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AN INTERMITTANT $MI^{(X)}/G^{(Y)}/1$ SYSTEM WITH MULTIPHASED CAPACITY OF THE SERVICE CHANNEL

by R. S. GAUR (*)

Résumé. — *The paper studies the behaviour of an $MI^{(X)}/G^{(Y)}/1$ system in which the capacity of the server varies, in phases, with the length of queue. The server is made intermittently available. Probability generating functions of the queue length when the server is free and busy are obtained.*

INTRODUCTION

Making investigations in bulk queues on the lines of Bailey (1954), Jaiswal (1960), Conolly (1960) and Foster (1961), Sharda (1968) finds the transient state queue length probabilities of an intermittent $MI^{(X)}/G^{(Y)}/1$ system assuming the capacity of the server, a random variable in the model, to be fixed. According to the assumption, the size of the batch is determined at the beginning of each service and is either equal to the total number of units waiting or to the capacity of the service channel, determined afresh before each service, whichever is less.

The probability that the service channel can serve j units is u_j where $\sum_{j=1}^M u_j = 1$.

The idea that the capacity may vary, in phases, with the queue length provides grounds for further investigations made in this paper. Sharda's results have been shown to follow as a particular case. As is evident from its nature the model will be found to apply to many practical situations.

The Model Description

The system is described as follows :

Customers arrive in a Poisson process with parameter λt in groups of size C_n with the distribution

$$Pr.(C_n = i) = c_i, \quad (i = 1, 2, \dots)$$

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where $\sum_{i=1}^{\infty} c_i = 1$. The batches are preordered for service.

2. The queue discipline is first come first served.

3. The capacity of the service channel is a random variable and has $k + 1$ phases, each phase operating over a definite length-range of the queue. A phase is designated as the l th phase ($l = 0, 1, \dots, k$), the 0th phase being the lower most. When the capacity is in the l th phase the maximum size of the service batch is M_l . The capacity remains in the l th phase so long as the queue length is less than M_{l+1} ($l = 0, 1, \dots, k - 1$) and steps to the $l + 1$ th phase as soon as the queue length becomes M_{l+1} . Of course $M_1 < M_{l+1}$. The capacity operates in the highest phase, that is, the k th phase for a queue $\geq M_k$. The size of the batch is determined at the beginning of each service and is either equal to the total number of units waiting or to the capacity of the service channel determined afresh before each service, whichever is less. The probability that the service channel can serve j units in the l th phase of its capacity is $u_j^{(l)}$ where $\sum_{j=1}^{M_l} u_j^{(l)} = 1$.

4. Service time of each batch is general with the probability density $D(x)$. $\eta(x)\Delta$ is the first order conditional probability that the corresponding service will be completed in time x and $x + \Delta$ if the same has not been completed till time x . $\eta(x)$ is related to $D(x)$ by the relation

$$D(x) = \eta(x) \exp. \left(- \int_0^x \eta(x) dx \right).$$

4. The server is intermittently available, the availability distribution being negatively exponential with mean v^{-1} .

Formulation of Equations

Define :

$P_{n,F}^{(t)}$ \equiv probability that there are n units in the queue at time t and the server is «free» in relation to the queue, i.e., there is neither a unit in service nor is any to be taken at that instant.

$P_{n,B}^{(x,t)}$ \equiv probability that there are n units waiting in the queue at time t and the server is « Busy » in relation to the queue and that the time passed since the service started lies between x and $x + \Delta$. « Busy » in relation to the queue means that a batch is being served or else the same is to be taken for service just then.

For the initial condition, we decide to reckon time from the instant when there is no unit in the queue and the server is free in relation to the queue. Thus

(1) $P_{0,F}^{(0)} = 1.$

From the probability considerations of the process we have the following difference-differential equations governing the system :

$$(2) \quad \frac{\partial P_{0,F}^{(t)}}{\partial t} + \lambda P_{0,F}^{(t)} = \int_0^\infty P_{0,B}^{(x,t)} \eta(x) dx.$$

$$(3) \quad \frac{\partial P_{n,F}^{(t)}}{\partial t} + (\lambda + v) P_{n,F}^{(t)} = \lambda \sum_{i=1}^n c_i P_{n-i,F}^{(t)} + \int_0^\infty P_{n,B}^{(x,t)} \eta(x) dx, \quad 1 \leq n < \infty.$$

$$(4) \quad \frac{\partial P_{0,B}^{(x,t)}}{\partial x} + \frac{\partial P_{0,B}^{(x,t)}}{\partial t} + (\lambda + \eta(x)) P_{0,B}^{(x,t)} = 0.$$

$$(5) \quad \frac{\partial P_{n,B}^{(x,t)}}{\partial x} + \frac{\partial P_{n,B}^{(x,t)}}{\partial t} + (\lambda + \eta(x)) P_{n,B}^{(x,t)} = \lambda \sum_{i=1}^n c_i P_{n-i,B}^{(x,t)}, \quad 1 \leq n < \infty.$$

The relevant boundary condition is obtained from the fact that at the moment of taking a batch for service, there should be n units waiting in the queue and the server should be free in relation to the queue. Thus we have

$$(6) \quad P_{0,B}^{(0,t)} = v \sum_{r=1}^{M_0} P_{r,F}^{(t)} \sum_{j=r}^{M_0} u_j^{(0)} + v \sum_{i=1}^k P_{M_i,F}^{(t)} u_{M_i}^{(i)},$$

$$P_{n,B}^{(0,t)} = v \sum_{j=1}^{M_0} P_{n+j,F}^{(t)} u_j^{(0)} + v \sum_{i=1}^k \sum_{j=M_i-n}^{M_i} P_{n+j,F}^{(t)} u_j^{(i)},$$

$1 \leq n \leq M_1 - M_0 - 1;$

$$P_{n,B}^{(0,t)} = v \sum_{j=1}^{M_i} P_{n+j,F}^{(t)} u_j^{(i)} + v \sum_{l=i+1}^k \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(t)} u_j^{(l)},$$

$$(7) \quad M_i - 1 \leq n \leq M_{i+1} - M_i - 1,$$

$1 \leq i \leq k-1;$

$$P_{n,B}^{(0,t)} = v \sum_{j=1}^{M_{i-n}-1} P_{n+j,F}^{(t)} u_j^{(i-1)} + v \sum_{l=i}^k \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(t)} u_j^{(l)},$$

$M_i - M_{i-1} \leq n \leq M_i - 2,$

$1 \leq i \leq k;$

$$P_{n,B}^{(0,t)} = v \sum_{j=1}^{M_k} P_{n+j,F}^{(t)} u_j^{(k)},$$

$M_k - 1 \leq n < \infty.$

We define the Laplace transforms as follows :

$$(8) \quad P_{n,F}^{(s)} = \int_0^\infty e^{-st} P_{n,F}^{(t)} dt, \quad P_{n,B}^{(x,s)} = \int_0^\infty e^{-st} P_{n,B}^{(x,t)} dt.$$

Taking the Laplace transforms of (2)-(7) and using (1) we have :

$$(9) \quad (s + \lambda + v)P_{n,F}^{(s)} = \lambda \sum_{i=1}^n c_i P_{n-i,F}^{(s)} + \int_0^\infty P_{n,B}^{(x,s)} \eta(x) dx.$$

$$(10) \quad (s + \lambda + v)P_{0,F}^{(s)} = 1 + vP_{0,F}^{(s)} + \int_0^\infty P_{0,B}^{(x,s)} \eta(x) dx.$$

$$(11) \quad \frac{\partial P_{0,B}^{(x,s)}}{\partial x} + (s + \lambda + \eta(x))P_{0,B}^{(x,s)} = 0.$$

$$(12) \quad \frac{\partial P_{n,B}^{(x,s)}}{\partial x} + (s + \lambda + \eta(x))P_{n,B}^{(x,s)} = \lambda \sum_{i=1}^n c_i P_{n-i,B}^{(x,s)}.$$

$$(13) \quad P_{0,B}^{(0,s)} = v \sum_{r=1}^{M_0} P_{r,F}^{(s)} \sum_{j=r}^{M_0} u_j^{(0)} + v \sum_{l=1}^k P_{M_l,F}^{(s)} u_{M_l}^{(l)}.$$

$$P_{n,B}^{(0,s)} = v \sum_{j=1}^{M_0} P_{n+j,F}^{(s)} u_j^{(0)} + v \sum_{l=1}^k \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(s)} u_j^{(l)},$$

$$1 \leq n \leq M_1 - M_0 - 1;$$

$$P_{n,B}^{(0,s)} = v \sum_{j=1}^{M_1} P_{n+j,F}^{(s)} u_j^{(1)} + v \sum_{l=i+1}^k \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(s)} u_j^{(l)},$$

$$(14) \quad M_i - 1 \leq n \leq M_{i+1} - M_i - 1,$$

$$1 \leq i \leq k - 1;$$

$$P_{n,B}^{(0,s)} = v \sum_{j=1}^{M_i-n-1} P_{n+j,F}^{(s)} u_j^{(i-1)} + v \sum_{l=i}^k \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(s)} u_j^{(l)},$$

$$M_i - M_{i-1} \leq n \leq M_i - 2,$$

$$1 \leq i \leq k;$$

$$P_{n,B}^{(0,s)} = v \sum_{j=1}^{M_k} P_{n+j,F}^{(s)} u_j^{(k)},$$

$$M_k - 1 \leq n < \infty.$$

Define the following generating functions :

$$(15) \quad P_F^{(a,s)} = \sum_{n=0}^\infty P_{n,F}^{(s)} a^n, \quad P_B^{(x,a,s)} = \sum_{n=0}^\infty P_{n,B}^{(x,s)} a^n, \quad C(a) = \sum_{i=1}^\infty c_i a^i$$

Multiplying (9) by a^n , summing over all n and using (10) and (15) we have :

$$(16) \quad (s + \lambda + v - \lambda C(a))P_F^{(a,s)} = 1 + vP_{0,F}^{(s)} + \int_0^\infty P_B^{(x,a,s)} \eta(x) dx.$$

Multiplying (12) by a^n , summing over all n and using (11) and (15) we have :

$$(17) \quad \frac{\partial}{\partial x} P_B^{(x,a,s)} + (s + \lambda + \eta(x) - \lambda C(a))P_B^{(x,a,s)} = 0.$$

Multiplying (14) by a^n , summing over all n and using (13) and (15) we have

$$\begin{aligned}
 (18) \quad P_B^{(0,s,a)} = v \left\{ \sum_{j=1}^{M_k} P_F^{(s,a)} \frac{u_j^{(k)}}{a^j} + P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \right. \\
 \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \\
 \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \\
 \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) - \\
 \left. \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} P_{0,F}^{(s)} \right\}
 \end{aligned}$$

Integrating (17) w.r.t.x. between 0 and x and using (18) we get

$$\begin{aligned}
 (19) \quad P_B^{(x,a,s)} = v \left\{ v \sum_{j=1}^{M_k} P_F^{(s,a)} \frac{u_j^{(k)}}{a^j} + P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \right. \\
 \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \\
 \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \\
 \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) - \\
 \left. \sum_{j=1}^{M_k} P_{0,F}^{(s)} \frac{u_j^{(k)}}{a^j} \right\} e^{-(s+\lambda-\lambda C(a))x - \int_0^x \eta(x) dx}
 \end{aligned}$$

Putting the value of $P_B^{(x,a,s)}$ in (16) and integrating we get

$$\begin{aligned}
 1 + v P_{0,F}^{(s)} \left\{ 1 - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} D(s + \lambda - \lambda C(a)) \right\} + v \left\{ P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \right. \\
 + \sum_{r=2}^{M_0} P_{r,F}^{(0)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\
 + \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\
 \left. + \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \right\}
 \end{aligned}$$

$$(20) \quad P_F^{(a,s)} = \frac{D(s + \lambda - \lambda - \lambda C(a))}{s + \lambda + v - \lambda C(a) - v \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} D(s + \lambda - \lambda C(a))}$$

where

$$D(s + \lambda - \lambda C(a)) = \int_0^\infty e^{-(s+\lambda-\lambda C(a))x} D(x) dx.$$

Also

$$(21) \quad P_B^{(a,s)} = \int_0^\infty P_B^{(x,a,s)} dx = v \left\{ \sum_{j=1}^{M_k} P_F^{(s,a)} \frac{u_j^{(k)}}{a^j} \right. \\ + P_{1,F}^{(s)} \left(1 - a' \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=2}^{M_0} u_j^{(0)} \right. \\ \left. + a' \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a' \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\ + \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\ + \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1, l}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\ \left. - \sum_{j=1}^{M_k} P_{0,F}^{(s)} \frac{u_j^{(k)}}{a^j} \right\} \left\{ \frac{1 - D(s + \lambda - \lambda C(a))}{s + \lambda - \lambda C(a)} \right\}.$$

To find $P_{r,F}^{(s)}$ ($r = 0, 1, 2, \dots, M_k - 1$) in (20) we apply Rouché's theorem to its denominator. Let

$$f(a) = a^{M_k}, g(a) = \frac{vD(s + \lambda - \lambda C(a)) \sum_{j=1}^{M_k} u_j^{(k)} a^{M_k-j}}{s + \lambda + v - \lambda C(a)}.$$

Now $f(a)$ and $g(a)$ are analytic inside and on the contour $|a| = 1$. Also on the contour $|a| = 1$, $|f(a)| = 1$ and

$$|g(a)| = v \left| \frac{D(s + \lambda - \lambda C(a)) \sum_{j=1}^{M_k} u_j^{(k)} a^{M_k-j}}{s + \lambda + v - \lambda C(a)} \right|_{|a|=1} \\ = v \left| \frac{\int_0^x D(x) e^{-(s+\lambda-\lambda C(a))x} dx \sum_{j=1}^{M_k} u_j^{(k)} a^{M_k-j}}{s + \lambda + v - \lambda C(a)} \right|_{|a|=1} \\ \leq \frac{v}{\xi + v} \int_0^\infty |D(x) e^{-(s+\lambda-\lambda C(a))x}| |dx|,$$

since $|s + \lambda + v - \lambda C(a)|_{|a|=1} \geq \xi + \lambda + v - \lambda |C(a)|_{|a|=1} \geq \xi + v$. Here $s = \xi + i\eta$.

Because $|e^{sx}| > 1$, if $\xi > 0$, we have

$$|g(a)|_{|a|=1} \leq \frac{v}{\xi + v} \int_0^\infty \frac{D(x)}{e^{(s+\lambda-\lambda C(a))x}} |dx| < \frac{v}{\xi + v} < 1 = |f(a)|_{|a|=1}.$$

Therefore on the contour $|a| = 1$, $|f(a)| > |g(a)|$. Hence all the conditions of Rouché's theorem are satisfied. Therefore $f(a) - g(a)$ and $f(a)$ have the same number of zeros inside the unit circle $|a| = 1$. But $f(a)$ has M_k zeros inside the unit circle so that $f(a) - g(a)$ also has M_k zeros inside this circle.

Since $P_F^{(a,s)}$ is singular inside and on the contour $|a| = 1$, the numerator of (20) must vanish for these M_k zeros of the denominator. Thus the M_k unknowns $P_{n,F}^{(s)} (r = 0, 1, 2, \dots, M_k - 1)$ can be uniquely determined. Hence $P_F^{(a,s)}$ can be completely determined and $P_{n,F}^{(s)}$ can be picked up as the coefficient of a^n in the expansion of $P_F^{(a,s)}$. Substituting the values of $P_F^{(a,s)}$ and $P_{r,F}^{(s)} (r = 0, 1, 2, \dots, M_k - 1)$ in (21) $P_B^{(a,s)}$ can be determined and $P_{n,B}^{(s)}$ can be picked up as the coefficient of a^n in the expansion of $P_B^{(a,s)}$. Hence the queue length distributions when the server is « free » and « busy » are completely determined.

Particular Cases

1. When the service time distribution is exponential with mean u^{-1}

$$D(s + \lambda - \lambda C(a)) = \frac{u}{s + \lambda + u - \lambda C(a)}.$$

Hence $P_F^{(a,s)}$ and $P_B^{(a,s)}$ are given by

$$\begin{aligned} & 1 + v P_{0,F}^{(s)} \left\{ 1 - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \frac{u}{s + \lambda + u - \lambda C(a)} \right\} \\ & + v \left\{ P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \right. \\ & + \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\ & + \sum_{M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\ & \left. + \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \right\} \end{aligned}$$

$$(22) \quad P_F^{(a,s)} = \frac{u}{s + \lambda + u - \lambda C(a)} \frac{1}{s + \lambda + v - \lambda C(a) - uv \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j}}$$

and

$$(23) \quad P_B^{(a,s)} = \frac{v}{s + \lambda + u - \lambda C(a)} \left\{ \sum_{j=1}^{M_k} P_F^{(s,a)} \frac{u_j^{(k)}}{a^j} + P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \right. \\ + \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\ + \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\ + \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(1)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \\ \left. - P_{0,F}^{(s)} \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right\}.$$

For applying Rouché's theorem to the denominator of (22) take

$$f(a) = a^{M_k}, g(a) = \frac{uv \sum_{j=1}^{M_k} u_j^{(k)} a^{M_k-j}}{(s + \lambda + v - \lambda C(a))(s + \lambda + u - \lambda C(a))}.$$

$f(a)$ and $g(a)$ are analytic inside and on the contour $|a| = 1$. Also because

$$\left| \frac{u}{s + \lambda + u - \lambda C(a)} \right| < 1 \quad \text{and} \quad \left| \frac{v}{s + \lambda + v - \lambda C(a)} \right| < 1, \\ |g(a)|_{|a|=1} = \left| \frac{uv \sum_{j=1}^{M_k} a^{M_k-j}}{(s + \lambda + v - \lambda C(a))(s + \lambda + u - \lambda C(a))} \right|_{|a|=1} \\ \leq \frac{uv \sum_{j=1}^{M_k} |a^{M_k-j}|_{|a|=1}}{|s + \lambda + v - \lambda C(a)|_{|a|=1} |s + \lambda + u - \lambda C(a)|_{|a|=1}} \\ < 1 = |f(a)|_{|a|=1}$$

Therefore $|f(a)| > |g(a)|$ on $|a| = 1$.

Thus all the conditions of Rouché's theorem are satisfied and so $f(a)$ and $f(a) - g(a)$ have the same number of zeros inside the contour $|a| = 1$. But since $f(a)$ has M_k zeros inside the unit circle $|a| = 1$, $f(a) - g(a)$ has only M_k

zeros inside this circle. As the numerator of (22) must vanish for these M_k zeros, the M_k unknowns $P_{r,F}^{(s)}$ ($r = 0, 1, 2 \dots M_k - 1$) can be uniquely determined and hence $P_F^{(a,s)}$ is completely known.

Putting the values of $P_F^{(a,s)}$ and $P_{r,F}^{(s)}$ ($r = 0, 1, 2 \dots M_k - 1$) in (23) $P_B^{(a,s)}$ can also be completely determined. $P_{n,F}^{(s)}$ and $P_{n,B}^{(s)}$ can be piecked up as coefficients of a^n in the expensions of $P_F^{(a,s)}$ and $P_B^{(a,s)}$ respectively.

(2) When the capacity of the service channel has only one phase we get the generating functions by taking $k = 0$ and consequently

$$u_j^{(l)} = u_j^{(0)} = u_j M_1 = M_0 = M$$

$$(l = 1, 2, \dots k; j = 1, 2, \dots M_0), \text{ in (20) and (21).}$$

Now since

$$\begin{aligned} 1 - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} D(s + \lambda - \lambda C(a)) &= 1 - \sum_{j=1}^M \frac{u_j}{a^j} D(s + \lambda - \lambda C(a)), \\ P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} \right. \\ &- a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \Big) = P_{1,F}^{(s)} \left(\sum_{j=1}^M u_j - a \sum_{j=1}^M \frac{u_j}{a^j} \right) \\ &+ \sum_{r=2}^M P_{r,F}^{(s)} \left(\sum_{j=r}^M u_j + a^r \sum_{j=1}^{r-1} \frac{u_j}{a^j} - a^r \sum_{j=1}^M \frac{u_j}{a^j} \right) \\ &= P_{1,F}^{(s)} \left(\sum_{j=1}^M u_j - a^r \sum_{j=1}^M \frac{u_j}{a^j} \right) \\ &+ \sum_{r=2}^M P_{r,F}^{(s)} \left(\sum_{j=r}^M u_j - a^r \sum_{j=r}^M \frac{u_j}{a^j} \right) \\ &= \sum_{r=1}^M P_{r,F}^{(s)} \left(\sum_{j=r}^M u_j - a^r \sum_{j=r}^M \frac{u_j}{a^j} \right) \\ &= \sum_{r=1}^M P_{r,F}^{(s)} \sum_{j=r}^M u_j \left(1 - \frac{a^r}{a^j} \right) = \sum_{r=1}^{M-1} P_{r,F}^{(s)} \sum_{j=r+1}^M u_j \left(1 - \frac{a^r}{a^j} \right), \end{aligned}$$

and the terms

$$\begin{aligned} \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right), \\ \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \end{aligned}$$

cease to exist we have

$$(24) \quad P_F^{(a,s)} = \frac{1 + vP_{0,F}^{(s)} \left\{ 1 - \sum_{j=1}^M \frac{u_j}{a^j} D(s + \lambda - \lambda C(a)) \right\} + v \sum_{r=1}^{M-1} P_{r,F}^{(s)} \sum_{j=r+1}^M u_j \left(1 - \frac{a^r}{a^j} \right) D(s + \lambda - \lambda C(a))}{s + \lambda + v - \lambda C(a) - v \sum_{j=1}^M \frac{u_j}{a^j} D(s + \lambda - \lambda C(a))}$$

and

$$(25) \quad P_B^{(a,s)} = v \left\{ \sum_{j=1}^M P_F^{(s,a)} \frac{u_j}{a^j} + \sum_{r=1}^{M-1} \sum_{j=r+1}^M u_j \left(1 - \frac{a^r}{a^j} \right) - P_{0,F}^{(s)} \sum_{j=1}^M \frac{u_j}{a^j} \right\} \left\{ \frac{1 - D(s + \lambda - \lambda C(a))}{s + \lambda - \lambda C(a)} \right\}.$$

As shown for the denominator of (20), the denominator (24) has only M zeros inside the unit circle $|a| = 1$ and since the numerator of (24) must vanish for these M zeros, the M unknowns $P_{r,F}^{(s)}$ ($r = 0, 1, 2 \dots M - 1$) can be uniquely determined and hence $P_F^{(a,s)}$ is completely known.

Putting the values of $P_F^{(a,s)}$ and $P_{r,F}^{(s)}$ ($r = 0, 1, \dots M - 1$) in (25) $P_B^{(a,s)}$ can also be completely determined. The $P_{n,F}^{(s)}$ and $P_{n,B}^{(s)}$ can be picked up as coefficients of a^n in the expansions of $P_F^{(a,s)}$ and $P_B^{(a,s)}$ respectively.

The results (24) and (25) agree with those obtained by Sharda (1968).

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