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OPTIMUM PREVENTIVE MAINTENANCE AND REPAIR LIMIT POLICIES MAXIMIZING THE EXPECTED EARNING RATE (*)

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Abstract. — *Optimum preventive maintenance policies and optimum repair time limit policies are derived for a system which generates earnings and is subject to failure and repair.*

1. INTRODUCTION

Consider a system which is repaired upon failure and then returned to operation. Morse [9] defined the preventive maintenance policy of a single operating system, in which it is repaired at failure or at age t_0, whichever occurs first. Barlow and Hunter [1] obtained the optimum policy maximizing the steady-state availability (or limiting efficiency) for an infinite time span. They also pointed out that this problem is formally the same as an age replacement problem if a mean time to repair is replaced by a replacement cost of a failed unit and a mean time to maintain preventively by an exchange cost of a nonfailed unit. Barlow and Proschan [3] derived the optimum age replacement policy minimizing the expected cost per unit of time for an infinite time span. They also summarized earlier results of the other replacement problems in their book [3]. Several interesting variations of such an age replacement problem were considered by Fox [6], Scheaffer [12], Cleroux and Hanscom [4], Nakagawa and Osaki [10], and others. In earlier contributions, almost all models introduced only maintenance costs in preventive maintenance problems (or replacement costs in age replacement problems), whereas we should also consider earnings that the system produces.

This paper considers preventive maintenance of the system which is repaired at failure or at age t_0. Then, introducing both maintenance costs and a net earning of the working system, we obtain an optimum policy maximizing the expected earning rate under suitable conditions. Such a policy is given by a unique solution of an integral equation. We also give an upper limit of an optimum preventive maintenance time t_0^*.

An alternative technique of preventive maintenance of the system is the repair limit policies of a failed unit discussed by Hastings *et al.* [5, 7, 8], and Nakagawa and Osaki [11]. A failed unit is replaced if the repair is not completed within a repair limit time t_0. We introduce maintenance costs and a net earning rate, and give an optimum repair limit policy maximizing

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the expected earning rate. An optimum policy is obtained in a similar manner to the previous results, exchanging the failure rate for the repair rate. Hastings *et al.* considered these costs and obtained the optimum repair limit policies by three methods of simulation, hill-climbing, and dynamic programming.

2. OPTIMUM PREVENTIVE MAINTENANCE POLICY

Consider a one-unit system. If the unit fails then it undergoes repair. Assume that the failure time distribution of the unit is an arbitrary $F(t)$ with finite mean $1/\lambda$ and the repair time distribution is an arbitrary $G(t)$ with finite mean $1/\mu$.

Consider a preventive maintenance policy. When the unit works for a specified time t_0 without failure, we stop the operation of the working unit for preventive maintenance. The distribution of the time to preventive maintenance completion is assumed to be the same as the repair time distribution $G(t)$. We assume that the unit is as good as new after the repair or the preventive maintenance.

Let c_0 be the net earning rate per unit of time made by the production of the working system. Let c_1 be the earning rate per unit of time while the unit is under preventive maintenance, and c_2 the earning rate per unit of time while the unit is under repair. Both c_1 and c_2 will usually be negative. We also assume that $c_0 > c_1 > c_2$.

Let $P_0(t)$ be the probability that the unit is operating at time t, $P_1(t)$ be the probability that the unit is under preventive maintenance and $P_2(t)$ be the probability that the unit is under repair. Then, the expected earning rate is given by, from Barlow and Hunter [1, 2],

$$C_1(t_0) = \lim_{t\to\infty} \frac{1}{t}\left[c_0\int_0^t P_0(t)\,dt + c_1\int_0^t P_1(t)\,dt + c_2\int_0^t P_2(t)\,dt\right]$$

$$= \frac{c_0\int_0^{t_0}\bar{F}(t)\,dt + (c_1/\mu)\bar{F}(t_0) + (c_2/\mu)F(t_0)}{\int_0^{t_0}\bar{F}(t)\,dt + 1/\mu}, \tag{1}$$

where $\bar{F}(t) \equiv 1-F(t)$ and which becomes equal to the expected earning per one cycle from the begining of the working unit to the completion of the repair (or the preventive maintenance).

We wish to obtain an optimum preventive maintenance time t_0^* maximizing the expected earning rate $C_1(t_0)$ in (1). Let $f(t)$ be the density of the failure time distribution $F(t)$. Let $r(t) \equiv f(t)/\bar{F}(t)$ be the failure rate. Then, we have the following theorem:

THEOREM 1: 1) *Suppose that the failure rate $r(t)$ is continuous and monotonely increasing.*

(i) *If* $r(0) < m_1$ *and* $r(\infty) > M_1$ *then there exists a finite and unique optimum preventive maintenance time* t_0^* *which satisfies the following equation:*

$$r(t_0)\left[\int_0^{t_0}\bar{F}(t)\,dt+1/\mu\right]+\bar{F}(t_0)=(c_0-c_2)/(c_1-c_2), \qquad (2)$$

and

$$C_1(t_0^*)=c_0-r(t_0^*)\,[(c_1-c_2)/\mu]. \qquad (3)$$

(ii) *If* $r(\infty) \leqq M_1$, *then the optimum preventive maintenance time is* $t_0^* = \infty$, *i. e., no preventive maintenance, and*

$$C_1(\infty)=(c_0/\lambda+c_2/\mu)/(1/\lambda+1/\mu). \qquad (4)$$

(iii) *If* $r(0) \geqq m_1$, *then the optimum preventive maintenance time is* $t_0^* = 0$, *i. e., preventive maintenance is made just upon repair completion or preventive maintenance completion, and* $C_1(0) = c_1$, *where*

$$m_1=(c_0-c_1)/[(c_1-c_2)/\mu], \qquad (5)$$

$$M_1=(c_0-c_2)/[(1/\lambda+1/\mu)(c_1-c_2)]. \qquad (6)$$

2) *Suppose that the failure rate* $r(t)$ *is continuous and non-increasing.*

(i) *If* $(c_0-c_1)/\lambda > (c_1-c_2)/\mu$ *then* $t_0^* = \infty$.

(ii) *If* $(c_0-c_1)/\lambda \leqq (c_1-c_2)/\mu$ *then* $t_0^* = 0$.

Proof: Differentiating $C_1(t_0)$ with respect to t_0 and putting it equal to zero imply equation (2). Define

$$q(t_0)\equiv r(t_0)\left[\int_0^{t_0}\bar{F}(t)\,dt+1/\mu\right]+\bar{F}(t_0). \qquad (7)$$

First suppose that $r(t)$ is monotonely increasing. Then, it is easily seen that $q(t_0)$ is also monotonely increasing. If $r(0) < m_1$ and $r(\infty) > M_1$ then $q(0) < k < q(\infty)$, where $k = (c_0-c_2)/(c_1-c_2)$. Thus, from the monotonicity and the continuity of $q(t_0)$, there exists a unique and finite t_0^* $(0 < t_0^* < \infty)$ satisfying equation (2), which maximizes the expected earning rate $C_1(t_0)$. Further, substituting (2) into (1) and rearranging it, we have (3).

If $r(\infty) \leqq M_1$, then $q(\infty) \leqq k$. Thus, $q(t_0) < k$ for any finite t_0, which implies $t_0^* = \infty$, i. e., no preventive maintenance.

If $r(0) \geqq m_1$, then $q(0) \geqq k$. Thus, $q(t_0) > k$ for any positive t_0, which implies $t_0^* = 0$.

Next suppose that $r(t)$ is non-increasing. Then, $q(t_0)$ is also non-increasing. Thus, it is easily seen that $C_1(0)$ or $C_1(\infty)$ is not less than $C_1(t_0)$ with any t_0. Therefore, we have $t_0^* = \infty$ if $C_1(\infty) > C_1(0)$, i. e., $(c_0-c_1)/\lambda > (c_1-c_2)/\mu$, and vice versa.

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In a case of (i) of Theorem 1, we have an upper limit of the optimum preventive maintenance time t_0^* which helps us to compute t_0^*.

THEOREM 2: *Suppose that the failure rate* $r(t)$ *is continuous and monotonely increasing, and* $r(0) < m_1$ *and* $r(\infty) > M_1$.

(i) *If* $(c_0 - c_1)/\lambda > (c_1 - c_2)/\mu$ *then* $t_0^* < \hat{t}_0$, *which is a unique solution of* $r(t_0) = M_1$.

(ii) *If* $(c_0 - c_1)/\lambda \leqq (c_1 - c_2)/\mu$ *then* $t_0^* < \tilde{t}_0$, *which is a unique solution of* $r(t_0) = m_1$.

Proof: From the assumption that $r(t)$ is monotonely increasing, we easily have

$$\bar{F}(t_0) \Big/ \int_{t_0}^{\infty} \bar{F}(t)\,dt > r(t_0) \qquad \text{for} \quad 0 \leqq t_0 < \infty$$

and

$$F(t_0) \Big/ \int_{0}^{t_0} \bar{F}(t)\,dt < r(t_0) \qquad \text{for} \quad 0 < t_0 \leqq \infty.$$

Thus,

$$q(t_0) > r(t_0)(1/\lambda + 1/\mu) \qquad \text{for} \quad 0 \leqq t_0 < \infty, \tag{8}$$

$$q(t_0) > r(t_0)(1/\mu) + 1 \qquad \text{for} \quad 0 < t_0 \leqq \infty. \tag{9}$$

Let $\hat{t}_0$ satisfy the equation $r(t_0) = M_1$. Then, there exists a $\hat{t}_0$ uniquely from the assumption that $r(0) < m_1$ and $r(\infty) > M_1$, and hence, $t_0^* < \hat{t}_0$ from the inequality (8). On the other hand, if a $\tilde{t}_0$ is a solution of the equation $r(t_0) = m_1$ or $\tilde{t}_0 = \infty$ when there does not exist a solution, then $t_0^* < \tilde{t}_0$ from the inequality (9).

Therefore, we have $t_0^* < \hat{t}_0$ if $M_1 < m_1$, i. e., $(c_0 - c_1)/\lambda > (c_1 - c_2)/\mu$. If $M_1 \geqq m_1$, i. e., $(c_0 - c_1)/\lambda \leqq (c_1 - c_2)/\mu$ then $t_0^* < \tilde{t}_0$, where in this case there always exists the solution $\tilde{t}_0$ of $r(t_0) = m_1$ uniquely.

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We have discussed the optimum preventive maintenance policy under certain assumptions. We can relax some of these conditions. For instance, if the time to preventive maintenance completion is different from the repair time and has a distribution $G_1(t)$ with mean $1/\mu_1$. Then, the expected earning rate is

$$C_1(t_0) = \frac{c_0 \int_0^{t_0} \bar{F}(t)\,dt + (c_1/\mu_1)\bar{F}(t_0) + (c_2/\mu) F(t_0)}{\int_0^{t_0} \bar{F}(t)\,dt + (1/\mu_1)\bar{F}(t_0) + (1/\mu) F(t_0)}. \tag{10}$$

3. OPTIMUM REPAIR LIMIT POLICY

Consider a one-unit system which is repaired or replaced when it fails. If the unit fails, the repair of the failed unit is started immediately and if it is not completed within a time t_0, it is replaced by a new one. The fixed time t_0

is called *the repair limit time* of the failed unit. Let c_3 be the earning that is gained for replacing a failed unit, which is not repaired within a time t_0; this includes all earnings from its replacement and is negative. Both c_0 and c_2 are defined as the same earning rates in Section 2. Then, from Nakagawa and Osaki [11], the expected earning rate is

$$C_2(t_0)=\frac{c_0/\lambda+c_2\int_0^{t_0}\bar{G}(t)\,dt+c_3\bar{G}(t_0)}{\int_0^{t_0}\bar{G}(t)\,dt+1/\lambda}. \tag{11}$$

We also wish to obtain an optimum repair limit time t_0^* maximizing the expected earning rate $C_2(t_0)$ in (11). Let $g(t)$ be the density of the repair time distribution $G(t)$. Let $h(t)\equiv g(t)/\bar{G}(t)$ be the repair rate.

THEOREM 3: 1) *Suppose that the repair rate $h(t)$ is continuous and monotonely decreasing.*

(i) *If $h(0)>m_2$ and $h(\infty)<M_2$ then there exists a finite and unique optimum repair limit time t_0^* which satisfies the following equation:*

$$h(t_0)\left[\int_0^{t_0}\bar{G}(t)\,dt+1/\lambda\right]+\bar{G}(t_0)=(c_0-c_2)/[\lambda(-c_3)], \tag{12}$$

and

$$C_2(t_0^*)=c_2-c_3\,h(t_0^*). \tag{13}$$

(ii) *If $h(\infty)\geqq M_2$ then $t_0^*=\infty$, i.e., the optimum repair limit policy is only repair (no replacement) and*

$$C_2(\infty)=(c_0/\lambda+c_2/\mu)/(1/\lambda+1/\mu). \tag{14}$$

(iii) *If $h(0)\leqq m_2$ then $t_0^*=0$, i. e., the optimum repair limit policy is only replacement (no repair) and*

$$C_2(0)=(c_0/\lambda+c_3)/(1/\lambda), \tag{15}$$

where

$$m_2=[(c_0-c_2)/\lambda+c_3]/[(-c_3)/\lambda], \tag{16}$$

$$M_2=[(c_0-c_2)/\lambda]/[(1/\lambda+1/\mu)(-c_3)]. \tag{17}$$

2) *Suppose that the repair rate $h(t)$ is continuous and non-decreasing.*

(i) *If $(c_0-c_2)/(\lambda\mu)<(-c_3)(1/\lambda+1/\mu)$ then $t_0^*=\infty$.*

(ii) *If $(c_0-c_2)/(\lambda\mu)\geqq(-c_3)(1/\lambda+1/\mu)$ then $t_0^*=0$.*

The above theorem is proved in a similar way Theorem 1. Note that $C_2(\infty)$ agrees with $C_1(\infty)$ in (1).

THEOREM 4: *Suppose that the repair rate $h(t)$ is continuous and monotonely decreasing, and $h(0)>m_2$ and $h(\infty)<M_2$.*

(i) *If* $(c_0-c_2)/(\lambda\mu) < (-c_3)(1/\lambda+1/\mu)$ *then* $t_0^* < \hat{t}_0$ *which is a unique solution of* $h(t_0) = M_2$.

(ii) *If* $(c_0-c_2)/(\lambda\mu) \geqq (-c_3)(1/\lambda+1/\mu)$ *then* $t_0^* \leqq \tilde{t}_0$ *which is a unique solution of* $h(t_0) = m_2$.

We omit the proof of the above theorem because this is easily shown by the following two inequalities:

$$h(t_0)\left[\int_0^{t_0} \bar{G}(t)\,dt+1/\lambda\right]+\bar{G}(t_0) < h(t_0)(1/\lambda+1/\mu) \qquad \text{for} \quad 0 \leqq t_0 < \infty, \tag{18}$$

$$h(t_0)\left[\int_0^{t_0} \bar{G}(t)\,dt+1/\lambda\right]+\bar{G}(t_0) < h(t_0)(1/\mu)+1 \qquad \text{for} \quad 0 < t_0 \leqq \infty. \tag{19}$$

We have discussed the case that the repair time of a failed unit is not estimated when the unit fails. However, if the repair time can be estimated when the operative unit fails and we can estimate whether it should be repaired or replaced, then

$$C_2(t_0) = \frac{c_0/\lambda+c_2\displaystyle\int_0^{t_0} t\,dG(t)+c_3\bar{G}(t_0)}{\displaystyle\int_0^{t_0} t\,dG(t)+1/\lambda}. \tag{20}$$

REFERENCES

1. R. E. Barlow and L. C. Hunter, *Optimum Preventive Maintenance Policies*, Opns. Res., Vol. 8, 1960, pp. 90-100.
2. R. E. Barlow and L. C. Hunter, *Reliability Analysis of a One-Unit System*, Opns. Res., Vol. 9, 1961, pp. 200-208.
3. R. E. Barlow and F. Proschan, *Mathematical Theory of Reliability*, Wiley, New York, 1965.
4. R. Cleroux and M. Hanscom, *Age Replacement with Adjustment and Depreciation Costs and Interest Charges*, Technometrics, Vol. 16, 1974, pp. 235-239.
5. R. W. Drinkwater and N. A. J. Hastings, *An Economic Replacement Model*, Opnl. Res. Q., Vol. 18, 1967, pp. 121-138.
6. B. Fox, *Age Replacement with Disconuting*, Opns. Res., Vol. 14, 1966, pp. 533-537.
7. N. A. J. Hastings, *Some Notes on Dynamic Programming and Replacement*, Opnl. Res. Q., Vol. 19, 1968, pp. 453-464.
8. N. A. J. Hastings, *The Repair Limit Replacement Method*, Opnl. Res. Q., Vol. 20, 1969, pp. 337-349.
9. P. M. Morse, *Queues, Inventories, and Maintenance*, Wiley, New York, 1958.
10. T. Nakagawa and S. Osaki, *Optimum Replacement Policies with Delay*, J. Appl. Prob., Vol. 11, 1974, pp. 102-110.
11. T. Nakagawa and S. Osaki, *The Optimum Repair Limit Replacement Policies*, Opnl. Res. Q., Vol. 25, 1974, pp. 311-317.
12. R. L. Scheaffer, *Optimum Age Replacement Policies with an Increasing Cost Factor*, Technometrics, Vol. 13, 1971, pp. 139-144.