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EXTREMUM CONDITIONS FOR NONLINEAR FUNCTIONS ON CONVEX SETS

by Stefan MITITELU (1)

Abstract. — The paper establishes necessary and sufficient extremum conditions for a function f that is nonlinear, differentiable on a closed convex set generated by quasi-convex and quasi-concave differentiable functions. Conditions of the same type are also established for the case when nonlinear function f is quasi-convex.

* * *

Let C be a convex set in the real Euclidean space R^n .

Definition 1. Any function $f : C \rightarrow R$ is called quasi-convex in C if for any points $x^1, x^2 \in C$ and any number $\lambda \in [0, 1]$ we have

$$(1) \quad f(\lambda x^1 + (1 - \lambda)x^2) \leq \max \{f(x^1), f(x^2)\}.$$

If for any points $x^1, x^2 \in C$, $x^1 \neq x^2$, and any number $\lambda \in (0, 1)$ relation (1) exists with the sign of strict inequality, then function f is called strictly quasi-convex on the set C .

Any strictly quasi-convex function is quasi-convex [7].

Definition 2. The function $f : X \subseteq R^n \rightarrow R$ is called positive definite in the point $x^0 \in X$ if its hessian matrix in x^0 ,

$$H_f(x^0) = \left(\frac{\partial^2 f}{\partial x_i^0 \partial x_j^0} \right)_{1 \leq i, j \leq n}$$

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is positive definite, i.e. if the following conditions are satisfied :

$$\frac{\partial^2 f}{\partial x_1^{02}} > 0, \left[\begin{array}{c} \frac{\partial^2 f}{\partial x_1^{02}} \quad \frac{\partial^2 f}{\partial x_1^0 \partial x_2^0} \\ \frac{\partial^2 f}{\partial x_2^0 \partial x_1^0} \quad \frac{\partial^2 f}{\partial x_2^{02}} \end{array} \right] > 0, \dots, \left[\begin{array}{c} \frac{\partial^2 f}{\partial x_1^{02}} \quad \frac{\partial^2 f}{\partial x_1^0 \partial x_2^0} \quad \frac{\partial^2 f}{\partial x_1^0 \partial x_n^0} \\ \frac{\partial^2 f}{\partial x_2^0 \partial x_1^0} \quad \frac{\partial^2 f}{\partial x_2^{02}} \quad \frac{\partial^2 f}{\partial x_2^0 \partial x_n^0} \\ \dots \dots \dots \\ \frac{\partial^2 f}{\partial x_n^0 \partial x_1^0} \quad \frac{\partial^2 f}{\partial x_n^0 \partial x_2^0} \quad \frac{\partial^2 f}{\partial x_n^{02}} \end{array} \right] > 0.$$

Theorem 1. Let be function f quasi-convex and differentiable on the convex set $C \subseteq R^n$ and a point $x^0 \in C$ where $\nabla f(x^0) = 0$. If function f is positive definite in x^0 , then x^0 is the global minimum of the function f on C .

Proof. Since f is positively defined in $x^0 \in C$ it results that x^0 is a point of strict local minimum of the function f in C . However, for a quasi-convex function any strict local minimum is the global minimum of the function [2], [8].

Theorem 2. Let be functions $f, g = (g_1, \dots, g_m)$ and $h = (h_1, \dots, h_p)$ differentiable on the convex set $K \subseteq R^n$, the convex set $C = \{x \in K/g(x) \leq 0, h(x) = 0\}$, a point $x^0 \in C$ and the set $J^0 = \{i/g_i(x^0) = 0\}$.

We assume that function $g_i (i \in J^0)$ are quasi-convex on K , while h is quasi-convex and quasi-concave on K .

If $u_i^0 \geq 0 (i \in J^0)$ and $v^0 = (v_1^0, \dots, v_p^0) \in R^p$ exist, such that

$$(2) \quad (x - x^0)' \nabla_x L(x^0, u^0, v_0) \geq 0, (\forall) x \in C,$$

where

$$L(x, u, v) = \epsilon f(x) + \sum_{i \in J^0} u_i^0 g_i(x) + v'h(x), \epsilon = +1 [\epsilon = -1]$$

and if

$$(3) \quad (a) \text{ for a } u_{i_0}^0 > 0 \quad (i_0 \in J^0) \text{ we have} \\ (x - x^0)' \nabla g_{i_0}(x^0) < 0, (\forall) x \in C - \{x^0\},$$

or at least for a $v_{j_0}^0 > 0$ we have

$$(x - x^0)' \nabla h_{j_0}(x^0) < 0, \quad (\forall) x \in C - \{x^0\},$$

or

$$(x - x^0)' \nabla h_{j_0}(x^0) > 0, \quad (\forall) x \in C - \{x^0\},$$

then x^0 is a local minimum [maximum] point of the function f on C .

$$(4) \quad (b) \quad C \cap \{x \in R^n / (x - x^0)' \nabla f(x^0) = 0\} = \{x^0\},$$

then x^0 is a local minimum [maximum] of the function f on C .

(c) function f is quasi-convex continuously differentiable in C and $\nabla f(x^0) \neq 0$, then x^0 is the global minimum [maximum] of the function f on C .

Proof. (a) Since relation $h(x) = 0$ is equivalent to relations $h(x) \leq 0$ and $-h(x) \leq 0$ it is enough to prove the theorem for the case when we have only $g(x) \leq 0$. Then only rel. (3) is necessary for the proof.

Since functions g_i are quasi-convex on C , relations

$$g_i(x) \leq g_i(x^0) = 0 (i \in J^0), \quad (\forall)x \in C,$$

imply relations [7] :

$$(5) \quad (x - x^0)' \nabla g_i(x^0) \leq 0, \quad i \in J^0, \quad (\forall)x \in C,$$

where for $i = i_0$ there exists the sign of strict inequality, according to (3).

For $u_i^0 \geq 0, i \in J^0$, where $u_{i_0}^0 > 0$, from (5) it results

$$(6) \quad (x - x^0)' \sum_{i \in J^0} u_i^0 \nabla g_i(x^0) < 0, \quad (\forall)x \in C - \{x^0\}.$$

From rels. (2) and (6) it results

$$(x - x^0)' \nabla f(x^0) > 0, \quad (\forall)x \in C - \{x^0\},$$

and according to [3], P. 36, point x^0 is a local minimum [maximum] of the function f in the set C .

(b) From relations (2) and (5) where $u_i^0 \geq 0, i \in J^0$, it results

$$(7) \quad (x - x^0)' \nabla f(x^0) \geq 0, \quad (\forall)x \in C$$

and taking account of (4), from (7) it results

$$(x - x^0)' \nabla f(x^0) > 0, \quad (\forall)x \in C - \{x^0\}.$$

According to point (a) x^0 is a point of local minimum [maximum] of the function f in C .

(c) Relation (7) result, where f is quasi-convex on C . If $\nabla f(x^0) \neq 0$ from (7) :

$$(8) \quad x^0' \nabla f(x^0) = \min_{x \in C} x' \nabla f(x^0)$$

will result and according to proposition 5 in [5] rel. (8) is equivalent to

$$f(x^0) = \min_{x \in C} f(x)$$

Theorem 3. Let be functions f differentiable and $g = (g_1, \dots, g_m)$ differentiable and quasi-convex on the convex set $K \subseteq R^n$, point $x^0 \in C = \{x \in K/g(x) \leq 0, x \geq 0\}$ with nonempty interior and the set

$$J^0 = \{i/g_i(x^0) = 0\}.$$

(n) A necessary condition that point x^0 should be a local minimum [maximum] of the function f in C is that $u^0 \in R^m$ should exist, so that

$$(9) \quad \begin{aligned} x^0 \geq 0, y^0 &= \varepsilon \nabla f(x^0) + \sum_{i=1}^m u_i^0 \nabla g_i(x^0) \geq 0, x^0, y^0 = 0 \\ g(x^0) &\leq 0, \quad u^0 \geq 0, \quad u^0, g(x^0) = 0, \end{aligned}$$

where $\varepsilon = +1$ [$\varepsilon = -1$] or equivalently :

$$(9') \quad \begin{aligned} x^0 \geq 0, \quad \nabla_x L(x^0, u^0) &\geq 0, \quad x^0, \nabla_x L(x^0, u^0) = 0 \\ u^0 \geq 0, \quad \nabla_u L(x^0, u^0) &\leq 0, \quad u^0, \nabla_u L(x^0, u^0) = 0 \end{aligned}$$

where

$$L(x, u) = \varepsilon f(x) + u'g(x)$$

(S₁) If for a $u_{i_0}^0 > 0$ ($i_0 \in J^0$) we have

$$(10) \quad (x - x^0)' \nabla g_{i_0}(x^0) < 0, (\forall) x \in C - \{x^0\},$$

then relations (9) or (9') are also sufficient so that x^0 should be a point of local minimum [maximum] of the function f in C .

(S₂) If $C \cap \{x \in R^n / (x - x^0)' \nabla f(x^0) = 0\} = \{x^0\}$,

then relations (9) or (9') are also sufficient so that point x^0 should be a local minimum [maximum] of the function f in C .

(S₃) If function f is quasi-convex in C and $\nabla f(x^0) \neq 0$, then relations (9) or (9') are also sufficient so that point x^0 should be the global minimum [maximum] of the function f on C .

Proof. (n) If x^0 is a point of local minimum of the function f in C , then functions f , g and $h = -x$ verify the Kuhn-Tucker conditions in x^0 [9] : there are $u^0 \in R^m$ and $y^0 \in R^n$ such that

$$\begin{aligned}
 \varepsilon \nabla f(x^0) + u^0 \nabla g(x^0) + y^0 \nabla h(x^0) &= 0 \quad (1) \\
 u^0 g(x^0) &= 0 \\
 y^0 h(x^0) &= 0 \\
 u^0 &\geq 0 \\
 y^0 &\geq 0
 \end{aligned}
 \tag{11}$$

However $\nabla h(x^0) = (\underbrace{-1, \dots, -1}_n)$ and relations (11) lead immediately to relations (9).

(S₁) From relations (9') it results

$$(x - x^0)' \nabla_x L(x^0, u^0) \geq 0, \quad (\forall) x \in C$$

The hypotheses of theorem 2 (a) are verified for $h \equiv 0$ and hence x^0 is a point of local minimum [maximum] of the function f in C .

(S₂) The hypotheses of the theorem 2 (b) with $h \equiv 0$ are verified.

(S₃) The hypotheses of the theorem 2 (c) with $h \equiv 0$ are verified.

From theorem 3 the following corollary will result.

Corollary. Let be functions $f, g = (g_1, \dots, g_m)$ and $h = (h_1, \dots, h_p)$ differentiable on an open convex set $K \subseteq R^{n_1} \times R^{n_2}$.

Besides, we assume that function g is quasi-convex on K , while h is quasi-convex and quasi-concave on K .

(n) A necessary condition so that point $(x^0, y^0) \in K$ should be a point of local minimum [maximum] for the function f on the set

$$C = \{ (x, y) \in K / g(x, y) \leq 0, \quad h(x, y) = 0, \quad y \geq 0 \},$$

with nonempty interior is that $u^0 \in R^m$ and $v^0 \in R^p$ should exist so that

$$\begin{aligned}
 x^0 \in R^{n_1}, \quad \nabla_x L(x^0, y^0, u^0, v^0) &= 0 \\
 y^0 \geq 0, \quad \nabla_y L(x^0, y^0, u^0, v^0) \geq 0, \quad y^0 \nabla_y L(x^0, y^0, u^0, v^0) &= 0 \\
 u^0 \geq 0, \quad \nabla_u L(x^0, y^0, u^0, v^0) \leq 0, \quad u^0 \nabla_u L(x^0, y^0, u^0, v^0) &= 0 \\
 v^0 \in R^p, \quad \nabla_v L(x^0, y^0, u^0, v^0) &= 0,
 \end{aligned}
 \tag{12}$$

where

$$L(x, y, u, v) = \varepsilon f(x, y) + u'g(x, y) + v'h(x, y),$$

$$\begin{aligned}
 (1) \quad \nabla g(x^0) &= \begin{pmatrix} \nabla g_1(x^0) \\ \vdots \\ \nabla g_m(x^0) \end{pmatrix}
 \end{aligned}$$

while $\varepsilon = +1$ [$\varepsilon = -1$].

(s₁) If for a $u_{i_0}^0 > 0$ we have

$$(x - x^0)' \nabla_x g_{i_0}(x^0, y^0) + (y - y^0)' \nabla_y g_{i_0}(x^0, y^0) < 0, (\forall)(x, y) \in C - \{(x^0, y^0)\}$$

or at least for a $v_{j_0}^0 > 0$ we have

$$(x - x^0)' \nabla_x h_{j_0}(x^0, y^0) + (y - y^0)' \nabla_y h_{j_0}(x^0, y^0) < 0, (\forall)(x, y) \in C - \{(x^0, y^0)\}$$

or

$$(x - x^0)' \nabla_x h_{j_0}(x^0, y^0) + (y - y^0)' \nabla_y h_{j_0}(x^0, y^0) > 0, (\forall)(x, y) \in C - \{(x^0, y^0)\}$$

the relations (12) are also sufficient, such that point (x^0, y^0) should be a local minimum [maximum] of the function f on C .

(s₂) If

$$C \cap \{ (x, y) \in R^{n_1} \times R^{n_2} / (x - x^0)' \nabla_x f(x^0, y^0) + \\ + (y - y^0)' \nabla_y f(x^0, y^0) = 0 \} = \{ (x^0, y^0) \},$$

the relations (12) are also sufficient such that point (x^0, y^0) should be a local minimum [maximum] of the function f on C .

(s₃) If function f is quasi-convex in C and $\nabla f(x^0, y^0) \neq 0$ then relations (12) are also sufficient such that point (x^0, y^0) should be the global minimum [maximum] of the function f on C .

REMARK. If the vectorial function $g(x)$ is quasi-concave then there are the theorems dual to theorem 2 and 3.

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