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Lajos Takács

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A QUEUING MODEL WITH FEEDBACK (*)

by Lajos TAKÁCS ([1])

Abstract. — *In the time interval* $(0, \infty)$ *customers arrive at a service system in accordance with a Poisson process and form a queue in a waiting room. The customers are served by a single server in a service room in order of arrival. The service times are mutually independent random variables having a common exponential distribution. After each service a customer may return to the waiting room with a constant probability. Every time the service room becomes empty all the customers in the waiting room and additional r customers* ($r \geqq 1$) *enter the service room. In this paper the distributions of the queue size, the waiting time and the total time spent in the system by a customer are determined for a stationary process.*

1. INTRODUCTION

The object of this paper is to find the limit distributions (stationary distributions) of the queue size, the waiting time and the total time spent in the system by a customer for a single-server queue with feedback. It is assumed that in the time interval $(0, \infty)$ customers arrive at a service system in accordance with a Poisson process of density λ and form a queue in a waiting room. The customers are served by a single server in a service room in order of arrival. The service times are mutually independent random variables having the same distribution function

$$H(x) = \begin{cases} 1 - e^{-\mu x} & \text{for} \quad x \geqq 0, \\ 0 & \text{for} \quad x < 0, \end{cases} \tag{1}$$

and are independent of the arrival times. After each service a customer may return to the waiting room with probability p where $0 \leqq p < 1$ or may depart permanently with probability $q = 1 - p$. Every time the service room becomes empty, all the customers in the waiting room and additional r customers ($r \geqq 1$) enter the service room.

Denote by $\xi_1(t)$ the number of customers in the waiting room and by $\xi_2(t)$ the number of customers in the service room at time t. The possible values of $\xi_1(t)$ are $0, 1, 2, \ldots$ and the possible values of $\xi_2(t)$ are $1, 2, \ldots$. We assume that the arrival times, the service times and the events of returns are independent of $\xi_1(0)$ and $\xi_2(0)$.

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([1]) Case Western Reserve University, Cleveland, Department of Mathematics and Statistics.

In the particular case when $p = 0$, that is, when there is no feedback the aforementioned model is equivalent to a counter model introduced by D. G. Lampard [1]. See also R. M. Phatarfod [2], [3], and the author [5].

2. THE LIMIT DISTRIBUTION OF THE QUEUE SIZE

We have the following result.

THEOREM 1 : *If* $\lambda < \mu q$, *then the limit distribution*

$$\lim_{t \to \infty} \mathbf{P}\{\xi_1(t) = i, \xi_2(t) = j\} = P_{ij} \tag{2}$$

exists for $i \geqq 0$ *and* $j \geqq 1$ *and is independent of the joint distribution of* $\xi_1(0)$ *and* $\xi_2(0)$. *The generating function*

$$P(w, z) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{ij} w^i z^j \tag{3}$$

is given by

$$P(w, z) = \frac{(\mu q - \lambda) z \{[Q(w)]^r - [zQ(z)]^r\}}{r[\mu q + \mu p w - (\lambda + \mu) z + \lambda w z]} \tag{4}$$

for $|w| \leqq 1$ *and* $|z| \leqq 1$ *where*

$$Q(z) = \sum_{n=0}^{\infty} A_n (\lambda z - \lambda)^n (\lambda z - \mu q)^{-n}, \tag{5}$$

$A_0 = 1$ *and*

$$A_n = A_{n-1}\left[p + \frac{\lambda}{\mu} - \frac{\mu q}{\lambda}\left(p + \frac{\lambda}{\mu}\right)^n\right]\left[1 - \left(p + \frac{\lambda}{\mu}\right)^n\right]^{-1} \tag{6}$$

for $n \geqq 1$. *If* $\lambda \geqq \mu q$, *then* $\lim_{t \to \infty} \mathbf{P}\{\xi_1(t) = i, \xi_2(t) = j\} = 0$ *regardless of the joint distribution of* $\xi_1(0)$ *and* $\xi_2(0)$.

Proof: The random variables $\{\xi_1(t), \xi_2(t)\}$ form a homogeneous and irreducible Markov process with state space

$$\{(i, j) : i = 0, 1, 2, \ldots, j = 1, 2, \ldots\}.$$

If the process $\{\xi_1(t), \xi_2(t)\}$ has a stationary distribution, then the limit distribution (2) exists and is identical with the stationary distribution. If $\{\xi_1(t), \xi_2(t)\}$ has no stationary distribution, then

$$\lim_{t \to \infty} \mathbf{P}\{\xi_1(t) = i, \xi_2(t) = j\} = 0$$

for all $i \geqq 0$ and $j \geqq 1$.

If we suppose that $\{ P_{ij} \}$ is a stationary distribution, then it satisfies the following system of equations

$$(\lambda + \mu)P_{ij} = \lambda P_{i-1,j} + \mu p P_{i-1,j+1} + \mu q P_{i,j+1} \tag{7}$$

for $i \geqq 1, j \geqq 1$,

$$(\lambda + \mu)P_{0j} = \mu q P_{0,j+1} \tag{8}$$

for $1 \leqq j < r$,

$$(\lambda + \mu)P_{0r} = \mu q P_{01} + \mu q P_{0,r+1} \tag{9}$$

$$(\lambda + \mu)P_{0,j+r} = \mu p P_{j-1,1} + \mu q P_{j,1} + \mu q P_{0,j+r+1} \tag{10}$$

for $j \geqq 1$ and

$$\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{ij} = 1. \tag{11}$$

Define $P(w, z)$ by (3). By (7), (8), (9), and (10) we obtain that

$$P(w, z) = \frac{\mu z[(q + pw)G(w) - (q + pz)z^r G(z)]}{\mu q - (\lambda + \mu)z + (\lambda z + \mu p)w} \tag{12}$$

for $|w| \leqq 1$ and $|z| \leqq 1$ where

$$G(w) = \sum_{i=0}^{\infty} P_{i1} w^i \tag{13}$$

for $|w| \leqq 1$.

If a stationary distribution $\{ P_{ij} \}$ exists, then by (11) we have $P(1, 1) = 1$, and (12) implies that

$$\lim_{z \to 1} P(1, z) = (p + r)G(1) + G'(1) = 1 \tag{14}$$

and

$$\lim_{w \to 1} P(w, 1) = \frac{\mu p G(1) + \mu G'(1)}{\lambda + \mu p} = 1. \tag{15}$$

Hence we get

$$G(1) = \frac{\mu q - \lambda}{\mu r}. \tag{16}$$

This shows at once that a stationary distribution cannot exist if $\lambda \geqq \mu q$. If $\lambda < \mu q$, then a stationary distribution exists and its generating function is given by (12) where $G(w)$ is still to be determined. In (12) we can determine $G(w)$ for $|w| \leqq 1$ by the requirement that $|P(w, z)| \leqq 1$ for $|w| \leqq 1$ and $|z| \leqq 1$. If $|w| \leqq 1$, then the denominator of (12) has a zero

$$z = \frac{\mu(q + pw)}{\mu + \lambda(1 - w)} \tag{17}$$

in the unit circle $|z| \leq 1$. If z is equal to (17), then the numerator of (12) should vanish too, that is,

$$(q + pw)G(w) = (q + pz)z^r G(z) \tag{18}$$

whenever z is defined by (17) and $|w| \leq 1$.

Let

$$R(w) = (q + pw)G(w)/G(1) \tag{19}$$

for $|w| \leq 1$. Then $R(1) = 1$ and by (18) we have

$$R(w) = z^r R(z) \tag{20}$$

whenever z is defined by (17) and $|w| \leq 1$. Hence it follows that

$$R(w) = [Q(w)]^r \tag{21}$$

where $Q(1) = 1$,

$$Q(w) = U(w)Q(U(w)) \tag{22}$$

and

$$U(w) = \frac{\mu(q + pw)}{\mu + \lambda(1 - w)} \tag{23}$$

for $|w| \leq 1$.

By (12), (19) and (21) we get (4) where only $Q(w)$ remains to be determined. If we define $U_1(w) = U(w)$ and $U_{n+1}(w) = U(U_n(w))$ for $n = 1, 2, \ldots$, then by (22) it follows that

$$Q(w) = \prod_{n=1}^{\infty} U_n(w) \tag{24}$$

for $|w| \leq 1$. We can easily see that

$$U_n(w) = \frac{\alpha_n w + \beta_n}{\gamma_n w + \delta_n} \tag{25}$$

for $n = 1, 2, \ldots$ where

$$\left\| \begin{matrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{matrix} \right\| = C_n \left\| \begin{matrix} \mu p & \mu q \\ -\lambda & \mu + \lambda \end{matrix} \right\|^n \tag{26}$$

and $C_n \neq 0$. If we choose $C_n = \mu q - \lambda$, then we have

$$\begin{aligned} \alpha_n &= \mu q(\lambda + \mu p)^n - \lambda\mu^n, \ \beta_n = \mu q[\mu^n - (\lambda + \mu p)^n] \\ \gamma_n &= \lambda[(\lambda + \mu p)^n - \mu^n], \ \delta_n = \mu q \mu^n - \lambda(\lambda + \mu p)^n. \end{aligned} \tag{27}$$

However, we can also determine $Q(w)$ in a simpler way. Let us write

$$Q(w) = A\left(\frac{\lambda w - \lambda}{\lambda w - \mu q}\right) \tag{28}$$

for $|w| \leqq 1$. Then by (22) we get

$$A(z)\left[1 - \left(p + \frac{\lambda}{\mu}\right)z\right] = A\left(pz + \frac{\lambda z}{\mu}\right)\left[1 - \frac{\mu q}{\lambda}\left(p + \frac{\lambda}{\mu}\right)z\right] \tag{29}$$

and $A(0) = Q(1) = 1$. By expanding $A(z)$ into a Taylor series at $z = 0$,

$$A(z) = \sum_{n=0}^{\infty} A_n z^n, \tag{30}$$

the coefficients A_n $(n = 0, 1, 2, \ldots)$ can be determined by (29). If we form the coefficient of z^n in (29), then we get

$$A_n - \left(p + \frac{\lambda}{\mu}\right)A_{n-1} = \left(A_n - \frac{\mu q}{\lambda} A_{n-1}\right)\left(p + \frac{\lambda}{\mu}\right)^n \tag{31}$$

for $n = 1, 2, \ldots$ and $A_0 = 1$. This proves formulas (5) and (6).

3. THE LIMIT DISTRIBUTION OF THE WAITING TIME

Denote by ξ_n the queue size, that is, the number of customers in the system, immediately before the *n-th* customer arrives. We can easily see that if $\lambda < \mu q$, then the limit distribution

$$\lim_{n\to\infty} \mathbf{P}\{\xi_n = j\} = P_j \tag{32}$$

exists for $j = 1, 2, \ldots$ and is independent of the joint distribution of $\xi_1(0)$ and $\xi_2(0)$. If $\lambda \geqq \mu q$, then $\lim_{n\to\infty} \mathbf{P}\{\xi_n = j\} = 0$ for $j = 1, 2, \ldots$.

Il $\lambda < \mu q$, then the generating function

$$\Phi(z) = \sum_{j=1}^{\infty} P_j z^j \tag{33}$$

is given by

$$\Phi(z) = P(z, z) = [Q(z)]^r \left(\frac{1 - z^r}{r - rz}\right)\left(\frac{\mu q z - \lambda z}{\mu q - \lambda z}\right) \tag{34}$$

for $|z| \leqq 1$ where $Q(z)$ is defined by (5) and (6).

Denote by η_n the waiting time of the *n-th* customer.

THEOREM 2 : *If* $\lambda < \mu q$, *then the limiting distribution*

$$\lim_{n\to\infty} \mathbf{P}\{\eta_n \leqq x\} = W(x) \tag{35}$$

exists and is independent of the joint distribution of $\xi_1(0)$ *and* $\xi_2(0)$. *The Laplace-Stieltjes transform*

$$\Omega(s) = \int_0^{\infty} e^{-sx}\, dW(x) \tag{36}$$

is given by

$$\Omega(s) = \Phi\left(\frac{\mu}{\mu + s}\right) \tag{37}$$

for $Re(s) \geqq 0$ *where* $\Phi(z)$ *is defined by* (34). *If* $\lambda \geqq \mu q$, *then* $\lim_{n \to \infty} \mathbf{P}\{\eta_n \leqq x\} = 0$ *for all* x.

Proof : Since η_n can be represented as a sum of ξ_n independent and identically distributed random variables with distribution function (1), (35) and (37) follow immediately from (32) and (33).

Define

$$M_\nu = \sum_{j=1}^{\infty} j^\nu P_j \tag{38}$$

for $\nu = 1, 2, \ldots$ and write $M = M_1$ and $D = M_2 - M_1^2$.

THEOREM 3 : *If* $\lambda < \mu q$, *then*

$$M = \frac{r(\lambda + \mu p)}{\mu q - \lambda} + \frac{r+1}{2} + \frac{\lambda}{\mu q - \lambda} \tag{39}$$

and

$$D = \frac{r\mu(\lambda + \mu q)(\lambda + \mu p)}{(\mu q - \lambda)^2(\lambda + \mu + \mu p)} + \frac{r^2 - 1}{12} + \frac{\lambda\mu q}{(\mu q - \lambda)^2}. \tag{40}$$

Proof : By (34) we obtain that

$$M = \Phi'(1) = rQ'(1) + \frac{r+1}{2} + \frac{\lambda}{\mu q - \lambda} \tag{41}$$

and

$$\begin{aligned} D &= \Phi''(1) + \Phi'(1) - [\Phi'(1)]^2 \\ &= r[Q''(1) + Q'(1) - (Q'(1))^2] + \frac{r^2 - 1}{12} + \frac{\lambda\mu q}{(\mu q - \lambda)^2}. \end{aligned} \tag{42}$$

Il we form the derivatives of (22) at $w = 1$, then we get

$$Q'(1) = \frac{U'(1)}{1 - U'(1)} \tag{43}$$

and

$$Q''(1) + Q'(1) - [Q'(1)]^2 = \frac{U''(1) + U'(1) - [U'(1)]^2}{[1 - U'(1)]^2[1 + U'(1)]}. \tag{44}$$

Since $U'(1) = (\lambda + \mu p)/\mu$ and $U''(1) = 2\lambda(\lambda + \mu p)/\mu^2$ by the above formulas we get (39) and (40).

From (37) it follows that if $\lambda < \mu q$, then

$$\int_0^\infty x \, dW(x) = M/\mu \tag{45}$$

and

$$\int_0^\infty \left(x - \frac{M}{\mu}\right)^2 dW(x) = (D + M)/\mu^2. \tag{46}$$

4. THE LIMIT DISTRIBUTION OF THE SOJOURN TIME

Denote by θ_n the total time spent in the system by the *n-th* customer. Obviously, the distribution of θ_n depends solely on the distribution of ξ_n, the number of customers in the system immediately before the arrival of the *n-th* customer. If $\lambda < \mu q$, then ξ_n has a limit distribution which does not depend on the initial distribution of the process. Consequently, if $\lambda < \mu q$, then θ_n also has a limit distribution which does not depend on the initial distribution of the process. The limit distribution of θ_n is evidently the same as the distribution of θ_n in the case where $\mathbf{P}\{\xi_n = j\} = P_j$ $(j = 1, 2, \ldots)$ defined by (32). Thus we assume that $\lambda < \mu q$ and that $\mathbf{P}\{\xi_n = j\} = P_j$ $(j = 1, 2, \ldots)$ is given by (33) and (34), and give a method of finding the distribution of θ_n.

Denote by $\theta_n^{(k)}$ the total time spent in the system by the *n-th* customer until his *k-th* service ends if he joins the queue at least k times. Denote by $\zeta_n^{(k)}$ the number of customers in the system immediately after the *k-th* service of the *n-th* customer ends. The *n-th* customer is not included in $\zeta_n^{(k)}$ even if he returns at least $k + 1$ times. If ξ_n has a stationary distribution, then the expectation

$$\Phi_k(s, z) = \mathbf{E}\{e^{-s\theta_n^{(k)}} z^{\zeta_n^{(k)}}\} \tag{47}$$

exists for $\operatorname{Re}(s) \geqq 0$ and $|z| \leqq 1$ and is independent of n. We can easily see that

$$\Phi_{k+1}(s, z) = \Phi_k\left(s, \frac{\mu(q + pz)}{\mu + s + \lambda(1 - z)}\right)\frac{\mu z^r}{\mu + s + \lambda(1 - z)} \tag{48}$$

for $k = 0, 1, 2, \ldots$ where $\Phi_0(s, z) = \Phi(z)$ is defined by (34). The proof of (48) follows the same lines as the proof of (28) in reference [4]. Define

$$\Phi(s, z) = q \sum_{k=1}^{\infty} p^{k-1}\Phi_k(s, z) \tag{49}$$

for $\operatorname{Re}(s) \geqq 0$ and $|z| \leqq 1$. The function $\Phi(s, z)$ is completely determined by the recurrence formula (48) and the initial condition $\Phi_0(s, z) = \Phi(z)$.

THEOREM 4 : *If $\lambda < \mu q$, then the limit distribution*

$$\lim_{n\to\infty} \mathbf{P}\{\theta_n \leqq x\} = K(x) \tag{50}$$

exists and is independent of the initial distribution of the process. If Re $(s) \geqq 0$, *then*

$$\int_0^\infty e^{-sx}\, dK(x) = \Phi(s, 1) \tag{51}$$

where $\Phi(s, z)$ *is defined by* (49).

Proof : The probability that the *n-th* customer joins the queue exactly k times (including his original arrival) is qp^{k-1} for $k = 1, 2, \ldots$. Thus we have

$$\mathbf{E}\{ e^{-s\theta_n} \} = q \sum_{k=1}^{\infty} p^{k-1} \mathbf{E}\{ e^{-s\theta_n^{(k)}} \} \tag{52}$$

for Re $(s) \geqq 0$. If ξ_n has a stationary distribution defined by (33), then (52) reduces to $\Phi(s, 1)$ and this proves (51).

We note that in (49) we can express $\Phi_k(s, z)$ in an explicit form. Let us write

$$U(s, z) = \frac{\mu(q + pz)}{\mu + s + \lambda(1 - z)} \tag{53}$$

and define $U_n(s, z)$ $(n = 0, 1, 2, \ldots)$ by the recurrence formula

$$U_{n+1}(s, z) = U(s, U_n(s, z)) = U_n(s, U(s, z)) \tag{54}$$

for $n = 0, 1, 2, \ldots$ where $U_0(s, z) = z$. It is easy to see that if we define

$$\Phi_k(s, z) = \Phi(U_k(s, z)) \prod_{j=0}^{k-1} \left(\frac{\mu[U_j(s, z)]^r}{\mu + s + \lambda[1 - U_j(s, z)]} \right) \tag{55}$$

for $k = 1, 2, \ldots$, and if $\Phi_0(s, z) = \Phi(z)$, then (48) is satisfied for $k = 0, 1, 2, \ldots$. In (55) we have

$$U_k(s, z) = \frac{\alpha_k(s)z + \beta_k(s)}{\gamma_k(s)z + \delta_k(s)} \tag{56}$$

for $k = 0, 1, 2, \ldots$ where

$$\left\| \begin{matrix} \alpha_k(s) & \beta_k(s) \\ \gamma_k(s) & \delta_k(s) \end{matrix} \right\| = \left\| \begin{matrix} \mu p & \mu q \\ -\lambda & \mu + \lambda + s \end{matrix} \right\|^k . \tag{57}$$

From (48) it follows immediately that

$$[\mu + s + \lambda(1 - z)]\Phi(s, z) = \mu q z^r \Phi(z) + \mu p z^r \Phi\left(s, \frac{\mu(q + pz)}{\mu + s + \lambda(1 - z)}\right) \tag{58}$$

for Re $(s) \geqq 0$ and $|z| \leqq 1$. By (58) we can easily determine the moments

$$K_m = \int_0^\infty x^m\, dK(x) \tag{59}$$

for $m = 1, 2, \ldots$. Define

$$\Phi_{ij} = \frac{1}{i!\,j!}\left(\frac{\partial^{i+j}\Phi(s, z)}{\partial s^i\,\partial z^j}\right)_{s=0,z=1} \tag{60}$$

for $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$. If we form the mixed partial derivatives of order m of (58), then we obtain $m + 1$ linear equations for the determination of $\Phi_{0m}, \Phi_{1,m-1}, \ldots, \Phi_{m0}$. These derivatives can be determined step by step for $m = 1, 2, \ldots$ and

$$K_m = (-1)^m m!\,\Phi_{m0} \tag{61}$$

yields the *m-th* moment of $K(x)$.

Introduce the notation

$$a_{ij} = \frac{1}{i!\,j!}\left[\frac{\partial^{i+j}}{\partial s^i\,\partial z^j}\left(\frac{\mu(q + pz)}{\mu + s + \lambda(1 - z)}\right)\right]_{s=0,z=1} \tag{62}$$

for $i \geqq 0$ and $j \geqq 0$. Then $a_{00} = 1$, $a_{10} = -1/\mu$, $a_{01} = (\lambda + \mu p)/\mu$, $a_{20} = 1/\mu^2$, $a_{11} = -(2\lambda + \mu p)/\mu^2$ and $a_{02} = \lambda(\lambda + \mu p)/\mu^2$.

By (58) we obtain

$$1 + \mu\Phi_{10} = \mu p\Phi_{10} - p\Phi_{01}, \tag{63}$$

and

$$\mu\Phi_{01} - \lambda = \mu r + \mu q\Phi'(1) + (\lambda + \mu p)p\Phi_{01}. \tag{64}$$

From (63)

$$\Phi_{10} = -(1 + p\Phi_{01})/\mu q \tag{65}$$

and from (64)

$$\Phi_{01} = [\lambda + \mu r + \mu q\Phi'(1)][\mu - (\lambda + \mu p)p]^{-1} \tag{66}$$

where $\Phi'(1) = M$ is given by (39). Since $K_1 = -\Phi_{10}$, by (65) and (66) we get

$$K_1 = \frac{1}{\mu q} + \frac{p}{\mu q}\left[\frac{2(\mu q - \lambda)(\lambda + \mu r) + \mu q\lambda(r + 1) + \mu^2 q(2r - (r - 1)q)}{2(\mu q - \lambda)[\mu - (\lambda + \mu p)p]}\right] \tag{67}$$

If we form the partial derivatives of order two of (58), then we obtain the following equations for the determination of $K_2 = 2\Phi_{20}$:

$$\mu\Phi_{20} + \Phi_{10} = \mu p a_{20}\Phi_{01} + \mu p\Phi_{20} + \mu p a_{10}\Phi_{11} + \mu p a_{10}^2\Phi_{02}, \tag{68}$$

$$\Phi_{01} - \lambda\Phi_{10} + \mu\Phi_{11} = \\ = \mu p[a_{11}\Phi_{01} + a_{01}\Phi_{11} + 2a_{10}a_{01}\Phi_{02} + r\Phi_{10} + ra_{10}\Phi_{01}] \tag{69}$$

and

$$\mu\Phi_{02} - \lambda\Phi_{01} = \mu q\left[\frac{\Phi''(1)}{2} + r\Phi'(1) + \binom{r}{2}\right] + \mu p\left[a_{02}\Phi_{01} + a_{01}^2\Phi_{02} + ra_{01}\Phi_{01} + \binom{r}{2}\right]. \quad (70)$$

Now Φ_{02} can be determined by (70), Φ_{11} by (69) and finally Φ_{20} by (68).

5. AN EXAMPLE

Let us suppose that in the queuing process, $p = 1/3$ and $\lambda/\mu = 1/6$. Then $\lambda/\mu q = 1/4$ and ξ_n, η_n and θ_n have a limit distribution. The probabilities $\lim_{n\to\infty} \mathbf{P}\{\xi_n = j\} = P_j$ $(j = 1, 2, \ldots)$ are determined by (34) where now

$$Q(z) = 3(z + 2)(4 - z)^{-2}. \quad (71)$$

This follows from (5) where now $A_0 = 1$, $A_1 = -3$, $A_2 = 2$ and $A_n = 0$ for $n \geqq 3$. In this case (34) reduces to

$$\Phi(z) = \left(\frac{1 - z^r}{r(1 - z)}\right)\left(\frac{3}{4 - z}\right)\left(\frac{3(z + 2)}{4 - z}\right)^r. \quad (72)$$

The Laplace-Stieltjes transform of the limiting distribution of η_n is given by (37), and the Laplace-Stieltjes transform of the limiting distribution of θ_n is given by (51).

By (39) and (40) we obtain that

$$M = (9r + 5)/6 \quad (73)$$

and

$$D = (3r^2 + 40r + 13)/36. \quad (74)$$

By (67) we get

$$\mu K_1 = (18r + 29)/15 \quad (75)$$

and by (68), (69) and (70) it follows that

$$\mu^2 K_2 = (16065r^2 + 37025r + 23827)/2475. \quad (76)$$

REFERENCES

1. D. G. LAMPARD, *A Stochastic Process whose Successive Intervals between Events Form a First order Markov Chain*. I. Journal of Applied Probability, 5 (1968), p. 648-668.
2. R. M. PHATARFOD, *Note on the Reversible Counters System of Lampard*. Journal of Applied Probability, 11 (1974), p. 624-628.
3. R. M. PHATARFOD, *On the Reversible Counters System of Lampard*. Journal of Applied Probability, 12 (1975), 639-646.
4. L. TAKÁCS, *A Single-Server Queue with Feedback*. Bell System Technical Journal, 42, 1963, p. 505-519.
5. L. TAKÁCS, *Some Remarks on a Counter Process*. Journal of Applied Probability, 13, 1976, p. 623-627.