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*Revue française d'automatique, d'informatique et de recherche
opérationnelle. Recherche opérationnelle*, tome 8, n° V2 (1974),
p. 119-124.

http://www.numdam.org/item?id=RO_1974__8_2_119_0

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FIXED COST MINIMIZATION OVER A LEONTIEF SUBSTITUTION SYSTEM ⁽¹⁾

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Résumé. — Using Bender's decomposition method and properties of a Leontief substitution system, a formulation for solving fixed cost minimizations over Leontief substitution systems is presented.

It has been pointed out in the literature [2] that Bender's decomposition method appears useful as a solution strategy. Using Bender's method and the properties of a Leontief substitution system, a solution procedure is specified for solving fixed cost minimization problems over a Leontief substitution system. Preliminary results indicate that for some large problems, this procedure is a viable alternative to procedures specified elsewhere [4].

A Leontief substitution system is characterized by a constraint matrix having exactly one positive element per column and non-trivial rows [3]. Substitute activities are identified by columns whose positive elements fall in the same row. A salient feature of Leontief substitution systems is that one and only one activity is optimal for each set of substitute activities.

Consider the problem

$$\begin{aligned} & \text{Min } f(x) \\ & \text{subject to} \\ & \quad Ax = b \\ & \quad x \geq 0 \\ & \quad b \geq 0 \end{aligned}$$

where A is m by n and Leontief, x is n by 1, b is m by 1, and $f(\cdot)$ is a quasi-concave function of the form :

$$f(x) = \sum_{i=1}^n f_i(x_i)$$

(1) This research was supported by the Office of Army Research under contract # DA-31-124-ARO-D-477.

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where

$$f_i(x_i) = \begin{cases} c_i x_i + d_i & x_i > 0 \\ 0 & x_i = 0 \end{cases} \quad d_i \geq 0$$

The above problem is termed a fixed cost minimization over a Leontief substitution system.

Let the Set J_i contain the column identifications for all columns whose positive coefficients fall in row i . Associate with each continuous variable x_i a binary variable y_i and reformulate the fixed cost problem as :

$$(1) \quad \text{Min } c'x + d'y$$

subject to

$$\begin{aligned} Ax &= b \\ x - ly &\leq 0 \\ x &\geq 0 \\ b &\geq 0 \\ y &\in Y \end{aligned}$$

where

$$Y = \{ y : y \text{ is binary and } \sum_{j \in J_i} y_j = 1, i = 1, 2, \dots, m \}$$

and

$$l > \bar{x}_i$$

for all possible x_i . The requirement that

$$\sum_{j \in J_i} y_j = 1 \quad i = 1, 2, \dots, m$$

is a result of the substitution property of Leontief substitution systems.

The problem given in equation (1) may be reformulated as :

$$\text{Min } d'y + \left[\begin{array}{l} \text{Min } c'x \\ x \\ \text{subject to} \\ Ax = b \\ x \leq ly \\ x \geq 0 \\ b \geq 0 \end{array} \right]$$

Using Bender's decomposition method [1], an approximation to the outer problem is now given. Let N denote the number of times the inner problem has been solved. Also, let π_1^i be the dual multipliers to the constraint set

$$Ax = b$$

at the solution of the inner problem at the i^{th} iteration. Likewise, let π_2^i be the dual multiplier to the constraint set

$$x \leq ly$$

at the i^{th} iteration. The approximation to the outer problem is

$$(2) \quad \text{Min } z + d'y$$

subject to

$$z \geq b'\pi_1^i + ly'\pi_2^i \quad i = 1, 2, \dots, N$$

$$y \in Y$$

Given a $y \in Y$, the inner problem is easily solved. Let y^i be the fixed y vector supplied to the inner problem at the beginning of iteration i . Then the solution to the inner problem is :

$$(3) \quad \pi_1^i = (A'_K)^{-1}c_K$$

where

$$K = \{j : y_j^i = 1\}$$

and A'_K is the transpose of the submatrix of A consisting of the columns in the set K . Also

$$(4) \quad \pi_2^i = [c - A'\pi_1^i]^+$$

where $[a]^+$ is a vector of $[a_i]^+$ $i = 1, 2, \dots, n$, and $[a_i]^+ = \min [a_i, 0]$.

For computational purposes, it is beneficial to note that if A_K is recursive [4], as is usually the case, then $[A'_K]^{-1}$ is easily computed.

As an example, consider an arborescence-structured production and inventory system. Let x_i^K , y_i^K , and r_i^K be, respectively, the production, end-of-period inventory, and known requirements at facility K in period i . For illustrative purposes let the number of facilities be 2 and the number of periods be 3. The arborescence model is

$$\text{Min } f(x, y)$$

subject to

$$\begin{aligned} y_{i-1}^1 + x_i^1 - y_i^1 - x_i^2 &= r_i^1 & i = 1, 2, 3 \\ y_{i-1}^2 + x_i^2 - y_i^2 &= r_i^2 & i = 1, 2, 3 \\ y_0^1 = y_0^2 = y_3^1 = y_3^2 &= 0 \\ x_i^K, y_i^K &\geq 0 \end{aligned}$$

where

$$f(x, y) = \sum_{K=1}^2 \sum_{i=1}^3 \{f_i^K(x_i^K) + f_i^K(y_i^K)\}$$

and

$$f_i^K(x_i^K) = \begin{cases} c_i^K x_i^K + d_i^K & x_i^K > 0 & d_i^K \geq 0 \\ 0 & x_i^K = 0 \end{cases}$$

and $f_i^K(y_i^K)$ is similarly defined. Let

$$\begin{aligned} r^1 &= \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} & C_x^2 &= \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} & C_y^2 &= \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} & d_x^1 &= \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} & d_y^1 &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\ C_x^1 &= \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} & C_y^1 &= \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} & r^2 &= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} & d_x^2 &= \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} & d_y^2 &= \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \end{aligned}$$

By the structure of the problem, the set of all feasible basis is

$$K = \bigtimes_{i=1}^6 J_i$$

where

$$\begin{aligned} J_1 &= \{x_1^1\} & J_4 &= \{x_1^2\} \\ J_2 &= \{y_1^1, x_2^1\} & J_5 &= \{y_1^2, x_2^2\} \\ J_3 &= \{y_2^1, x_3^1\} & J_6 &= \{y_2^2, x_3^2\} \end{aligned}$$

To start the solution procedure, an arbitrary initial feasible solution is specified. Let the optimal solution to the continuous portion of the problem be the initial starting basis. Then

$$K_1 = \{x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, y_2^2\}$$

and the outer problem can be written as follows. Let the continuous variable identifications, x_i^K and y_i^K represent their binary counterparts.

Then equation (2) becomes :

$$\text{Min } z + d'_x x + d'_y y$$

subject to

$$z \geq 54$$

Then

$$K_2 = \{ x_1^1, y_1^1, y_2^1, x_1^2, y_1^2, x_3^2 \}$$

or

$$K_2 = \{ x_1^1, y_1^1, x_3^1, x_1^2, y_1^2, x_3^2 \}$$

and the outer problem becomes (using the first K_2) :

$$\text{Min } z + d'_x x + d'_y y$$

subject to

$$z \geq 54$$

$$z \geq 87 - 200x_2^1 - 300x_3^1 - 500x_2^2$$

where $l = 100$.

Continuing in like manner and using all alternative optimal points to form constraints, the following was found.

Iteration

3	$x_1^1, y_1^1, x_3^1, x_1^2, y_1^2, x_3^2$	$z \geq 78 - 200x_2^1 - 500x_2^2$
4	$x_1^1, x_2^1, x_3^1, x_1^2, y_1^2, x_3^2$	$z \geq 70 - 700x_2^2$
	$x_1^1, x_2^1, y_2^1, x_1^2, y_1^2, x_3^2$	$z \geq 73 - 100x_3^1 - 700x_2^2$
5	$x_1^1, y_1^1, y_2^1, x_1^2, x_2^2, x_3^2$	$z \geq 77 - 200x_2^1 - 300x_3^1 - 300y_2^2$
	$x_1^1, y_1^1, x_3^1, x_1^2, x_2^2, x_3^2$	$z \geq 68 - 200x_2^1$
6	$x_1^1, x_2^1, y_2^1, x_1^2, x_2^2, x_3^2$	$z \geq 59 - 100x_3^1 - 300y_2^2$
	$x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, x_3^2$	$z \geq 56 - 200y_2^2$
7	$x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, x_3^3$	

STOP

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