\
&+\int_0^{\tau_i\wedge T}\sigma_{j2}\frac{\partial V_j}{\partial L_j}\,\mathrm{d}W_2(v)+\int_0^{\tau_i\wedge T}\sigma S^{\beta+1}(v)\frac{\partial V_j}{\partial s}\,\mathrm{d}W_1(v).
\end{aligned}
\tag{B.3}
$$

For a specific policy u_j^* meets (20), that is, $\mathcal{L}^{u_j}V_j(u,\hat{Y}_j(u),L_j(u),S(u))=0$. In equation (B.3), the last six terms on the right side of the equality are martingale, so that the expectation is taken on the left and right sides of the equality, and we have

$$
E\Big[V_j\Big(\tau_i\wedge T,\hat{Y}_j(\tau_i\wedge T),L_j(\tau_i\wedge T),S(\tau_i\wedge T)\Big)\Big]=V_j(0,\hat{y}_{j0},l_{j0},s_0)<\infty.
$$

(iii) $J_j(t,\hat{y}_j,l_j,s)=V_j(t,\hat{y}_j,l_j,s)$ for a specific policy u_j^*.
Applying Ito formula for $V_j(t,\hat{y}_j,l_j,s)$ on $[t,\tau_i\wedge T]$ once again, we obtain

$$
\begin{aligned}
V_j\Big(\tau_i\wedge T,\hat{Y}_j(\tau_i\wedge T),L_j(\tau_i\wedge T),S(\tau_i\wedge T)\Big) &= V_j(t,\hat{y}_j,l_j,s)\\
&+\int_t^{\tau_i\wedge T}\mathcal{L}^{u_j}V_j\Big(v,\hat{Y}_j(v),L_j(v),S(v)\Big)\,\mathrm{d}v+\int_t^{\tau_i\wedge T}(p_j(v)-\kappa_j p_k(v))\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_0(v)\\
&+\int_t^{\tau_i\wedge T}[(\sigma\pi_j(v)-\sigma_{j1})-\kappa_j(\sigma\pi_k(v)-\sigma_{k1})]S^{\beta}(v)\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_1(u)\\
&-\int_t^{\tau_i\wedge T}(\sigma_{j2}-\kappa_j\sigma_{k2})\frac{\partial V_j}{\partial \hat{y}_j}\,\mathrm{d}W_2(v)+\int_t^{\tau_i\wedge T}\sigma_{j1}S^{\beta}(v)\frac{\partial V_j}{\partial L_j}\,\mathrm{d}W_1(v)\\
&+\int_t^{\tau_i\wedge T}\sigma_{j2}\frac{\partial V_j}{\partial L_j}\,\mathrm{d}W_2(v)+\int_t^{\tau_i\wedge T}\sigma S^{\beta+1}(v)\frac{\partial V_j}{\partial s}\,\mathrm{d}W_1(v).
\end{aligned}
\tag{B.4}
$$

Taking the conditional expectation for the left and right sides of the above equation, we have

$$
V_j(t,\hat{y}_j,l_j,s)=E\Big[V_j\Big(\tau_i\wedge T,\hat{Y}_j(\tau_i\wedge T),L_j(\tau_i\wedge T),S(\tau_i\wedge T)\Big)\Big|\hat{Y}_j(t)=\hat{y}_j,L_j(t)=l_j,S(t)=s\Big].
$$

By taking the limit, we get

$$
V_j(t,\hat{y}_j,l_j,s)=\lim_{i\to\infty}E\Big[V_j\Big(\tau_i\wedge T,\hat{Y}_j(\tau_i\wedge T),L_j(\tau_i\wedge T),S(\tau_i\wedge T)\Big)\Big|\hat{Y}_j(t)=\hat{y}_j,L_j(t)=l_j,S(t)=s\Big].
$$

Furthermore, we obtain

$$
\begin{aligned}
J_j(t,\hat{y}_j,l_j,s) &= \sup_{u_j\in U_j}\Big\{E_{t,\hat{y}_j,l_j,s}\Big[\mathcal{U}_j\Big(\widehat{Y}_j^{u_j,u_k}(T)\Big)\Big]\Big\}\\
&=\lim_{i\to\infty}E\Big[V_j\Big(\tau_i\wedge T,\hat{Y}_j(\tau_i\wedge T),L_j(\tau_i\wedge T),S(\tau_i\wedge T)\Big)\Big|\hat{Y}_j(t)=\hat{y}_j,L_j(t)=l_j,S(t)=s\Big]\\
&=V_j(t,\hat{y}_j,l_j,s).
\end{aligned}
\tag{B.5}
$$

That is to say, $u_1^*(t)$ and $u_2^*(t)$ are indeed the optimal investment-reinsurance strategies for the Problem 3.