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A LIMITED QUEUING PROBLEM WITH ANY NUMBER OF ARRIVALS AND DEPARTURES

by R. S. GAUR (*)

Abstract. — *A limited queueing problem with any number of Poisson arrivals and departures, less than a certain fixed number, has been considered. Probabilities for the queue length and the steady state mean queue length have been obtained.*

INTRODUCTION

The paper studies the behaviour of a single channel queueing process with a limited space of N. The queue discipline is first come first served and arrivals and departures may be in batches. A batch of arrivals is assumed to be at the most of l ($l \leqslant N$) units and that of departures at the most of l' ($l' \leqslant N$) units. If a batch of j units arrives at N-i ($i < j$), $j - i$ units balk with probability 1.

Two cases have been considered. In case I, l and l' are taken to be equal and in case II, they are different.

Cases of such queues are very common. For example, any number of persons may board a bus from a queue at a bus stop or any number of them may arrive to join the queue.

The distribution function for the queue lengths in terms of Laplace Transforms have been found out in the two cases. Steady state mean queue lengths have also been obtained.

NOTATION

We adopt the following notation :

$P_n(t)$ = the probability that at time t there are n units in the queue including the one in the channel.

$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$ is the Laplace transform (L. T.) of a function $f(t)$.

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$F = \lim_{t \to \infty} F(t) = \lim_{s \to \infty} s\bar{F}(s)$ is the steady state notation corresponding to a function $F(t)$ or $\bar{F}(s)$ of the unsteady state.

L = The mean queue length for the steady state.

FORMULATION OF EQUATIONS AND THEIR SOLUTIONS

Case I

The model consists of a first come first served single service channel where units arrive with Poisson mean rate λ and form a queue. The units are served exponentially at the head of the line basis with rate μ. The arrivals and departures are in batches of maximum size $l (l \leqslant N)$. The probability that a batch of arrivals will consist of exactly i units $(1 \leqslant i \leqslant l)$ is $p_i\lambda$ where $\sum_{i=1}^{l} p_i = 1$. The probability that a batch of departures will consist of exactly i units is $k_i\mu$ $(1 \leqslant i \leqslant l)$ where $\sum_{i=1}^{l} k_i = 1$

The probability considerations lead to the following set of difference equations governing the system :

$$P_0'(t) + \lambda P_0(t) = \sum_{i=1}^{l} P_i(t) k_i \mu$$

$$P_n'(t) + \left(\sum_{i=1}^{m} p_i\lambda + \sum_{i=1}^{r} k_i\mu \right) P_n(t) = \sum_{i=1}^{r} \underset{n-i}{P}(t) p_i\lambda + \sum_{i=1}^{m} P_{n+i}(t) k_i\mu$$

(1)
$$n = 1, 2, \ldots, N-1$$

$$\left[\left. \begin{aligned} m &= l && \text{for } n \leqslant N-l \\ &= N-n && \text{for } n > N-l \end{aligned} \right\} , \left. \begin{aligned} r &= n \text{ for } n < l \\ &= l \text{ for } n \geqslant l \end{aligned} \right\} \right]$$

$$P_N'(t) + \mu P_N(t) = \sum_{i=1}^{l} \underset{N-i}{P}(t) p_i\lambda$$

Initially we assume that

(2)
$$P_0(0) = 1,\ P_n(0) = 0,\ n > 0.$$

Taking L. T. of the equations (1) and using (2) we have

$$(s + \lambda)\bar{P}_0(s) = \sum_{i=1}^{l} \bar{P}_i(s) k_i\mu + 1$$

$$(3) \qquad \left(s + \sum_{i=1}^{m} p_i\lambda + \sum_{i=1}^{r} k_i\mu\right)\bar{P}_n(s) = \sum_{i=1}^{r} \bar{P}_{n-i}(s)\, p_i\lambda + \sum_{i=1}^{m} \bar{P}_{n+i}(s)\, k_i\mu.$$

$$n = 1, 2, \ldots\ldots, N-1.$$

$$(s + \mu)\bar{P}_N(s) = \sum_{l=1}^{l} \bar{P}_{N-i}(s)\, p_i\lambda.$$

We define the following generating function :

$$\bar{A}(\theta, s) = \sum_{n=0}^{N} \bar{P}_n(s)\theta^n$$

Multplying equations (3) by appropriate powers n of θ and summing over n we have

$$(4) \qquad \bar{A}(\theta, s) = \frac{\displaystyle\sum_{j=0}^{l-1} \bar{P}_j(s)\theta^j \sum_{i=j+1}^{l} \left(k_i\mu - k_i \frac{\mu}{\theta^i}\right) + \sum_{j=N-l+1}^{N} \bar{P}_j(s)\theta^j \sum_{i=N-j+1}^{l} (p_i\lambda - p_i\lambda\theta^i) + 1}{\displaystyle s + \sum_{i=1}^{l} (p_i\lambda - p_i\lambda\theta^i) + \sum_{i=1}^{l} \left(k_i\mu - k_i \frac{\mu}{\theta^i}\right)}$$

Since $\bar{A}(\theta, s)$ is a polynomial the zeros of the denominator of (4) must vanish its numerator. If θ_x $(x = 1, 2, \ldots, 2l)$ be the zeros of the denominator of (4) we have.

$$(5) \qquad \sum_{j=0}^{l-1} \bar{P}_j(s)\theta_x^j \sum_{i=j+1}^{l} \left(k_i\mu - k_i \frac{\mu}{\theta_x^i}\right) + \sum_{j=N-l+1}^{N} \bar{P}_j(s)\theta_x^j \sum_{i=N-j+1}^{l} (p_i\lambda - p_i\lambda\theta_x^i) + 1 = 0.$$

$$\bar{P}_j(s), (j = 1, 2, \ldots, l-1, N-l+1, N-l+2, \ldots, N)$$

can be determined from the equations (5). Hence $\bar{A}(\theta, s)$, the distribution fuction for the queue length in terms of L. T. is completely known.

It is to be noted that this complete solution of $\bar{A}(\theta, s)$ is valid only so long as $l \leqslant \frac{N+1}{2}$. When $l > \frac{N+1}{2}$, the number of zeros in the denominator of (4) exceed the number of unknowns in its numerator. The latter case needs further investigations. These investigations are being carried out and will be communicated in the next paper.

STEADY STATE SOLUTION

Using the well known property of the Laplace Transforms, viz,

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} s\bar{F}(s)$$

if the limit on the left exists, we have from (4),

$$\bar{A}(\theta) = \frac{\sum_{j=0}^{l-1} P_j\theta^j \sum_{i=j+1}^{l} \left(k_i\mu - k_i\frac{\mu}{\theta^i}\right) + \sum_{j=N-l+1}^{N} P_j\theta^j \sum_{i=N-j+1}^{l} (p_i\lambda - p_i\lambda\theta^i)}{\sum_{i=1}^{l} (p_i\lambda - p_i\lambda\theta^i) + \sum_{i=1}^{l}\left(k_i\mu - k_i\frac{\mu}{\theta^i}\right)}$$

This on simplification yields

$$(6)\qquad A(\theta) = \frac{\sum_{j=N-l+1}^{N} P_j\theta^j \sum_{i=N-j+1}^{l} p_i\lambda \sum_{n=0}^{i-1}\theta^n - \sum_{j=0}^{l-1} P_j\theta^j \sum_{i=j+1}^{l} k_i\frac{\mu}{\theta^i}\sum_{n=0}^{i-1}\theta^n}{\sum_{i=1}^{l}\{p_i\lambda\sum_{n=0}^{i-1}\theta^n - k_i\frac{\mu}{\theta^i}\sum_{n=0}^{i-1}\theta^n\}}$$

A (θ) being a polynomial, if $\theta_x (x = 1, 2, \ldots, 2l - 1)$ are the zeros of the denominator of (6) we have

$$(7)\qquad \sum_{j=N-l+1}^{N} P_j\theta_x^j \sum_{i=N-j+1}^{l} p_i\lambda \sum_{n=0}^{i-1}\theta_x^n - \sum_{j=0}^{l-1} P_j\theta_x^j \sum_{i=j+1}^{l} k_i\frac{\mu}{\theta_x^i}\sum_{n=0}^{i-1}\theta_x^n = 0$$

And since $A(1) = 1$, we have

$$(8)\qquad \sum_{j=N-l+1}^{N} P_j \sum_{i=N-j+1}^{l} p_i\lambda i - \sum_{j=0}^{l-1} P_j \sum_{i=j+1}^{l} k_i\mu_i = \sum_{i=1}^{l}(p_i\lambda - k_i\mu)i$$

From (7) and (8) P_j $(j = 0, 1, 2, \ldots, l-1, N-l+1, \ldots, N)$ can be determined and hence $A(\theta)$ is completely known. Differentiating (6) at $\theta = 1$, we have

$$L = \frac{\left[\sum_{i=1}^{l}(p_i\lambda - k_i\mu)i\left\{\sum_{j=N-l+1}^{N}\sum_{i=N-j+1}^{l} P_jp_i\lambda(2j+i-1)\frac{i}{2} - \sum_{i=1}^{l-1}\sum_{i=j+1}^{l} P_jk_i\mu(2j-i-1)\frac{i}{2}\right\} - \left\{\sum_{j=N-l+1}^{N}\sum_{i=N-j+1}^{l} P_jp_i\lambda i - \sum_{j=0}^{l-1}\sum_{i=j+1}^{l} P_jk_i\mu i\right\} \times \sum_{i=1}^{l}\left\{p_i\lambda(i-1) + k_i\mu(i+1)\right\}\frac{i}{2}\right]}{\left\{\sum_{i=1}^{l}(p_i\lambda - k_i\mu)i\right\}^2}$$

Case II

The Model, as before consists of a first come first served single service channel where units arrive with Poisson mean rate λ and form a queue. The units are served exponentially at the head of the line basis with rate μ. The arrivals are in batches of maximum size $l(l \leqslant N)$. Departures are also in batches of maximum size $l'(l' \leqslant N)$. The probability that a batch of arrivals will consist of exactly $i(1 \leqslant i \leqslant l)$ units is $p_i\lambda$ so that $\sum_{i=1}^{l} p_i = 1$ whereas the probability that a batch of departures will consist of exactly i units $(1 \leqslant i \leqslant l')$ is $k_i\mu$ so that $\sum_{i=1}^{l'} k_i = 1$.

Here we assume that

$$P_0(0) = 1,\ P_n(0) = 0,\ n > 0.$$

and define the following generating function

$$\bar{A}(\theta, s) = \sum_{n=0}^{N} \bar{P}_n(s)\theta^n$$

Probability considerations lead to the following set of difference equations governing the system.

$$P_0'(t) + \sum_{i=1}^{l} p_i\lambda P_0(t) = \sum_{i=1}^{l'} k_i\mu P_i(t)$$

$$P_n'(t) + \left(\sum_{i=1}^{m} p_i\lambda + \sum_{i=1}^{r} k_i\mu \right) P_n(t) = \sum_{i=1}^{j} k_i\mu P_{n+i}(t) + \sum_{i=1}^{g} p_i\lambda P_{n-i}(t)$$

$$n = 1, 2, \ldots\ldots, N-1$$

$$\left[\begin{array}{ll} \left.\begin{array}{lll} m = l & \text{for } n \leqslant N-l \\ \quad = N-n & \text{for } n > N-l \end{array}\right\}, & \left.\begin{array}{ll} r = n \text{ for } n < l' \\ \quad = l' \text{ for } n \geqslant l' \end{array}\right\} \\ \left.\begin{array}{lll} j = l' & \text{for } n \leqslant N-l' \\ \quad = N-n & \text{for } n > N-l' \end{array}\right\}, & \left.\begin{array}{ll} g = n \text{ for } n < l \\ \quad = l \text{ for } n \geqslant l \end{array}\right\} \end{array}\right]$$

$$P_N'(t) + \mu P_N(t) = \sum_{i=1}^{l} p_i\lambda P_{N-i}(t)$$

Proceeding on the lines similar to those of case I, we get

$$(9) \quad \bar{A}(\theta, s) = \frac{\sum_{j=0}^{l'-1} \bar{P}_j(s)\theta^j \sum_{i=j+1}^{l'} \left(k_i\mu - k_i \frac{\mu}{\theta^i}\right) + \sum_{j=N-l+1}^{N} \bar{P}_j(s)\theta^j \sum_{i=N-j+1}^{l} (p_i\lambda - p_i\lambda\theta^i) + 1}{s + \sum_{i=1}^{l} (p_i\lambda - p_i\lambda\theta^i) + \sum_{i=1}^{l'} \left(k_i\mu - k_i \frac{\mu}{\theta^i}\right)}$$

$$\bar{P}_j(s) \quad (j = 0, 1, 2, \ldots, l'-1, N-l+1, N-l+2, \ldots, N.)$$

are determined from the equations

$$\sum_{j=0}^{l'-1} P_j(s)\theta_x^j \sum_{i=j+1}^{l'} \left(k_i\mu - k_i \frac{\mu}{\theta_x^i}\right) + \sum_{j=N-l+1}^{N} \bar{P}_j(s)\theta_x^j \sum_{i=N-j+1}^{l} (p_i\lambda - p_i\lambda\theta_x^i) + 1 = 0$$

Where $\theta_x (x = 1, 2, \ldots, l + l')$ are the zeros of the denominator of (9).

Here again, as in case I, $\bar{A}(\theta, s)$ is completely solvable so long as $l + l' \leqslant N + 1$. The case when $l + l' > N + 1$ needs further investigation and will be communicated in the next paper.

The corresponding results for the steady state are

$$A(\theta) = \frac{\sum_{j=0}^{l'-1} P_j\theta^j \sum_{i=j+1}^{l'} \left(k_i\mu - k_i \frac{\mu}{\theta^i}\right) + \sum_{j=N-l+1}^{N} P_j\theta^j \sum_{i=N-j+1}^{l} (p_i\lambda - p_i\lambda\theta^i)}{\sum_{i=1}^{l} (p_i\lambda - p_i\lambda\theta^i) + \sum_{i=1}^{l'} \left(k_i\mu - k_i \frac{\mu}{\theta^i}\right)}$$

This on simplification yields

$$\text{(10)} \qquad A(\theta) = \frac{\sum_{j=N-l+1}^{l} P_j\theta^j \sum_{i=N-j+1}^{l} p_i\lambda \sum_{n=0}^{i-1} \theta^n - \sum_{j=0}^{l'-1} P_j\theta^j \sum_{i=j+1}^{l'} k_i \frac{\mu}{\theta^i} \sum_{n=0}^{i-1} \theta^n}{\sum_{i=1}^{l} p_i\lambda \sum_{n=0}^{i-1} \theta^n - \sum_{i=1}^{l'} k_i \frac{\mu}{\theta^i} \sum_{n=0}^{i-1} \theta^n}$$

$P_j (j = 0, 1, 2, \ldots, l' - 1, N - l + 1, N - l + 2, \ldots, N)$ are determined from the equations

$$\sum_{j=N-l+1}^{N} P_j\theta_x^j \sum_{i=N-j+1}^{l} p_i\lambda \sum_{n=1}^{i-1} \theta_x^n - \sum_{j=0}^{l'-1} P_j\theta_x^j \sum_{i=j+1}^{l'} k_i \frac{\mu}{\theta_x^i} \sum_{n=0}^{i-1} \theta_x^n = 0$$

and

$$\sum_{j=N-l+1}^{N} P_j \sum_{i=N-j+1}^{l} p_i\lambda i - \sum_{j=0}^{l'-1} P_j \sum_{i=j+1}^{l'} k_i\mu i = \sum_{i=1}^{l} p_i\lambda i - \sum_{i=1}^{l'} k_i\mu i$$

Where $\theta_x (x = 1, 2 \ldots, l + l' - 1)$ are the zeros of the denominator of (10).

The steady state mean queue length is given by

$$L = \frac{\left[\left(\sum_{i=1}^{l} p_i\lambda i - \sum_{i=1}^{l'} k_i\mu i\right)\left\{\sum_{j=N-l+1}^{N}\ \sum_{i=N-j+1}^{l}(2j+i-1)\frac{i}{2}P_jp_i\lambda - \sum_{j=0}^{l'-1}\ \sum_{i=j+1}^{l'}(2j-i-1)\frac{i}{2}P_jk_i\mu\right\} - \left(\sum_{j=N-l+1}^{N}\ \sum_{i=N-j+1}^{l} P_jp_i\lambda i - \sum_{j=0}^{l'-1}\ \sum_{i=j+1}^{l'} P_jk_i\mu i\right)\left\{\sum_{i=1}^{l} p_i\lambda(i-1)\frac{i}{2} + \sum_{i=1}^{l'} k_i\mu(i+1)\frac{i}{2}\right\}\right]}{\left(\sum_{i=1}^{l} p_i\lambda i - \sum_{i=1}^{l'} k_i\mu i\right)^2}$$

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