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0-1 OPTIMIZATION AND NON-LINEAR PROGRAMMING

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Abstract. — *The optimization problem for pseudo-Boolean functions is : Find the points of minima of f on the vertices of the n-dimensional unit cube where f is a real polynomial linear in each variable. This discrete problem is equivalent to the continuous problem : Find the points of minima of f on the n-dimensional unit cube.*

1. INTRODUCTION

It is well known [1] that many problems in operations research, switching theory, combinatorics, graph theory etc. can be reduced to the following problem P : Let $f(X)$ be a real polynomial with n variables which is linear in each variable. Find the minimum of f on the set $\{0, 1\}^n$. There are ways (at least at the theoretical level) to reduce any 0-1 program to this problem [3]. In this note we present the following (apparently so far unrecorded) simple fact : In P we can replace $\{0, 1\}^n$ by $[0, 1]^n$ in other words, P can be treated as a continuous non-linear problem : Minimize $f(X)$ subject to very simple constraints $0 \leqslant x_i \leqslant 1$ ($i = 1, \ldots, n$). It is hoped that this problem will be easier to solve than P.

Using an idea of Picard and Ratliff we show that for such polynomials of an even degree it suffices to investigate only those without linear terms provided $\{0,1\}$ is replaced by $\{-1, 1\}$. It has been shown in [5] that P can be reduced to a similar problem with a quadratic polynomial provided that a sufficient number of slack variables are introduced. In view of this the following problem is of prime interest : Find the minimum of $g(X)$ on $[-1, 1]^n$, where g is a quadratic polynomial without linear terms or squares.

A special case of Proposition 1 was found also by P. L. Hammer (oral communication).

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2. MAIN RESULT

We will need the following result :

Lemma : *Let $p(x_1, \ldots, x_n)$ be a polynomial linear in each variable. Let $a_i < b_i$ $(i = 1, \ldots, n)$, let $U = [a_1, b_1] \times \ldots \times [a_n, b_n]$, and let*

$$X^* = (x_1^*, \ldots, x_n^*) \in U.$$

If $m = f(X^*)$ is the minimum of f on U, then f is constant and equal to m on

$$W_{X^*} = \{ (w_1, \ldots, w_n) \in U \mid x_i^* \in \{ a_i, b_i \} \Rightarrow w_i = x_i^* \}.$$

Proof : Let $I = \{ 1 \leqslant i \leqslant n \mid x_i^* \in \{ a_i, b_i \} \}$ and let $J = \{1, \ldots, n\} \setminus I = \{j_1, \ldots, j_k\}$. Let $g(y_1, \ldots, y_k)$ be the function obtained from P by setting $x_i = x_i^*$ $(\forall\, i \in I)$ and $x_{j_i} = y_i$ $(i = 1, \ldots, k)$. In other words, we have fixed all variables for which $x_i^* \in \{ a_i, b_i \}$ and kept all variables for which $a_i < x_i^* < b_i$. Since m is the minimum of p ou U, m is also the minimum of p on $W_{X^*} \subseteq U$ and therefore $m = g(x_{j_1}^*, \ldots, x_{j_k}^*)$ is the minimum of g on

$$V = [a_{j_1}, b_j] \times \ldots \times [a_{j_k}, b_{j_k}].$$

The function g is clearly linear in y_1; hence $g(y_1, x_{j_2}^*, \ldots, x_{j_k}^*) = ay_1 + b$. Since $m = ax_{j_1}^* + b$ is the minimum of g, it follows that $a = 0$ and $g(y_1, x_{j_2}^*, \ldots, x_{j_k}^*)$ is constant and equal m. Continuing in the same way we get that

$$g(y_1, y_2, x_{j_3}^*, \ldots, x_{j_k}^*)$$

is constant and equal m and finally we obtain that g is constant and equal m on V. But this proves the lemma.

Let $S \subseteq R^n$ and let $f : R^n \to R$ (R reals). We set $\Omega_S^f = \{ X \in S \mid f(X)$ is minimum of f on $S \}$. Now we have :

Proposition 1. *Let f be a polynomial linear in each of its n variables and let $S = [a_1, b_1] \times \ldots \times [a_n, b_n]$ and $T = \{a_1, b_1\} \times \ldots \times \{a_n, b_n\}$. Then Ω_S^f is the set of all faces C of S satisfying*

$$C \cap T \subseteq \Omega_T^f,$$

and Ω_T^f is the set of the integer points of Ω_S^f.

Proof : If $X^* \in \Omega_S^f$, then, by the lemma, f is constant and equal $m = f(X^*)$ on W_{X^*}; in particular f takes the value m on $W_{X^*} \cap T$. Since $T \subseteq S$, m is also the minimum of f on T and this, in fact, proves the statement.

Corollary 1. *If f is linear in each of its variables then $\Omega_{\{0,1\}^n}^f$ is the set of all integer points of $\Omega_{[0,1]^n}^f$.*

3. REMOVAL OF LINEAR TERMS

In the problem P the simplest nontrivial case is the case of a quadratic polynomial. This is of special interest because the general case can be the transformed to the quadratic one by adding enough slack variables [5]. Also there are some results concerning the quadratic case [2]. If we wish to use the theory of quadratic forms we have to eliminate the linear terms. The ordinary approach ($x_i = y_i + \alpha_i$) requires solution of a linear system and therefore if it can be used at all, presents practical obstacles. Picard and Ratliff [4] have indicated a simple method which can be slightly generalized as follows :

Let

$$x_i = \tfrac{1}{2}[1 - \xi_0 \xi_i]. \qquad (i = 1, \ldots, n) \tag{1}$$

It is easy to check that for each i (1) defines a mapping of $\{-1, 1\}^2$ onto $\{0, 1\}$. Moreover if $\xi_0 \in \{-1, 1\}$ and $1 \leqslant i_1 < \ldots < i_k \leqslant$ then

$$x_{i_1} \ldots x_{i_k} = \frac{1}{2^k}[r - \xi_0 s] \tag{2}$$

where r and s are polynomials in $\xi_{i_1}, \ldots, \xi_{i_k}$ containing only terms of even and odd degrees, respectively. Since the degrees of r and s do not exceed k we have :

Proposition 2. *Let f be a polynomial linear in each variable. If f has an even degree, then the substitution* (1) *together with $\xi_0^2 = 1$ yields a polynomial g of the same degree and without linear terms such that the points of*

$$\Omega^f_{\{0,1\}^n} \textit{ and } \Omega^g_{\{-1,1\}^{n+1}}$$

are related by (1).

Now we can apply Proposition 1 :

Corollary 2. *The set $\Omega^g_{\{-1,1\}^n}$ is the set of all integer points from $\Omega^g_{[-1,1]^n}$.*

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