REVUE FRANÇAISE D'AUTOMATIQUE, D'INFORMATIQUE ET DE RECHERCHE OPÉRATIONNELLE. RECHERCHE OPÉRATIONNELLE

CHARLES S. TAPIERO

Optimal simultaneous replacement and maintenance of a machine with process discontinuities

Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle, tome 5, nº V2 (1971), p. 79-86.

<http://www.numdam.org/item?id=RO_1971__5_2_79_0>

R.I.R.O.
(5e année, V-2, 1971, p. 79-86)

OPTIMAL SIMULTANEOUS REPLACEMENT AND MAINTENANCE OF A MACHINE WITH PROCESS DISCONTINUITIES

by Charles S. TAPIERO [1]

Abstract. — *In general, most pieces of equipment tend to become less efficient productively as their age increases, until they reach a point where it is no longer deemed profitable to use them. However, the obsolescence of a machine is usually limited to a particular production process. This paper is concerned with productive obsolescence of machines, which is defined as process discontinuities. These usually occur when, at some point in time t_1, a machine changes its economic function and is retired at another point in time T_1. In this case the salva value at T_1 is controlled by the production processes during the time intervals* $[0, t_1)$ *and* $[t_1, T_1)$.

INTRODUCTION

An ever-present problem faced by almost all businessmen is that of maximizing the value of a machine or piece of equipment used in the operation of their business. In general, most pieces of equipment tend to become less efficient productively as their age increases until they reach a point where it is no longer profitable to use them. However, the obsolescence of a machine is usually limited to a particular production process. For example, passenger trains can at some period T be transferred to some other function within the same company. Similarly, commercial airlines can assign new planes to lucrative passenger transportation, and assign the old planes to freight transportation. The plane itself would not be retired but simply put to some other use. Retirement would come only when it is no longer deemed profitable to maintain the plane.

This problem is defined here as *process discontinuities,* and usually occurs when at some time t_1 a machine changes function, and is retired only at another point in time T_1. In this case there is a salvage value at T_1 controlled by the

(1) Assistant Professor of Business Columbia university. This research was supported in part by a grant from the Graduate School of Business.

processes at which the machine was operating in the time intervals $[0, t_1)$ and $[t_1, T_1)$.

The purpose of this paper is to analyse simultaneous replacement and maintenance policies with process discontinuities. In a previous paper (4) we generalized the Thompson model (5) for simultaneous maintenance and retirement policies from one machine to n machines. Here we accept process changes for machines so that they are actually not retired but used in some other way. As a result, accumulation of machinery in a firm can be explained by replacement policies, obsolescence and technical progress. The replacement problem we are confronted with is :

Consider a machine put to some production process during an interval $[0, t_1)$. During this interval, the machine's salvage value-maintenance relation is defined by a differential equation with specified obsolescence and effectiveness-maintenance functions. At t_1 the machine is not retired but put to some other production process and retired at T_1, $T_1 > t_1$. During the time interval $[t_1, T_1)$, the salvage value resulting from this second process is defined by a differential equation with different obsolescence and effectiveness-maintenance functions. Find the optimal replacement times t_1, T_1 as well as the optimal maintenance policies during both production processes.

THE MODEL

The simultaneous maintenance and replacement problem for one machine with no process discontinuities is (5) to determine a sequence of maintenance decisions, $m(t)$ which will maximize the present value of the machine. This value equals the salvage value of the machine plus the anticipated stream of revenues minus the operating and maintenance costs, all discounted at the proper constant rate of interest. This is a constrained optimization problem in that the rate of change of the salvage value is a function of the deterioration of the machine and the maintenance policy employed. The model for determining the optimum maintenance policy and replacement time is a classic variational problem maximizing :

$$V(T) = S(T)\, e^{-rT} + \int_0^T Q(t)\, e^{-rt}\, dt \tag{1}$$

subject to :

$$0 \leqslant m(t) \leqslant M \tag{2}$$

$$Q(t) = p(t)\, S(t) - m(t)$$

and

$$dS(t)/dt = -a(t) + f(t)m(t) \qquad S(0) = K \tag{3}$$

where

T = sale date of the machine
$V(t)$ = present value of the machine at time t
$S(t)$ = salvage value of the machine at time t
$Q(t)$ = net operating receipts at time t
r = constant rate of interest
$m(t)$ = maintenance policy — hours of maintenance at time t
$f(t)$ = maintenance effectiveness function a time t — S added to $S(t)$/hrs· of maintenance
K = cost of the machine
$a(t)$ = obsolescence function at t — S subtracted from $S(t)$
$p(t)$ = production rate at time t; output value at $t/S(t)$
$V(t)$ *and* $S(t)$ are *state variables*, while $m(t)$ is the *control variable* under management discretion.

The resulting optimal maintenance policy for fixed p is [5]

$$(4) \qquad m(t) = \begin{cases} M & \text{if } f(t) > r/(p-(p-r)\,e^{-r(T-t)}) \\ \text{undetermined} & \text{if } \quad = \quad — \\ 0 & \text{if } \quad < \quad — \end{cases}$$

The implications of this maintenance policy are [5]

Spend money on maintenance at time t only if the machine is to be held for a sufficiently long time after t so that the maintenance expenditure will be recovered due to the increased profitability resulting from the maintenance before the machine is sold, p. 546.

The optimal retirement time T is found by maximizing (1) with respect to T. Then we have :

$$(5) \quad \frac{dV(T)}{dT} = 0 = S(T)\,e^{-rT} - rS(T)\,e^{-rT} + Q(T)\,e^{-rT} + \int_0^T \frac{\partial Q(t)}{\partial T}\,e^{-rT} dt$$

Since in this case $\partial Q(t)/\partial T = 0$ [5] and if at T, $m(T) = 0$, we find T which satisfies the equation below

$$(6) \qquad S(T) = a(T)/(p - r)$$

However, in reality, the retirement and maintenance policies are weighed against other machines that recently appeared on the market as well as other uses the original machine can be assigned to within the firm. This is the problem we are concerned with in this paper.

Assume that at time t_1, the machine is transferred to a different process with different characteristics and is retired only at T_1. The problem is then :

Maximize

$$(7) \qquad V(T_1) = S(T_1)\,e^{-rT_1} + \int_0^{T_1} Q_1(t)\,e^{-rT}dt + \int_{t_1}^{T_1} Q_{11}(t)\,e^{-rT}dt$$

subject to :

$$(8) \qquad \begin{aligned} Q_1(t) &= p_1(t)S(t) - m_1(t) && o \leqslant t < t_1 \\ Q_{11}(t) &= p_{11}(t)S(t) - m_{11}(t) && t_1 \leqslant t < T_1 \end{aligned}$$

$$o \leqslant m_1(t) \leqslant M_1$$
$$o \leqslant m_{11}(t) \leqslant M_{11}$$

$$(9) \qquad dS(t)/dt = -a_1(t) + f_1(t)m_1(t) \qquad o \leqslant t < t_1 \; S(o) = k_1$$

$$(10) \qquad dS(t)/dt = -a_{11}(t) + f_{11}(t)m_{11}(t) \qquad t_1 \leqslant t < T_1 \; S(t_1) = ?$$

This probem is a variational problem with discontinuities both in the cost and the system functions. However, dS/dt and the cost function are continuous in the state $S(t)$ and the controls $m_1(t)$ and $m_{11}(t)$. Therefore, a solution based on the Maximum principle of Pontryiagin [2, 3] can be obtained. Such a solution will be obtained by introducing a variable $k_1(t, t_1)$.

$$(11) \qquad k_1(t, t_1) = \begin{cases} 1 \text{ for} & 0 \leqslant t < t_1 \\ 0 \text{ elsewhere} \end{cases}$$

$k_1(t, t_1)$ is a piecewise constant function whose levels (1 and 0) specify that it exists only during the first process of the machine. This variable is introduced so that the integrand and the systems equations will be defined over the whole life of the machine. Using k_1, and introducing the salvage value of the machine at T_1 in the integral, we obtain :

Maximize :

$$(12) \qquad V(T_1) = \int_0^{T_1} \{ [Q_1(t) + \dot{S}(t) - rS(t)]\,e^{-rt}k_1(t, t_1) + [Q_{11}(t) + \dot{S}(t) - rS(t)]\,e^{-rt}(1 - k_1(t, t_1) \}\,dt$$

subject to :

$$(13) \qquad dS(t)/dt = [-a_1(t) + f_1(t)m_1(t)]k_1(t, t_1) + [-a_{11}(t) + f_{11}(t)m_{11}(t)]\,(1 - k_1)\,(t, t_1))$$

$$0 \leqslant t < T_1 \qquad \text{and} \qquad S(o) = K_1$$

and (8).

THE OPTIMAL MAINTENANCE POLICIES : ONE MACHINE

For (12), (13) and (8) a Hamiltonian H can be definied by (we drop here all the time subscripts)

$$(14) \quad H = -\{(Q_1 + \dot{S} - rS)k_1 + (Q_{11} + \dot{S} - rS)(1 - k_1)\} e^{-rt} + Z_1\{(-a_1 + f_1 m_1)k_1 + (-a_{11} + f_{11}m_{11})(1 - k_1)$$

$Q_1 = p_1 S - m_1$ and $Q_{11} = p_{11}S - m_{11}$ we observe that

$$(15) \quad \dot{Z}_1(t) = -\partial H/\partial S = e^{-rt}[(p_1 - r)k_1 + (p_{11} - r)(1 - k_1)] \quad Z_1(T_1) = 0$$

Thus for fixed production rates p_1 and p_{11}.

$$(16) \qquad Z_1(t) = ((p_1 - r)/r)(1 - e^{-rt}) + Z(o) \quad 0 \leqslant t < t_1$$

$$(16\,a) \qquad Z_1(t) = ((p_{11} - r)/r)(e^{-rt}1 - e^{-rt}) + Z(t_1) \quad t_1 \leqslant t < T_1$$

Since $Z_1(T_1) = 0$, we have

$$(16\,b) \qquad Z(t_1) = ((p_{11} - r)/r)(e^{-rT_1} - e^{-rt_1})$$

and

$$(16\,c) \qquad Z(o) = ((p_1 - r)/r)(e^{-rt_1} - 1)$$

Thus

$$(17) \quad Z_1(t) = ((p_1 - r)/r)(e^{-rt_1} - e^{-rt})k_1(t, t_1) + ((p_{11} - r)/r)(e^{-rT_1} - e^{-rt})(1 - k_1(t, t_1))$$

and

$$(18) \quad Z_1(t) = ((p_1 - r)/r)k_1\, e^{-rt}(e^{-r(t_1 - t)} - 1) + ((p_{11} - r)/r)(1 - k_1)\, e^{-rt}(e^{-r(T_1 - t)} - 1)$$

Introducing (18) into the Hamiltonian (14) and grouping all terms m_1 and m_{11}, we find that the control variables m_1 and m_{11} are linear. Therefore, the optimal maintenance (controls) policies are bang-bang (1) and equal

(19)

$$m_1(t) = \begin{cases} M_1 & \text{if } f_1(t) > r/[p_1 - (p_1 - r)\,e^{-r(t_1 - t)}] \\ \text{undetermined} & \text{if } f_1(t) = \quad - \\ 0 & \text{if } f_1(t) < \quad - \end{cases}$$

$$m_{11}(t) = \begin{cases} M_{11} & \text{if } f_{11}(t) > r/[p_{11} - (p_{11} - r)\,e^{-r(T_1 - t)}] \\ \text{undetermined} & \text{if } f_{11}(t) = \quad - \\ 0 & \text{if } f_{11}(t) < \quad - \end{cases}$$

THE PROCESS REPLACEMENT AND RETIREMENT TIMES. ONE MACHINE

Given the maintenance policy, the optimal process replacement time t_1 and retirement T_1 can be found by deriving (7) with respect to t_1 and T_1. Then since $\partial Q_1/\partial t_1 = 0$ and $\partial Q_{11}/\partial T_1 = 0$

(20) $$\partial V(T_1)/\partial t_1 = [Q_1(t_1) - Q_{11}(t_1)]\,e^{-rt_1} + \int_{t_1}^{T_1} \frac{\partial Q_{11}}{\partial t_1}\,e^{-rt}dt + [\partial S(T_1)/\partial T_1]\,e^{-rt} = 0$$

and

(21) $$\partial V(T_1)/\partial T_1 = Q_{11}(T_1)\,e^{-rT_1} + [\partial S(T_1)/\partial T_1]\,e^{-rT_1} - r\,e^{-rT_1}\,S(T_1) = 0$$

Since $\partial Q_{11}(t)/\partial t_1 = p_{11}\partial S(t)/\partial t$, and from (10),

$$\partial S(t)/\partial t_1 = \partial S(t_1)/\partial t_1 + a_{11}(t_1) - f_{11}(t_1)m_{11}(t_1)$$

Also, since, $\partial S(t_1)/\partial t_1 = -a_1(t_1) + f_1(t_1)m_1(t_1)$
and $\partial S(T_1)/\partial T_1 = -a_{11}(T_1) + f_{11}(T_1) + m_{11}(T_1)$, we obtain for (20) :

(22) $$\partial V(T_1)/\partial t_1 = [p_1S(t_1) - m_1(t_1) + p_{11}S(t_1) + m_{11}(t_1)]\,e^{-rt_1} + \frac{e^{-r(T_1 - t_1)}}{r}[a_{11}(t_1) - a_1(t_1) - f_{11}(t_1)m_{11}(t_1) + f_1(t_1)m_1(t_1)] + e^{-rT_1}[-a_1(t_1) + f_1(t_1)m_1(t_1) + a_{11}(t_1) - f_{11}(t_1)m_{11}(t_1)] = 0$$

In (22), all terms are either expressed as specific values of the depreciation, effectiveness and maintenance functions or can be specifically described as a function of these known terms. Similarly for (21), we can show that :

$$\text{(23)} \quad \partial V(T_1)/\partial T_1 = (p_1 - r)\left[K - \int_0^{t_1} a_1(t)\,\mathrm{d}t + \int_0^{t_1} f_1(t)m_1(t)\,\mathrm{d}t - \int_{t_1}^{T_1} a_{11}(t)\,\mathrm{d}t + \int_{t_1}^{T_1} f_{11}(t)m_{11}(t)\,\mathrm{d}t\right] - a_{11}(T_1) + (f_{11}(T_1) - 1)m_{11}(T_1) = 0$$

In (22) and (23) all terms are known except for t_1 and T_1. Furthermore, since the maintenance policies m_1 and m_{11} are functions of t_1 and T_1 respectively, computation of the integrals $\int_0^{t_1} f_1(t)m_1(t)\,\mathrm{d}t$ and $\int_{t_1}^{T} f_{11}(t)m_{11}(t)\,\mathrm{d}t$ create computational difficulties which can be resolved. Assume a certain value for T_1^* then compute the corresponding time t_1^*. Given t_1^* and T_1^*, the switching times for maintenance policies $m_1(t)$ and $m_{11}(t)$ can be computed according to (19). Therefore, the integrals of maintenance policies can also be computed and some number is obtained for $\partial V(T_1^*)/\partial T_1^*$ in (23). If $\partial V(T_1^*)/\partial T_1^* > 0$, then an increase in T_1^* to $T_1^{**} = T_1^* + \Delta t$ will have this value decrease. If $\partial V(T_1^*)/\partial T_1^* < 0$, we decrease instead T_1^* to $T_1^{**} = T_1^* - \Delta t$. This procedure is repeated until (23) is satisfied.

CONCLUSION

A machine operating in a certain production process is usually replaced because of deterioration, failure beyond repair or because of new competitive machines appearing in the marketplace. Also a machine can be « retired » from a production process not because of a failure to provide high quality products but merely because the opportunity cost of not buying the new machines is too high. However, in particular in big firms, a machine is not phased out of operation or sold, but is transferred to some other function within the firm. As an example, we have considered an airline buying new planes and phasing out older planes into other functions, such as freight transportation or an assignment to shorter and less glamorous routes. The question we have raised is then; given the characteristic forms of both production processes as well as the obsolescence functions in both processes, what should the switching-process time and the retirement time be. Simultaneously, we have provided conditions for the optimality of the maintenance policies to be applied on both production processes.

In our model $a(t)$ reflects the depreciation rate resulting from technical progress. Thus, high values for $a(t)$ will undermine the influences of the present machine and will favor the acquisition of new machines. However, technical progress alone is not a sufficient criterion. In addition, the retirement time of a machine in a particular production process will also be determined by other competitive production processes within the firm.

REFERENCES

1. R. E. Bellman, J. Glicksberg and D. A. Gross, *Some Aspects of the Mathematical Theory of Control Processes*, Santa Monica, California, Rand Corporation, 1958.
2. M. R. Hestenes, *Calculus of Variations and Optimal Control Theory*, New York, John Wiley and Sons Inc., 1966.
3. L. S. Pontryagin, V. G. Boltyonskii, P. V. Gamkrelidze and E. V. Mischenko. *The Mathematical Theory of Optimal Control Processes*, Interscience, New York, 1962.
4. C. S. Tapiero, « N-Machines Simultaneous Maintenance, Replacement and Technical Absolescence » *Columbia University*, 1971.
5. G. L. Thompson, « Optimal Maintenance Policy and Sale Date of a Machine », *Management Science*, 1968, 14, 543-550.