REVUE FRANÇAISE D'AUTOMATIQUE, D'INFORMATIQUE ET DE RECHERCHE OPÉRATIONNELLE. RECHERCHE OPÉRATIONNELLE

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Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle, tome 8, n° V1 (1974), p. 41-44.

<http://www.numdam.org/item?id=RO_1974__8_1_41_0>

R.A.I.R.O.
(8e année, V-1, 1974, p. 41-44)

ON THE BUSY PERIOD FOR THE M|G|1 QUEUE WITH FINITE WAITING ROOM

by E. J. VANDERPERRE (1)

Summary. — *The Laplace-Stieltjes transform (L.S.T.) of the probability distribution function of the busy period for the M|G|1 queueing system with finite waiting room can be obtained by using a supplementary variable method (a Markov characterization).*

In the present paper, it is shown that applying this method, it is rather easy to analyse queueing systems with a finite waiting room and with Poisson input.

1. INTRODUCTION

For the well known model of the M|G|1 queue with finite waiting room, Cohen [1] obtained the L.S.T. of the distribution function of the busy period in a rather simple way using a Markov renewal branching argument.

In this paper, we use a simple method based on the inclusion of a supplementary variable (Markov characterization). Moreover, the method is not restricted to queueing systems with Poisson input.

2. MATHEMATICAL ANALYSIS OF THE SYSTEM

Denote by λ^{-1} the mean interarrival time of customers and by F(.) the distribution function of the service times. Let $\underline{\theta}_K$ denote the duration of a busy period of the M|G|1 queue with K waiting places. A customer who finds all waiting places occupied, cannot enter the system. He departs and never returns (overflow).

Moreover, let $\underline{X}_t$ be the number of customers waiting in the system at any instant of time t and $\underline{x}_t$ the past service time of the customer being served at time t. It is assumed that a customer arrives at time $t = 0$ and that he meets an empty system.

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The busy period $\underline{\theta}_K$ is the time between $t = 0$ and the first instant at which the system becomes empty (the absorbing state).

For $i = 0, 1, ..., K$ and $t, x > 0$ we define

$$p_i(t, x)\mathrm{d}x \overset{\text{def}}{=} \Pr[\underline{X}_t = i, x \leqslant \underline{x}_t < x + \mathrm{d}x \mid x_0 = 0]$$

By simple probabilistic arguments, we have for $i = 0, 1, \ldots K - 1$

$$p_i(t, x) = \sum_{j=0}^{j=i} p_j(t - x, 0+)\mathrm{e}^{-\lambda x} \frac{(\lambda x)^{i-j}}{(i-j)!}[1 - F(x)]$$

$$p_K(t, x) = \sum_{j=0}^{j=K-1} p_j(t - x, 0+)\left\{1 - \sum_{l=0}^{l=K-1-j} \mathrm{e}^{-\lambda x}\frac{(\lambda x)^l}{l!}\right\}[1 - F(x)]$$

The Laplace transform with respect to t of $p_i\ (t, x)$ will be denoted by $p_i^*(s, x)$; Re $s \geqslant 0$.

If we consider the events in which $\underline{x}_t = 0+$, then we obtain the following set of recurrence relations for the boundary values of the functions $p_i^*(s, x)$; $i = 0, 1, ..., K - 1$,

$$(1)\quad \begin{cases} p_i^*(s, 0+) = \delta_{0i} + \displaystyle\sum_{j=0}^{j=i+1} p_j^*(s, 0+) \int_0^\infty \mathrm{e}^{-sx}\mathrm{e}^{-\lambda x}\frac{(\lambda x)^{i+1-j}}{(i+1-j)!}\,\mathrm{d}F(x) \\ p_{K-1}^*(s, 0+) = \displaystyle\sum_{j=0}^{j=K-1} p_j^*(s, 0+) \\ \qquad\qquad \displaystyle\int_0^\infty \mathrm{e}^{-sx}\left\{1 - \sum_{l=0}^{l=K-1-j}\mathrm{e}^{-\lambda x}\frac{(\lambda x)^l}{l!}\right\}\mathrm{d}F(x) \\ p_K^*(s, 0+) = 0 \end{cases}$$

where δ_{0i} is the Kronecker symbol.

Let

$$f(s) \overset{\text{def}}{=} \int_0^\infty \mathrm{e}^{-sx}\mathrm{d}F(x) \qquad \text{Re } s \geqslant 0$$

$$f_i(s) \overset{\text{def}}{=} \int_0^\infty \mathrm{e}^{-sx}\mathrm{e}^{-\lambda x}\frac{(\lambda x)^i}{i!}\,\mathrm{d}F(x) \qquad i \geqslant 0$$

The L.S.T. of the probability distribution function of the busy period $\underline{\theta}_K$ will be denoted by $E\{\mathrm{e}^{-s\underline{\theta}_K}\}$.

Clearly

$$(2)\qquad E\{\mathrm{e}^{-s\underline{\theta}_K}\} = f_0(s)p_0^*(s, 0+)$$

The following column K-vectors will be used.

$$\bar{p}^*(s, 0+) \stackrel{\text{def}}{=} [p_0^*(s, 0+), \ \ldots, p_i^*(s, 0+), \ \ldots, p_{K-1}^*(s, 0+)]^T$$

$$\bar{1} = [1, 0, \ \ldots, 0, \ \ldots, 0]^T$$

and the $K \times K$ matrix function

$$\mathbf{P}(s) \stackrel{\text{def}}{=} \begin{bmatrix} f_1(s) & f_0(s) & 0 & 0 & . & . & . & 0 & 0 & 0 \\ f_2(s) & f_1(s) & f_0(s) & 0 & . & . & . & 0 & 0 & 0 \\ f_3(s) & f_2(s) & f_1(s) & f_0(s) & 0 & . & . & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{i+1}(s) & f_i(s) & f_{i-1}(s) & \ldots & f_2(s) & f_1(s) & f_0(s) & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{K-1}(s) & f_{K-2}(s) & f_{K-3}(s) & . & . & . & . & f_2(s) & f_1(s) & f_0(s) \\ a_{K-1}(s) & a_{K-2}(s) & a_{K-3}(s) & . & . & . & . & a_2(s) & a_1(s) & a_0(s) \end{bmatrix}$$

where

$$a_i \stackrel{\text{def}}{=} f(s) - \sum_{l=0}^{l=i} f_l(s) \qquad i = 0, 1, \ \ldots, K-1.$$

By (1) we obtain

$$[I - \mathbf{P}(s)]\bar{p}^*(s, 0+) = \bar{1}$$

where I is the unit matrix.

It can be shown that the matrix $[I - \mathbf{P}(s)]$, $0 \leqslant \operatorname{Re} s < \infty$, is nonsingular. For $K \geqslant 0$ we define $\det [I - \mathbf{P}(s)] = \Delta_K(s)$ and $\Delta_0(s) = 1$. By (2) and Cramer's rule, we have

$$(3) \qquad E\{e^{-s\theta_K}\} = f_0(s) \frac{\Delta_{K-1}(s)}{\Delta_K(s)} \qquad K \geqslant 1.$$

But

$$(4) \qquad \Delta_K(s) = \Delta_{K-1}(s) - \sum_{j=1}^{j=K-1} f_0^{j-1}(s) f_j(s) \Delta_{K-j}(s) - f_0^{K-1}(s) a_{K-1}(s).$$

Let

$$(5) \qquad \gamma_{K+1-j}(s) \stackrel{\text{def}}{=} \frac{\Delta_{K-j}(s)}{f_0^{K-j}(s)} \qquad j = 0, 1, \ \ldots, K.$$

By (4) and (5) we obtain after some simplifications

$$\gamma_K(s) = \sum_{j=0}^{j=K-1} f_j(s)\gamma_{K+1-j}(s) + a_{K-1}(s) \qquad K \geqslant 1 \tag{6}$$

Let

$$\gamma(s, Z) \overset{\text{def}}{=} \sum_{K=1}^{K=\infty} \gamma_K(s) Z^K \qquad |Z| < |\rho(s)| \tag{7}$$

where $\rho(s)$ is the smallest root of the functional equation $Z - f(s + \lambda - \lambda Z) = 0$ $\operatorname{Re} s \geqslant 0$.

By (6), (7) and algebra, we find

$$\gamma(s, Z) = \frac{Z}{1 - Z}\left\{ 1 - \frac{Z - Zf(s)}{Z - f(s + \lambda - \lambda Z)} \right\} \tag{8}$$

If $\circlearrowleft$ denote a circle with center at the origin of the complex Z-plane and with radius $|Z| < |\rho(s)|$, Re $s \geqslant 0$, then by (3), (5) and (8) we obtain for $K = 1, 2, \ldots$ Re $s \geqslant 0$,

$$E\left\{e^{-s\underline{\theta}_K}\right\} = \frac{1 - \{1 - f(s)\}\dfrac{1}{2\pi i}\displaystyle\oint \frac{dZ}{Z^{K-1}}(1 - Z)^{-1}[Z - f(s + \lambda - \lambda Z)]^{-1}}{1 - \{1 - f(s)\}\dfrac{1}{2\pi i}\displaystyle\oint \frac{dZ}{Z^{K}}(1 - Z)^{-1}[Z - f(s + \lambda - \lambda Z)]^{-1}}$$

It is easy to show that the result agrees with Cohen's Theorem [1] p. 825.

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