Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle

K. Bidhi Singh

A queueing system with random additional service facility

Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle, tome 12, nº 3 (1978), p. 311-318.

<http://www.numdam.org/item?id=RO_1978__12_3_311_0>

R.A.I.R.O. Recherche opérationnelle/Operations Research
(vol. 12, n° 3, août 1978, p. 311 à 318)

A QUEUEING SYSTEM WITH RANDOM ADDITIONAL SERVICE FACILITY (*)

by K. BIDHI SINGH [1]

Abstract. — *This paper studies the transient and steady state behaviour of a limited waiting space queueing system wherein: (i) there is one regular service facility (r. s.f.) serving the units one by one; (ii) a search for an additional service facility (a. s.f.) for the service of a group of units is started when the queue length reaches the maximum size and is dropped when the queue length reduces to some tolerable fixed size; (iii) the availability time of an a. s.f. is a random variable; (iv) there are costs associated for providing an a. s.f. and the loss of the customers who go elsewhere because of the limited waiting space. Finally a relationship among the costs, traffic intensities and the queue size is obtained.*

INTRODUCTION

A large number of queueing problems with additional service channels have been solved by various authors. Romani (1955) and Phillips (1960) obtain the steady state probabilities of queueing problems with variable number of service channels assuming that when the waiting line size increases to some preassigned fixed number N, then with each arriving unit a new channel is made available and is cancelled at the termination of service if there is no unit waiting, with the exception of one channel which remains open at all times. Murari (1971) in his studies modifies the results due to Romani and Phillips as follows. When the queue length increases to some undesirable number m_1, then another channel is called for its help. If in spite of two service channels operating the queue length increases to some undesirable number $m_2 > m_1$, then third channel is called for their help and so on. In all these cases when there is a demand of an additional service facility (a. s. f.), it is made available instantaneously. The present study relates to the situation when the queue length (the number of units including one being served) reaches M (the maximum waiting space), then a search for an a. s. f. is started and the availability time of an a. s. f. is a random variable.

Because of the limited waiting space an arriving unit is considered lost for the system when there is no waiting space. To avoid the loss of these units and to win the good-will of the customers the manager of the firm would like to get the

(*) Received October 1977, revised February 1978.
(1) Department of Mathematics, B. N. Chakravarty University, Kurukshetra Haryana, Inde.

service of a group of units from another service facility in the market. When there is no waiting space he starts searching for an a. s. f. for the service of N units and the search continues till the queue length reduces to $M-N$. Whenever an a. s. f. is available, N of the units are sent to this service facility.

Thus, this paper studies the transient and the steady state behaviour of a queueing system under the following assumptions:

1° customers are arriving at a service facility in a Poisson stream with mean rate λ and form a queue. There is a limited waiting space for M customers, i. e. if at anytime the queue length is M, then an arriving unit is considered lost for the system;

2° the queue discipline is first-come-first served;

3° there is one regular service facility and the service time of a customer is exponentially distributed with parameter μ;

4° when the queue length reaches M a search is started for the service of $N (< M/2)$ units from a. s. f. The availability time (time for arriving an a. s. f.) is exponentially distributed with parameter V. The search continues till the queue length reduces to $M-N$. If during the small interval $[t, t+\Delta t)$ the search is fulfilled that is an a. s. f. is available, N of the units leave the system to get the service from the a. s. f.

Define:

$P_{n,A}(t) \equiv$ probability that at time t the queue length is n and a search for the service of N units from an a. s. f. is in progress;

$P_{n,B}(t) \equiv$ probability that at time t the queue length is n and no search for the service of N units from an a. s. f. is in progress.

$$R_n(t) = P_{n,A}(t) + P_{n,B}(t).$$

Thus by the assumptions imposed on the system

$$P_{n,A}(t) = 0, \qquad 0 \leqq n \leqq M-N,$$
$$P_{M,B}(t) = 0.$$

The state $(M-1, B)$ will change to (M, A) through an arrival. The state $(M-N+1, A)$ will change to $(M-N, B)$ through a service. The state $(n+N, A)$, $M-N < n+N \leqq M$, will change to (n, B) when an a. s. f. for N units becomes available.

The Laplace Transform of probability generating functions for various queue lengths are obtained and steady state results are explicitly derived therefrom. In particular, we obtain the explicit expressions for the probabilities, $P_{0,B}(t)$, $P_{M-N+1,A}(t)$, $P_{M-1,B}(t)$ and $P_{M,A}(t)$ in the limiting case when t tends to infinity.

Kolmogorov's forward equations governing the system are

$$\left.\begin{aligned}\frac{d}{dt}P_{M-N+1,A}(t)&=-(\lambda+\mu+V)P_{M-N+1,A}(t)+\mu P_{M-N+2,A}(t),\\ n&=M-N+1,\end{aligned}\right\}\quad(1)$$

$$\left.\begin{aligned}\frac{d}{dt}P_{n,A}(t)&=-(\lambda+\mu+V)P_{n,A}(t)+\mu P_{n+1,A}(t)+\lambda P_{n-1,A}(t),\\ M-N+1&<n<M,\end{aligned}\right\}\quad(2)$$

$$\frac{d}{dt}P_{M,A}(t)=-(\mu+V)P_{M,A}(t)+\lambda P_{M-1,A}(t)+\lambda P_{M-1,B}(t),\qquad n=M,\quad(3)$$

$$\frac{d}{dt}P_{0,B}(t)=-\lambda P_{0,B}(t)+\mu P_{1,B}(t),\qquad n=0,\quad(4)$$

$$\left.\begin{aligned}\frac{d}{dt}P_{n,B}(t)&=-(\lambda+\mu)P_{n,B}(t)+\lambda P_{n-1,B}(t)+\mu P_{n+1,B}(t),\\ 0&<n\leqq M-2N,\end{aligned}\right\}\quad(5)$$

$$\left.\begin{aligned}\frac{d}{dt}P_{n,B}(t)&=-(\lambda+\mu)P_{n,B}(t)+\lambda P_{n-1,B}(t)+\mu P_{n+1,B}(t)+VP_{n+N,A}(t),\\ M-2N&<n<M-N,\end{aligned}\right\}\quad(6)$$

$$\begin{aligned}\frac{d}{dt}P_{M-N,B}(t)=&-(\lambda+\mu)P_{M-N,B}(t)+\lambda P_{M-N-1,B}(t)+\mu P_{M-N+1,B}(t)\\ &+VP_{M-N+N,A}(t)+\mu P_{M-N+1,A}(t),\qquad n=M-N,\end{aligned}\quad(7)$$

$$\left.\begin{aligned}\frac{d}{dt}P_{n,B}(t)&=-(\lambda+\mu)P_{n,B}(t)+\lambda P_{n-1,B}(t)+\mu P_{n+1,B}(t),\\ M-N&<n<M-1,\end{aligned}\right\}\quad(8)$$

$$\frac{d}{dt}P_{M-1,B}(t)=-(\lambda+\mu)P_{M-1,B}(t)+\lambda P_{M-2,B}(t),\qquad n=M-1.\quad(9)$$

Let the time be reckoned from the instant when the queue length is zero, so that the initial condition becomes

$$P_{0,B}(0)=1.\quad(10)$$

Define the probability generating functions

$$P_A(t,\alpha)=\sum_{n=M-N+1}^{M}\alpha^n P_{n,A}(t),\quad(11)$$

$$P_B(t,\alpha)=\sum_{n=0}^{M-1}\alpha^n P_{n,B}(t), \tag{12}$$

$$R(t,\alpha)=\sum_{n=0}^{M}\alpha^n R_n(t). \tag{13}$$

Let $F(S)$ denote the Laplace transform (L.T.) of $F(t)$ defined by

$$\overline{F}(S)=\int_0^\infty e^{-St}F(t)\,dt.$$

Multiplying (1)-(9) by appropriate powers of α, using (11)-(12) and taking L.T.s, we have

$$\overline{P}_A(S,\alpha)=\frac{\lambda\alpha^{M+1}(1-\alpha)\overline{P}_{M,A}(S)-\mu\alpha^{M-N+1}\overline{P}_{M-N+1,A}(S)+\lambda\alpha^{M+1}\overline{P}_{M-1,B}(S)}{\alpha(\lambda+\mu+V+S)-\mu-\lambda\alpha^2}, \tag{14}$$

$$\overline{P}_B(S,\alpha)=\frac{\left\{\begin{array}{l}\alpha-\mu(1-\alpha)\overline{P}_{0,B}(S)-\lambda\alpha^{M+1}\overline{P}_{M-1,B}(S)\\ \quad+V\alpha^{-N+1}\overline{P}_A(S,\alpha)+\mu\alpha^{M-N+1}\overline{P}_{M-N+1,A}(S)\end{array}\right\}}{\alpha(S+\lambda+\mu)-\mu-\lambda\alpha^2}, \tag{15}$$

Substituting for $\overline{P}_A(S,\alpha)$ from (14) in (15):

$$\left.\begin{array}{l}\overline{P}_B(S,\alpha)=\dfrac{\begin{array}{l}[\alpha(\lambda+\mu+V+S)-\mu-\lambda\alpha^2][\alpha-\mu(1-\alpha)\overline{P}_{0,B}(S)]\\ \quad+[\alpha(\lambda+\mu+V+S)-\mu-\lambda\alpha^2-V\alpha^{-N+1}]\times\\ \qquad\left\{\begin{array}{l}[\mu\alpha^{M-N+1}\overline{P}_{M-N+1,A}(S)-\lambda\alpha^{M+1}\overline{P}_{M-1,B}(S)]\\ \quad+V\lambda(1-\alpha)\alpha^{M-N+2}\overline{P}_{M,A}(S)\end{array}\right\}\end{array}}{[\alpha(\lambda+\mu+V+S)-\mu-\lambda\alpha^2][\alpha(\lambda+\mu+S)-\mu-\lambda\alpha^2]},\end{array}\right\} \tag{16}$$

$$\overline{R}(S,\alpha)=\overline{P}_A(S,\alpha)+\overline{P}_B(S,\alpha).$$

The denominator in (16) has four roots in α. Since $\overline{P}_B(S,\alpha)$ is a polynomial, these roots must vanish the numerator in (16), giving rise to four equations involving four unknowns namely $\overline{P}_{0,B}(S)$, $\overline{P}_{M-N+1,A}(S)$, $\overline{P}_{M-1,B}(S)$, and $\overline{P}_{M,A}(S)$. Solving these four equations, we can determine all the unknowns. Thus $\overline{P}_B(S,\alpha)$, $\overline{P}_A(S,\alpha)$ and $\overline{R}(S,\alpha)$ are completely determined.

STEADY STATE SOLUTION

The steady state solution can be obtained by the well known property of L.T. viz.

$$\lim_{S\to 0} S\overline{F}(S)=\lim_{t\to\infty}F(t)=F, \tag{17}$$

if the limit on the right hand side exists, thus if

$$P_A(\alpha)=\sum_{n=M-N+1}^{M}\alpha^n P_{n,A},$$

$$P_B(\alpha)=\sum_{n=0}^{M-1}\alpha^n P_{n,B},$$

$$R(\alpha)=\sum_{n=0}^{M}\alpha^n R_n.$$

Applying the property (17) to equations (14), (16), we have

$$P_A(\alpha)=\frac{\rho_1\rho_2\alpha^{M+1}(1-\alpha)P_{M,A}-\rho_2\alpha^{M-N+1}P_{M-N+1,A}+\rho_1\rho_2\alpha^{M+1}P_{M-1,B}}{\alpha(\rho_1+\rho_2+\rho_1\rho_2)-\rho_2-\rho_1\rho_2\alpha^2} \quad (18)$$

$$P_B(\alpha)=\frac{\begin{aligned}&-(1-\alpha)[\alpha(\rho_1+\rho_2+\rho_1\rho_2)-\rho_2-\rho_1\rho_2\alpha^2]P_{0,B}+\rho_1^2(1-\alpha)\alpha^{M-N+2}P_{M,A}\\&+[\alpha^{M-N+1}P_{M-N+1,A}-\rho_1\alpha^{M+1}P_{M-1,B}]\times[\alpha(\rho_1+\rho_2+\rho_1\rho_2)-\rho_2-\rho_1\rho_2\alpha^2-\rho_1\alpha^{-N+1}]\end{aligned}}{\rho_1\rho_2(1-\alpha)(\alpha\rho_1-1)(\alpha-\alpha_1)(\alpha_2-\alpha)}, \quad (19)$$

where

$$\left.\begin{aligned}&\rho_1=\frac{\lambda}{\mu},\qquad \rho_2=\frac{\lambda}{V},\\ &\alpha_1+\alpha_2=\frac{\rho_1+\rho_2+\rho_1\rho_2}{\rho_1\rho_2},\\ &\alpha_1\alpha_2=\frac{1}{\rho_1},\\ &R(\alpha)=P_A(\alpha)+P_B(\alpha),\end{aligned}\right\} \quad (20)$$

$P_B(\alpha)$ is a polynomial, the roots of the denominator in $P_B(\alpha)$ must vanish its numerator, giving rise to the set of three equations. Solving these equations, we have

$$P_{0,B}=\frac{-1}{\rho_1^{M-N+1}(1-\rho_1)}\times\left[(1-\rho_1^{-N+1})\left(\frac{\alpha_1^{-N}-\alpha_2^{-N}}{\alpha_1-\alpha_2}\right)-(1-\rho_1^{-N})\left(\rho_1^{N+1}+\frac{\alpha_1^{-N-1}-\alpha_2^{-N-1}}{\alpha_1-\alpha_2}\right)\right]P_{M-N+1,A}, \quad (21)$$

$$P_{M,A} = -\left[\frac{(\alpha_1^{-N} - \alpha_2^{-N})}{\rho_1(\alpha_1 - \alpha_2)}\right] P_{M-N+1,A}, \tag{22}$$

$$P_{M-1,B} = -\left[\frac{(\alpha_1^{-N-1} - \alpha_2^{-N-1}) - \rho_1(\alpha_1^{-N} - \alpha_2^{-N})}{\rho_1^2(\alpha_1 - \alpha_2)}\right] P_{M-N+1,A}. \tag{23}$$

Now the normalising condition

$$R(1) = P_B(1) + P_A(1) - 1, \tag{24}$$

gives

$$(1-\rho_1) = P_{0,B} - \rho_1 P_{M,A} - N(\rho_1 P_{M-1,B} - P_{M-N+1,A}),$$

or

$$\begin{aligned}(1-\rho_1)[P_{M-N+1,A}]^{-1} &= \left(\frac{\alpha_1^{-N} - \alpha_2^{-N}}{\alpha_1 - \alpha_2}\right) - \frac{1}{\rho_1^{M+1}}\left[\frac{\rho_1^N(1-\rho_1^{-N+1})}{1-\rho_1}\left(\frac{\alpha_1^{-N} - \alpha_2^{-N}}{\alpha_1 - \alpha_2}\right)\right. \\ &\quad \left. + \left(\frac{1-\rho_1^N}{1-\rho_1}\right)\left(\rho_1^{N+1} + \frac{\alpha_1^{-N-1} - \alpha_2^{-N-1}}{\alpha_1 - \alpha_2}\right)\right] \\ &\quad + N\left[\frac{(\alpha_1^{-N-1} - \alpha_2^{-N-1}) - \rho_1(\alpha_1^{-N} - \alpha_2^{-N})}{\rho_1(\alpha_1 - \alpha_2)} + 1\right]. \end{aligned} \tag{25}$$

Thus all the four unknowns $P_{M,A}$, $P_{0,B}$, $P_{M-1,B}$ and $P_{M-N+1,A}$ in $P_A(\alpha)$ and $P_B(\alpha)$ are determined from (21)-(25).

The probability $P_A(1)$ of the availability of an a. s. f. is obtained by setting $\alpha = 1$ in (18) and using (23):

$$\begin{aligned} P_A(1) &= \frac{-\rho_2}{\rho_1^2(\alpha_1 - \alpha_2)} \\ &\quad \times [(\alpha_1^{-N-1} - \alpha_2^{-N-1}) - \rho_1(\alpha_1^{-N} - \alpha_2^{-N}) + \rho_1(\alpha_1 - \alpha_2)] P_{M-N+1,A}. \end{aligned} \tag{26}$$

COMPARISON WITH A LIMITED WAITING SPACE M/M/1 MODEL

Let T_M be the fraction of the customers who go elsewhere for the M/M/1 model with a limited waiting space for M units. It is given by

$$T_M = \frac{\rho_1^M(1-\rho_1)}{(1-\rho_1^{M+1})}, \tag{27}$$

where ρ_1 is the traffic intensity.

$P_{M,A}$ of our model denotes the fraction of the customers who go elsewhere when there is an a. s. f.

Assume that c_1 is the cost for providing an a. s. f. for the service of a group of N units and c_2, the loss per customer who goes elsewhere. It is obvious that employing an a. s. f. is profitable if

$$c_1 \, V P_A(1) < c_2 \, \lambda (T_M - P_{M,A}),$$

or

$$\overset{*}{c} < \left(\frac{T_M - P_{M,A}}{P_A(1)} \right) \rho_2,$$

where

$$\overset{*}{c} = \frac{c_1}{c_2}.$$

Using (22), (26) and (27), we have

$$\overset{*}{c} < \frac{-\rho_1}{(1-\rho_1^{M+1})} \left[\frac{\rho_1^{M+1}(1-\rho_1)(\alpha_1-\alpha_2)[P_{M-N+1,A}]^{-1} + (1-\rho_1^{M+1})(\alpha_1^{-N}-\alpha_2^{-N})}{(\alpha_1^{-N-1}-\alpha_2^{-N-1}) - \rho_1(\alpha_1^{-N}-\alpha_2^{-N}) + \rho_1(\alpha_1-\alpha_2)} \right].$$

The upper bound for $\overset{*}{c}$ is given by

$$\overset{*}{c} = \frac{-\rho_1}{(1-\rho_1^{M+1})} \left[\frac{\rho_1^{M+1}(1-\rho_1)(\alpha_1-\alpha_2)[P_{M-N+1,A}]^{-1} + (1-\rho_1^{M+1})(\alpha_1^{-N}-\alpha_2^{-N})}{(\alpha_1^{-N-1}-\alpha_2^{-N-1}) - \rho_1(\alpha_1^{-N}-\alpha_2^{-N}) + \rho_1(\alpha_1-\alpha_2)} \right].$$

It may be noted that the above relation depends only $\overset{*}{c}$, M, N, ρ_1 and ρ_2. Thus given any four, the value of the 5th can be computed. Given $M=40$, $\rho_1 = .7$ the upper bound for $\overset{*}{c}$ for various values of N and ρ_2 is given in the table.

N \ ρ_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.	1.00688	1.37842	1.69797	2.03493	2.35508	2.77325	3.11900	3.12513	3.57324
2.	1.43334	1.69658	1.91730	2.14197	2.35454	2.61333	2.83454	2.88732	3.16019
3.	1.75425	1.97732	2.15643	2.33266	2.49636	2.68910	2.85380	2.90338	3.10158
4.	1.98339	2.18969	2.34962	2.50220	2.64076	2.79950	2.93376	2.97774	3.13582
5.	2.14443	2.34274	2.49362	2.63453	2.75998	2.90039	3.01745	3.05724	3.19211
6.	2.25716	2.45093	2.59724	2.73230	2.85097	2.98156	3.08895	3.12601	3.24730
7.	2.33600	2.52686	2.67059	2.80257	2.91768	3.04305	3.14507	3.18046	3.29391
8.	2.39114	2.58003	2.72214	2.85237	2.96554	3.08809	3.18715	3.22156	3.33053
9.	2.42970	2.61722	2.75827	2.88741	2.99947	3.12045	3.21787	3.25169	3.35813
10.	2.45666	2.64323	2.78355	2.91199	3.02336	3.14343	3.23992	3.27340	3.37840
11.	2.47552	2.66141	2.80123	2.92920	3.04012	3.15964	3.25559	3.28887	3.39303
12.	2.48870	2.67413	2.81360	2.94124	3.05187	3.17104	3.26665	3.29981	3.40348
13.	2.49793	2.68303	2.82225	2.94967	3.06010	3.17903	3.27443	3.30751	3.41089
14.	2.50437	2.68925	2.82830	2.95556	3.06585	3.18463	3.27989	3.31292	3.41612
15.	2.50888	2.69360	2.83253	2.95968	3.06987	3.18854	3.28372	3.31672	3.41980
16.	2.51204	2.69664	2.83549	2.96256	3.07269	3.19129	3.28640	3.31938	3.42239
17.	2.51424	2.69876	2.83765	2.96457	3.07465	3.19320	3.28828	3.32124	3.42420
18.	2.51578	2.70025	2.83900	2.96598	3.07603	3.19454	3.28959	3.32254	3.42546
19.	2.51686	2.70129	2.84001	2.96697	3.07699	3.19548	3.29050	3.32345	3.42636

It may be remarked in passing that employing an a.s.f. is profitable if $c_1/c_2 <$ the upper bound as given in the table. For example, if $M=40$, $N=10$, $\rho_1=.7$, $\rho_2=.5$, then employing an a.s.f. is profitable only when $c_1/c_2 < 3.023{,}36$.

ACKNOWLEDGEMENTS

The author is grateful to Dr. K. Murari, Department of Mathematics, B.N. Chakravarty University, Kurukshetra for suggesting the problem and supervising the investigations and to Pr. C. Mohan for useful discussions. Thanks are also due to Dr. S. D. Chopra, Senior Professor and Head of the Department of Mathematics, B.N. Chakravarty University, Kurukshetra for his keen interest in the work.

REFERENCES

1. J. ROMANI, *La Teoria de las colas aplicada a un problema de production industrial*, Trabajos estadistica, Vol. 6, 1955, Cuaderno ill.
2. C. R. PHILLIPS, *A Variable Channel Queueing Model with a Limited number of Channels*, Thesis, Georgia Institute of Technology, 1960.
3. K. MURARI, *Some c-Additional Service Channels, Limited Space Queueing Problem*, Z.A.M.M., Vol. 51, 1971, pp. 315-318.
4. P. M. MORSE, *Queues, Inventories and Maintenance*, New York, Wiley, 1958.