REVUE FRANÇAISE D'AUTOMATIQUE, D'INFORMATIQUE ET DE RECHERCHE OPÉRATIONNELLE. RECHERCHE OPÉRATIONNELLE

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Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle, tome 12, n° 1 (1978), p. 93-99.

<http://www.numdam.org/item?id=RO_1978__12_1_93_0>

R.A.I.R.O. Recherche opérationnelle/Operations Research
(vol. 12, n° 1, février 1978, p. 93 à 99)

OPTIMUM ORDERING POLICIES WHEN ORDER COSTS DEPEND ON TIME (*)

by NAOTO KAIO ([1]) and SHUNJI OSAKI ([1])

Abstract. — *This paper considers optimum ordering policies minimizing the expected cost per unit time in the steady-state. It is assumed that a lead time for the spare is constant and order costs depend on time. It is shown in theorem that there exists a finite and unique optimum ordering policy under certain conditions.*

1. INTRODUCTION

When we discuss replacement policies, especially age ones, for a one-unit system subject to failure, we must first consider how to supply a new unit for replacement. That is, there exist the following two cases. One is that a new unit is always on hand and is available for replacement, and another is that a new unit is delivered after a lead time and an order cost is constant or dependent on time when we order. The former was discussed by Barlow and Proschan [1] and others [2, 3, 4, 5, 6], and the latter will be of interest in this paper as optimum ordering policy with lead time. In this paper we treat the optimum ordering policy with a constant lead time and two kind of order costs depending on time.

The analysis is done as follows. Introducing a constant lead time and three kind of costs, and noticing that every replacement time instant is a regeneration point, we derive the expected cost per unit time in the steady-state. We seek the optimum ordering time minimizing that expected cost. It is shown that there exists a unique and finite optimum ordering policy under certain conditions. It is further shown that there exists an upper limit of such an optimum policy.

(*) Reçu novembre 1976.
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2. MODEL AND ASSUMPTIONS

The original unit starts operating at time 0. If the unit does not fail up to a prespecified time $t_0 \in [0, \infty)$, then order for the spare is made at time t_0 regularly, and it is named the *regular order*. After a constant lead time L, the spare is delivered and immediately the original unit is replaced by the spare within negligible time, even if the original one is operating. On the other hand, if the original unit fails up to time t_0, immediately emergency order is made at the failure point, and it is named the *expedited order*. And the spare starts operating as soon as it is delivered after a lead time L. The similar cycles are repeated from time to time.

The failure time for each unit has an arbitrary distribution $F(t)$ with a finite mean $1/\lambda$ and a p.d.f. $f(t)$. Let us introduce the following three costs : The constant cost k_1 per unit time is suffered for failure, the cost $C_1(t)$ for expedited order made at time t before time t_0, and the cost $C_2(t_0)$ for regular order made at time t_0. Assume that $C_i(t)$ is differentiable twice, finite and positive, $C_1(\mathrm{t}) > C_2(t_0)$ for $t \in [0, t_0]$, and $dC_i(t)/dt \equiv c_i(t)$ $(i = 1, 2)$. Moreover, assume that $C_i(t)$ is a convex-decreasing function, $0 \geq c_1(t) \geq c_2(t)$. Of course, it is evident that $c_i'(t) \geqq 0$ $(i=1, 2)$.

Under the above assumptions, we define an interval from the beginning of the original unit (replacement) to the (next) replacement as one cycle.

3. ANALYSIS

Consider the following two expected costs : (i) When the original unit fails, if no spare is available, the system is under failure state until the spare is delivered. The expected cost during that period is

$$k_1\left[\int_0^{t_0} L\, dF(t) + \int_{t_0}^{t_0+L} (t_0 + L - t)\, dF(t)\right] = k_1 \int_{t_0}^{t_0+L} F(t)\, dt. \quad (1)$$

(ii) The expected order cost is

$$\int_0^{t_0} C_1(t)\, dF(t) + \int_{t_0}^{\infty} C_2(t_0)\, dF(t)$$
$$= C_1(t_0)F(t_0) - \int_0^{t_0} c_1(t)F(t)\, dt + C_2(t_0)\overline{F}(t_0), \quad (2)$$

where $\overline{F}(t) \equiv 1 - F(t)$. Moreover, the mean time of one cycle is

$$\int_0^{t_0} (t + L)\, dF(t) + \int_{t_0}^{\infty} (t_0 + L)\, dF(t) = L + \int_0^{t_0} \overline{F}(t)\, dt. \quad (3)$$

Thus, the expected cost per unit time in the steady-state is

$$K(t_0) = \frac{k_1 \int_{t_0}^{t_0+L} F(t)\,dt + C_1(t_0)F(t_0) - \int_0^{t_0} c_1(t)F(t)\,dt + C_2(t_0)\bar{F}(t_0)}{L + \int_0^{t_0} \bar{F}(t)\,dt} \tag{4}$$

(see Ross [7]), $K(t_0) > 0$,

$$K(\infty) = \frac{k_1 L + C_1(\infty) - \int_0^{\infty} c_1(t)F(t)\,dt}{L + 1/\lambda}, \tag{5}$$

and

$$K(0) = \frac{k_1 \int_0^{L} F(t)\,dt + C_2(0)}{L}. \tag{6}$$

Define the numerator of the derivative of the right-hand side in (4) as

$$\begin{aligned} q(t_0) = {} & [k_1 R(t_0) + (C_1(t_0) - C_2(t_0))r(t_0) \\ & + c_2(t_0)]\left[L + \int_0^{t_0} \bar{F}(t)\,dt\right] \\ & - \left[k_1 \int_{t_0}^{t_0+L} F(t)\,dt + C_1(t_0)F(t_0)\right. \\ & \left. - \int_0^{t_0} c_1(t)F(t)\,dt + C_2(t_0)\bar{F}(t_0)\right], \end{aligned} \tag{7}$$

where $R(t_0) \equiv [F(t_0 + L) - F(t_0)]/\bar{F}(t_0)$, $r(t_0) \equiv f(t_0)/\bar{F}(t_0)$ and we assume that these functions are differentiable. These functions, $R(t_0)$ and $r(t_0)$, are called *failure rates* and have the same monotone property (see Barlow and Proschan [1, p. 23]). Further,

$$\begin{aligned} q(\infty) = {} & [k_1 R(\infty) + (C_1(\infty) - C_2(\infty))r(\infty) + c_2(\infty)][L + 1/\lambda] \\ & - \left[k_1 L + C_1(\infty) - \int_0^{\infty} c_1(t)F(t)\,dt\right], \end{aligned} \tag{8}$$

and

$$\begin{aligned} q(0) = {} & [k_1 R(0) + (C_1(0) - C_2(0))r(0) + c_2(0)]L \\ & - \left[k_1 \int_0^{L} F(t)\,dt + C_2(0)\right]. \end{aligned} \tag{9}$$

Now, we have the following.

THEOREM 1 : (i) *If* $q(\infty) > 0$ *then there exists at least an optimum ordering time* t_0^* $(0 \leq t_0^* < \infty)$ *minimizing the expected cost* $K(t_0)$.

(ii) *If* $q(0) < 0$ *then there exists at least an optimum ordering time* t_0^* $(0 < t_0^* \leq \infty)$ *minimizing the expected cost* $K(t_0)$.

Proof: By differentiating $\log K(t_0)$ with respect to t_0, we have

$$\frac{d \log K(t_0)}{dt_0} = \bar{F}(t_0)\left[\frac{k_1 R(t_0) + (C_1(t_0) - C_2(t_0))r(t_0) + c_2(t_0)}{k_1 \int_{t_0}^{t_0+L} F(t)\,dt + C_1(t_0)F(t_0) - \int_0^{t_0} c_1(t)F(t)\,dt + C_2(t_0)\bar{F}(t_0)} - \frac{1}{L + \int_0^{t_0} \bar{F}(t)\,dt}\right]. \quad (10)$$

For large t_0, we have

$$\frac{d \log K(t_0)}{dt_0} \sim \bar{F}(t_0)\left[\frac{k_1 R(\infty) + (C_1(\infty) - C_2(\infty))r(\infty) + c_2(\infty)}{k_1 L + C_1(\infty) - \int_0^{\infty} c_1(t)F(t)\,dt} - \frac{1}{L + 1/\lambda}\right]. \quad (11)$$

Thus, if the bracket of the right-hand side is positive, i.e., $q(\infty) > 0$, then there exists at least an optimum ordering time t_0^* $(0 \leq t_0^* < \infty)$ minimizing the expected cost $K(t_0)$.

Also, for small to, we have

$$\frac{d \log K(t_0)}{dt_0} \sim \bar{F}(t_0)\left[\frac{k_1 R(0) + (C_1(0) - C_2(0))r(0) + c_2(0)}{k_1 \int_0^L F(t)\,dt + C_2(0)} - \frac{1}{L}\right]. \quad (12)$$

Thus, if $q(0) < 0$ then there exists at least an optimum ordering time t_0^* $(0 < t_0^* \leq \infty)$ minimizing the expected cost $K(t_0)$. Q.E.D.

The above theorem states that there is at least an optimum ordering time t_0^* (not necessarily unique). However, supposing the monotone property, especially the strictly increasing property, of the failure rate, we have the following.

THEOREM 2 : *Suppose that the failure rate is strictly increasing.*

(i) *If* $q(0) < 0$ *and* $q(\infty) > 0$ *then there exists a finite and unique optimum ordering time* t_0^* $(0 < t_0^* < \infty)$ *satisfying* $q(t_0) = 0$, *and the expected cost is*

$$K(t_0^*) = k_1 R(t_0^*) + (C_1(t_0^*) - C_2(t_0^*))r(t_0^*) + c_2(t_0^*). \quad (13)$$

(ii) If $q(\infty) \leq 0$ *then the optimum ordering time is* $t_0^* \to \infty$, *i.e., order for the spare is made at the same time as failure of the original unit, and the expected cost is given by* (5).

(iii) If $q(0) \geq 0$ *then the optimum ordering time is* $t_0^* = 0$, *i.e., order for the spare is made at the same time as the beginning of the original unit, and the expected cost is* (6).

Proof : By differentiating $K(t_0)$ with respect to t_0 and setting it equal to zero, we have the equation $q(t_0) = 0$. Further, we have

$$q'(t_0) = [k_1 R'(t_0) + (c_1(t_0) - c_2(t_0))r(t_0) + (C_1(t_0) - C_2(t_0))r'(t_0) + c_2'(t_0)]\left[L + \int_0^{t_0} \overline{F}(t)\,dt\right]. \quad (14)$$

Since the failure rate is strictly increasing, we have $q'(t_0) > 0$, i.e., $q(t_0)$ is strictly increasing.

If $q(0) < 0$ and $q(\infty) > 0$ then, since $q(t_0)$ is strictly increasing and continuous, there exists a finite and unique t_0^* $(0 < t_0^* < \infty)$ minimizing the expected cost $K(t_0)$ and satisfying $q(t_0) = 0$. By substituting the relation of $q(t_0^*) = 0$ into $K(t_0^*)$ in (4), we obtain the equation (13).

If $q(\infty) \leq 0$ then the optimum ordering time is $t_0^* \to \infty$, since for an arbitrary non-negative t_0 we have $K'(t_0) \leq 0$ and consequently $K(t_0)$ is a strictly decreasing function with respect to t_0.

If $q(0) \geq 0$ then the optimum ordering time is $t_0^* = 0$, since for an arbitrary non-negative t_0 we have $K'(t_0) \geq 0$ and consequently $K(t_0)$ is a strictly increasing function with respect to t_0. Q.E.D.

Moreover, in case of (i) in Theorem 2, we give an upper limit for the optimum ordering time t_0^*.

THEOREM 3 : *Suppose that the failure rate is strictly increasing,* $q(0) < 0$, $q(\infty) > 0$, *and* $L \neq 0$. *If* $\overline{t}_0$ *is a solution satisfying the equation* $h(t_0) = 0$, $\overline{t}_0$ *exists uniquely and* $t_0^* < \overline{t}_0$, *where*

$$h(t_0) = [k_1 R(t_0) + (C_1(t_0) - C_2(t_0))r(t_0)]L + c_2(t_0)\left[L + \int_0^{t_0} \overline{F}(t)\,dt\right] - \left[k_1 \int_0^L F(t)\,dt - \int_0^{t_0} c_1(t)F(t)\,dt + C_2(t_0)\right] \quad (15)$$

for $t_0 > 0$.

Proof : Since the failure rate is strictly increasing, we have the following two inequalities :

$$r(t_0) > \frac{F(t_0)}{\int_0^{t_0} \bar{F}(t)\,dt}, \tag{16}$$

and

$$R(t_0) > \frac{\int_{t_0}^{t_0+L} F(t)\,dt - \int_0^L F(t)\,dt}{\int_0^{t_0} \bar{F}(t)\,dt}. \tag{17}$$

Further,

$$q(t_0) - h(t_0) = \int_0^{t_0} \bar{F}(t)\,dt \Bigg[\{ k_1 R(t_0) + (C_1(t_0) - C_2(t_0)) r(t_0) \}$$

$$- \left\{ k_1 \frac{\int_{t_0}^{t_0+L} F(t)\,dt - \int_0^L F(t)\,dt}{\int_0^{t_0} \bar{F}(t)\,dt} + (C_1(t_0) - C_2(t_0)) \frac{F(t_0)}{\int_0^{t_0} \bar{F}(t)\,dt} \right\} \Bigg]. \tag{18}$$

Thus, from the inequalities (16) and (17), we obtain $q(t_0) > h(t_0)$ for $t_0 > 0$. If there exists a solution satisfying $h(t_0) = 0$, the solution $\bar{t}_0$ is a unique one and $t_0^* < \bar{t}_0$, since $h(t_0)$ is strictly increasing. Q.E.D.

4. CONCLUDING REMARKS

Especially, when $L \equiv 0$, $C_1(t) \equiv c_1$, $C_2(t_0) \equiv c_2$, the expected cost per unit time in the steady-state is

$$K(t_0) = \frac{c_1 F(t_0) + c_2 \bar{F}(t_0)}{\int_0^{t_0} \bar{F}(t)\,dt}, \tag{19}$$

and the model discussed here is identified with the age replacement model [1].

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