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ADJACENT VERTICES OF THE ALL 0-1 PROGRAMMING POLYTOPE (*) (1)

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Abstract. — *We give a constructive characterization of adjacency relations between vertices of the convex hull of feasible 0-1 points of an all 0-1 program. This characterization can be used, for instance, to generate all vertices of the convex hull, adjacent to a given vertex. As a by-product, we establish a strong bound on the diameter of the convex hull of feasible 0-1 points.*

Any linear 0-1 programming problem can be brought (by using binary expansion on the slack variables, when necessary, or other devices) to the form of an equality-constrained all 0-1 program:

(P) $$\min \{ cx \mid x \in X, x \text{ integer} \}$$

where

$$X = \{ x \in R^n \mid Ax = b, x \geq 0 \}$$

and where A is $m \times n$, and $Ax = b$ implies $x_j \leq 1$, $\forall j \in N = \{ 1, \ldots, n \}$.

We will assume, without loss of generality, that A has no identical columns or zero columns, and is of full row rank. The j-th column of A will be denoted a_j.

Let (P') denote the linear program associated with (P), i. e.,

(P') $$\max \{ cx \mid x \in X \}.$$

Further, let X_I be the convex hull of the feasible 0-1 points, i. e.,

$$X_I = \text{conv} \{ x \in X \mid x \text{ integer} \}$$

and let vert X (vert X_I) denote the set of vertices of X (of X_I). X_I is the all 0-1 programming polytope referred to in the title.

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It is a well-known property of 0-1 programs that every feasible integer solution is basic. Hence, vert $X_I \subseteq$ vert X. Further, a solution associated with a feasible basis B, whose columns are indexed by I, is integer if and only if $\sum_{i \in Q} a_i = b$ for some $Q \subseteq I$, and Q is unique whenever it exists.

Finally, if B is a feasible basis, I and J are the associated basic and nonbasic index sets, and $\bar{a}_j = B^{-1} a_j$. To simplify notation, we assume the components of x to have been ordered so that $I = \{1, \ldots, m\}$; thus the components of $\bar{a}_j$ are $\bar{a}_{ij}$, $i = 1, \ldots, m$. Observe that $\bar{a}_{ij} > 0$ for at least one $i \in I$ and every $j \in J$, since X is bounded. Further, we denote

$$\bar{a}^j = \begin{pmatrix} \bar{a}_j \\ -e_j \end{pmatrix} \tag{1}$$

where e_j is the $(n - m)$-dimensional unit vector whose j-th component is 1; i. e., the n-vector $\bar{a}^j$ is the j-th column of the Tucker-tableau. The k-th component of $\bar{a}^j$ is denoted by $\bar{a}^j_k$.

Given a linear program, two *bases* are called *adjacent* if they differ in exactly one column. Two *basic feasible solutions* are called *adjacent* if they are adjacent vertices of the feasible set (i. e., distinct vertices contained in an edge, or 1-dimensional face). Two adjacent bases may be associated with the same solution; while two adjacent basic feasible solutions may be associated with two bases that are not adjacent to each other.

The results of this paper were first shown in [1], [2] to hold for the set partitioning problem, a special case of the problem considered here. Most of the proofs given in [2] carry over to the general case with only minor changes, but the main result (the sufficiency part of Theorem 3) requires a different approach. For the sake of completeness, we give all the proofs.

THEOREM 1 : Let x^1 and x^2 be two feasible integer solutions to (P'). Let B be a basis matrix associated with x^1, let I and J be the index sets for the basic and nonbasic variables respectively, and for $i = 1, 2$, let

$$Q_i = \{ j \in N \mid x^i_j = 1 \}, \qquad \bar{Q}_i = N - Q_i .$$

Then, denoting $\bar{a}_j = B^{-1} a_j$, $j \in J$,

$$\sum_{j \in J \cap Q_2} \bar{a}_{kj} = \begin{cases} 1 & k \in Q_1 \cap \bar{Q}_2 \\ -1 & k \in Q_2 \cap \bar{Q}_1 \cap I \\ 0 & k \in (Q_1 \cap Q_2) \cup (\bar{Q}_1 \cap \bar{Q}_2 \cap I) . \end{cases} \tag{2}$$

Proof: From the definition of Q_i, $i = 1, 2$, we have

$$\sum_{k \in Q_1} a_k = \sum_{k \in Q_2} a_k ,$$

which implies

$$\sum_{j\in J_1\cap Q_2} a_j = \sum_{k\in Q_2} a_k - \sum_{k\in Q_2\cap I} a_k$$
$$= \sum_{k\in Q_1} a_k - \sum_{k\in Q_2\cap I} a_k$$
$$= \sum_{k\in Q_1\cap\bar{Q}_2} a_k - \sum_{k\in Q_2\cap\bar{Q}_1\cap I} a_k$$

(by subtracting and adding $\sum_{k\in Q_1\cap Q_2} a_k$).

Premultiplying the last equation by B^{-1} then produces (2), since the vectors a_k, $k\in Q_1$ and $k\in I$, are columns of B. *Q. E. D.*

Next we state the converse of Theorem 1.

THEOREM 2: Let x^1 be a feasible integer solution to (P'), let B, I, J, Q_1 and $\bar{a}_j$, $j\in J$, be defined as in Theorem 1. Further, let the index set $Q\subseteq J$ satisfy

$$\sum_{j\in Q}\bar{a}_{kj} = \begin{cases} 0 \text{ or } 1 & k\in Q_1 \\ 0 \text{ or } -1 & k\in I_1\cap\bar{Q}_1 . \end{cases} \tag{3}$$

Then x^2 defined by

$$x^2 = x^1 - \sum_{j\in Q}\bar{a}^j \tag{4}$$

is a basic feasible solution to (P'), and

$$x_j^2 = \begin{cases} 1 & j\in Q_2 = Q\cup S \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where

$$S = \{\, k\in Q_1 \mid \sum_{j\in Q}\bar{a}_{kj} = 0\,\} \cup \{\, k\in\bar{Q}_1\cap I \mid \sum_{j\in Q}\bar{a}_{kj} = -1\,\}.$$

Proof: Consider the problem ($\bar{P}'$) in $(n+1)$-space, obtained from (P') by augmenting A with the composite column $a_{j_*} = \sum_{j\in Q} a_j$. The transformed column $\bar{a}_{j_*} = B^{-1}a_{j_*}$ has an entry $\bar{a}_{kj_*} = 1$ for some $k\in Q_1$, for otherwise (3) implies $\bar{a}_{kj_*}\leq 0$, $\forall k\in I$, which is impossible in view of the boundedness of the solution set. Pivoting on $\bar{a}_{kj_*} = 1$ yields a feasible solution $\tilde{x}^2$ to ($\bar{P}'$), defined by

$$\tilde{x}_j^2 = \begin{cases} 1 & j\in\{j_*\}\cup S \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\sum_{j\in S} a_j + a_{j*} = \sum_{j\in S\cup Q} a_j = e$$

it follows that x^2 as defined by (5) is feasible for (P'). Since x^2 is integer, it is also basic. From Theorem 1, relation (4) follows with $Q = J_1\cap Q_2$. *Q. E. D.*

A set $Q\subset J$ for which (3) holds will be called *decomposable* if it can be partitioned into two subsets, Q^* and Q^{**}, such that (3) remains true when Q is replaced by Q^* and Q^{**} respectively.

We now give a necessary and sufficient condition for two integer vertices of X to be adjacent on X_I.

THEOREM 3: Let x^1 and x^2 be two feasible integer solutions to (P'), with B, I, J, Q_1 and $\bar{a}_j$, $j \in J$, defined as in Theorem 1, and $Q_2 = \{ j \in N \mid x_j^2 = 1 \}$. Then x^2 is adjacent to x^1 on X_I (i. e., x^1 and x^2 lie on an edge of X_I) if and only if $Q = J \cap Q_2$ is not decomposable.

Proof: (*i*) Necessity. Suppose Q is decomposable into Q^* and Q^{**}. Then the vectors

$$x^i = x^1 - \sum_{j \in S_i} \bar{a}^j, \qquad i = 2, 3, 4 \tag{6}$$

where $S_2 = Q$, $S_3 = Q^*$, $S_4 = Q^{**}$, and $\bar{a}^j$ is defined by (1), are all feasible integer solutions to (P'), hence vertices of X_I. Let $\pi x = \pi_0$ be a supporting hyperplane for X_I, such that $\pi x^i = \pi_0$ for $i = 1$, 2 and $\pi x \leq \pi_0$, $\forall x \in X_I$. (If no such hyperplane exists, then x^1 and x^2 are not adjacent on X_I, and the statement is proved.) Then from (6)

$$\pi x^1 = \pi x^2 = \pi x^1 - \pi\Big(\sum_{j \in Q} \bar{a}^j\Big) = \pi_0$$

or

$$\pi\Big(\sum_{j \in Q} \bar{a}^j\Big) = 0\,, \tag{7}$$

whereas

$$\pi x^3 = \pi x^1 - \pi\Big(\sum_{j \in Q^*} \bar{a}^j\Big) \leq \pi_0 = \pi x^1$$

$$\pi x^4 = \pi x^1 - \pi\Big(\sum_{j \in Q^{**}} \bar{a}^j\Big) \leq \pi_0 = \pi x^1$$

or

$$\pi\Big(\sum_{j \in Q^*} \bar{a}^j\Big) \geq 0\,, \qquad \pi\Big(\sum_{j \in Q^{**}} \bar{a}^j\Big) \geq 0\,. \tag{8}$$

Then from (7) and (8) we have

$$\pi\Big(\sum_{j \in Q^*} \bar{a}^j\Big) = 0\,, \qquad \pi\Big(\sum_{j \in Q^{**}} \bar{a}^j\Big) = 0$$

or $\pi x^3 = \pi x^4 = \pi_0$. Hence any supporting hyperplane for X_I that contains x^1 and x^2, also contains x^3 and x^4; i. e., x^1 and x^2 cannot lie on an edge of, or be adjacent on, X_I.

(*ii*) Sufficiency. Suppose x^1 and x^2 are not adjacent on X_I. Let F be the face of minimal dimension of X_I, which contains both x^1 and x^2 (F is clearly unique), and let $x^{11}, \ldots, x^{1p}$ be the vertices of F adjacent to x^1 on F. The (translated) convex polyhedral cone

$$C(x^1) = \{ x \mid x = x^1 + (x^{1i} - x^1)\lambda_i, \lambda_i \geq 0, i = 1, \ldots, p \}$$

is known (see for instance [3]) to be the intersection of those halfspaces H_i^+, $i = 1, \ldots, p$, such that $x^1 = \bigcap_{i=1}^{p} H_i$, where $H_i = bdH_i^+$. Since $\{ H_i^+ \}_{i=1}^{i=p}$ is a subset of the set of halfspaces whose intersection is X_I, clearly $X_I \subseteq C(x^1)$, and therefore every vertex x of F can be expressed as

$$x = x^1 + \sum_{i=1}^{p} (x^{1i} - x^1)\lambda_i, \qquad \lambda_i \geq 0, \qquad i = 1, \ldots, p. \tag{9}$$

Since x^2 is not adjacent to x^1, $p \geq 2$. From Theorems 1 and 2,

$$x^{1i} = x^1 - \sum_{j \in Q_{1i}} \bar{a}^j, \qquad i = 1, \ldots, p \tag{10}$$

and

$$x^2 = x^1 - \sum_{j \in Q} \bar{a}^j \tag{11}$$

where $Q_{1i} \subseteq J$, $i = 1, \ldots, p$, and $Q \subseteq J$.

Since F is the lowest-dimensional face of X_I containing both x^1 and x^2, there exist $\lambda_i > 0$ for $i = 1, \ldots, p$, such that (9) holds with $x = x^2$. For if not, then x^2 is contained in a face F' of $C(x^1)$ such that

$$\dim F' < \dim C(x^1) = \dim F .$$

But $F'' = \text{aff } F' \cap X_I$ is a face of X_I that contains both x^1 and x^2 and

$$\dim F'' = \dim F' < \dim F ,$$

which contradicts the assumption that F is the lowest-dimensional face of X_I containing both x^1 and x^2. Using (10) and the fact that (9) holds with $x = x^2$ for some $\lambda_i > 0$, $i = 1, \ldots$, p, we have

$$\begin{aligned} x^2 &= x^1 + \sum_{i=1}^{p} (x^{1i} - x^1)\lambda_i \\ &= x^1 - \sum_{i=1}^{p} \Big(\sum_{j \in Q_{1i}} \bar{a}^j \Big)\lambda_i \end{aligned}$$

and from (11)

$$\sum_{j \in Q} \bar{a}^j = \sum_{i=1}^{p} \Big(\sum_{j \in Q_{1i}} \bar{a}^j \Big)\lambda_i, \quad \text{with} \quad \lambda_i > 0, \qquad i = 1, \ldots, p,$$

which implies $Q = \bigcup_{i=1}^{p} Q_{1i}$.

We now partition Q into two subsets $Q^* = Q_{11}$ and $Q^{**} = Q - Q^*$. To complete the proof, we will show that (3) holds when Q is replaced by Q^{**} (for Q^* this follows from Theorem 1). This will be done by showing that x^{**} is a feasible integer solution to (P'), where

$$\begin{aligned} x^{**} &= x^1 - \sum_{j \in Q^{**}} \bar{a}^j \\ &= x^2 + \sum_{j \in Q^*} \bar{a}^j \end{aligned} \tag{12}$$

Theorem 1 then implies that (3) holds with Q replaced by Q^{**}.

First, from Theorem 1 and the definition of Q^*, x^{**} is integer. Next we show by contradiction that $x^{**} \geqq 0$. Suppose $x_k^{**} < 0$. Then from (12), $x_k^2 = 0$ and $\sum_{j \in Q_{11}} \bar{a}_k^j = -1$ (since $Q^* = Q_{11}$). But from (10), this implies (for $i = 1$) $x_k^1 = 0$, and hence

$$\sum_{j \in Q_{1i}} \bar{a}_k^j \leq 0, \qquad \forall i \in \{1, \ldots, p\} \tag{13}$$

But

$$x_k^2 = x_k^1 - \sum_{i=1}^{p} \Big(\sum_{j \in Q_{1i}} \bar{a}_k^j\Big)\lambda_i, \qquad \lambda_i > 0, \qquad i = 1, \ldots, p; \tag{14}$$

hence $x_k^2 > 0$, contradicting our earlier finding that $x_k^2 = 0$. Hence, $x^{**} \geq 0$.

Suppose on the other hand that $x_k^{**} > 1$. By (12), $x_k^2 = 1$ and $\sum_{j \in Q_{11}} \bar{a}_k^j = 1$ (since $Q_{11} = Q^*$). But from (10), this implies (for $i = 1$) that $x_k^1 = 1$, and hence that (13) holds with reversed inequality. Again from (14) we conclude that $x_k^2 < 1$, contradicting our earlier finding that $x_k^2 = 1$. Consequently, $0 \leq x_k^{**} \leq 1$ for all $k \in N$. Finally, $Ax^{**} = b$, since

$$\begin{aligned} A\bar{a}^j &= (B, R)\begin{pmatrix} B^{-1}a_j \\ -e_j \end{pmatrix} \\ &= a_j - a_j = 0, \qquad \forall j \in J \end{aligned}$$

where R is the submatrix of A consisting of the columns a_j, $j \in J$. Hence x^{**} is a feasible 0-1 point. *Q. E. D,*

COROLLARY 3.1 : Let x^1 and x^2 be two vertices of X_I, and let B, I, J and $\bar{a}^j$, $j \in J$, be defined as above. Then x^2 is not adjacent to x^1 on X_I, if and only if there exists a family of p sets $Q_{1i} \subseteq J$, $i = 1, \ldots, p$, such that

(*i*) $p \geq 2$;
(*ii*) $Q_{1i} \cap Q_{1k} = \emptyset$, $\forall i \neq k$;
(*iii*) the points

$$x^{1i} = x^1 - \sum_{j \in Q_{1i}} \bar{a}^j, \qquad i = 1, \ldots, p$$

are vertices of X_I, adjacent to x^1; and

(*iv*)
$$x^2 = x^1 - \sum_{i=1}^{p} \sum_{j \in Q_{1i}} \bar{a}^j$$
$$= x^1 + \sum_{i=1}^{p} (x^{1i} - x^1).$$

Proof: (α) Necessity. If x^1 and x^2 are not adjacent on X_I, then by Theorem 3 $Q = J \cap Q_2$ can be partitioned into two subsets Q^* and Q^{**} such that (3) holds with Q replaced by Q^* and Q^{**}. If

$$x^* = x^1 - \sum_{j \in Q^*} \bar{a}^j$$

and

$$x^{**} = x^1 - \sum_{j \in Q^{**}} \bar{a}^j$$

are both adjacent to x^1, the statement is proved; otherwise the reasoning can be applied to Q^* and/or Q^{**}, and can be repeated as many times as needed to obtain pairwise disjoint sets Q_{1i}, $i = 1, \ldots, p$, with $p \geq 2$, which are not decomposable.

(β) Sufficiency. If the condition holds, then $Q = \bigcup_{i=1}^{p} Q_{1i} = J \cap Q_2$. Furthermore, (7) is satisfied when Q is replaced by Q_{1i} for $i = 1, \ldots, p$.

From (*iii*) it follows that the vectors $\sum_{j \in Q_{1i}} \bar{a}^j$ and $\sum_{j \in Q_{1h}} \bar{a}^j$ are mutually orthogonal for all $i \neq h$, $i, h \in \{1, \ldots, p\}$. Consequently, (3) also holds when Q is replaced by $\bigcup_{i=2}^{p} Q_{1i}$. Thus Q is decomposable into Q_{11} and $\bigcup_{i=2}^{p} Q_{1i}$, hence x^1 and x^2 are not adjacent. *Q. E. D.*

Corollary 3.2 : If x^1 and x^2 are two non-adjacent vertices of X_I related to each other by (*iv*), then for any subset H of $\{1, \ldots, p\}$,

$$x^* = x^1 - \sum_{i \in H} \sum_{j \in Q_{1i}} \bar{a}^j$$
$$= x^1 + \sum_{i \in H} (x^{1i} - x^1)$$

is a vertex of X_I.

Proof: From (*iii*), the vectors $\sum_{j \in Q_{1i}} \bar{a}^j$ and $\sum_{j \in Q_{1h}} \bar{a}^j$ are pairwise orthogonal for all $i, h \in \{1, \ldots, p\}$, $i \neq h$; hence if (3) holds for $Q = \bigcup_{i=1}^{p} Q_{1i}$, then it also holds when Q is replaced by $\bigcup_{i \in H} Q_{1i}$. *Q. E. D.*

Corollary 3.2 can be given the following geometric interpretation. A *path* on X_I between two vertices x, y is a sequence of vertices $(x^1, x^2, \ldots, x^k)$, with $x^1 = x$, $x^k = y$, such that every pair of vertices x^i, x^{i+1}, $i = 1, \ldots, k-1$, is connected by an edge of X_I; the length of the path being $k - 1$. The *edge-distance* $d(x, y)$ between x and y is the length of a shortest path between x and y. The *diameter* $\delta(X_I)$ of X_I is the longest edge-distance between any two vertices of X_I.

Let $[a]$ denote the largest integer less than or equal to the real number a. For the next corollary, we assume that the matrix A defining X_I has no identical columns.

COROLLARY 3.3 : $\delta(X_I) \leq \left[\frac{z^*}{2}\right]$ where

$$z^* = \max \left\{ \sum_{j=1}^{n} x_j \mid x \in X_I \right\}.$$

Proof: Let x^1, x^2 be a pair of vertices of X_I which are at maximal edge-distance from each other, i. e., for which

$$d(x^1, x^2) = \delta(X_I).$$

Further, let B be a basis associated with x^1; let I, J, Q_1 and $\bar{a}^j$, $j \in J$, be defined as above.

From Corollary 3.1,

$$x^2 = x^1 - \sum_{i=1}^{p} \sum_{j \in Q_{1i}} \bar{a}^j \tag{15}$$

and from Corollary 3.2, (15) holds with $p \geq \delta(X_I)$, since the sequence of vertices $\{ x^{10}, x^{11}, \ldots, x^{1p} \}$, of X_I, where $x^{10} = x^1$ and $x^{1p} = x^2$, with

$$x^{1k} = x^1 - \sum_{i=1}^{k} \sum_{j \in Q_{1i}} \bar{a}^j \qquad k = 1, \ldots, p,$$

defines a path of length p between x^1 and x^2.

Now let $P = \{ 1, \ldots, p \}$, and let

$$P_1 = \{ i \in P \mid \sum_{j \in Q_{1i}} \bar{a}_k^j = 1 \text{ for exactly one } k \in N \}.$$

If $P_1 = \emptyset$, then from (15) and the definition of z^*,

$$p \leq \left[\frac{|Q_1|}{2}\right] \leq \left[\frac{z^*}{2}\right]$$

which, together with $\delta(X_I) \leq p$, proves the corollary. Suppose now that $P_1 \neq \emptyset$. Then for each $i \in P_1$, the vector $\sum_{j \in Q_{1i}} \bar{a}^j$ has at least two negative components.

For otherwise Q_{1i} is a singleton, say $Q_{1i} = \{ h \}$, and $\bar{a}^h$ is of the form

$$\bar{a}^h = \left(\frac{e_i^m}{-\ e_h^{n-m}} \right)$$

(where e_j^k is the k-dimensional unit vector whose j-th entry is 1); which implies that the nonbasic column a_h of A is identical to a basic column, contrary to our assumption. Now let

$$x^3 = x^1 - \sum_{i \in P_1} \sum_{j \in Q_{1i}} \bar{a}^j ,$$
$$x^4 = x^1 - \sum_{i \in P - P_1} \sum_{j \in Q_{1i}} \bar{a}^j ,$$

where both x^3 and x^4 are vertices of X_I (Corollary 3.2). Then

$$x^4 = x^3 - \sum_{i \in P_1} (- \sum_{i \in Q_{1i}} \bar{a}^j) - \sum_{i \in P - P_1} \sum_{j \in Q_{1i}} \bar{a}^j ; \qquad (16)$$

but in view of

$$\sum_{j \in Q_{1i}} \bar{a}_k^j \neq 0 \Rightarrow \sum_{j \in Q_{1h}} \bar{a}_k^j = 0 , \qquad \forall k \in N , \quad \forall i, h \in P, \quad i \neq h ,$$

(16) implies that $p \leq \left\lceil \frac{|Q_3|}{2} \right\rceil$, where $Q_3 = \{ j \in N \mid x_j^3 = 1 \}$. Hence, in view of $\delta(X_I) \leq p$ and $|Q_3| \leq z^*$, the corollary follows. *Q. E. D.*

REMARK: If in the definition of X_I, $A = (A_G, I_m)$ and $b = (e^m)$, where I_m is the identity matrix of order m, $e^m = (1, \ldots, 1) \in R^m$, and A_G is the $m \times \binom{m}{2}$ incidence matrix of the complete undirected graph with m vertices, then $\delta(X_I) = \left\lceil \frac{z^*}{1} \right\rceil$, since $\delta(X_I)$ is achieved by the minimum distance between the empty matching and any maximum matching on the matching polytope. In this sense the upper bound on $\delta(X_I)$ given in the above Corollary is a strongest possible one.

The property stated in the next Theorem, which does not hold for arbitrary integer programs, has some interesting algorithmic implications.

THEOREM 4: Let x^1 be a non-optimal vertex of X_I, let x^{1i}, $i = 1, \ldots, k$, be those vertices of X_I adjacent to x^1, and such that $cx^{1i} < cx^1$, $i = 1, \ldots, k$. Then the convex polyhedral cone

$$C = \{ x \mid x = x^1 + \sum_{i=1}^{k} (x^{1i} - x^1) \lambda_i , \lambda_i \geq 0, i = 1, \ldots, n \}$$

contains an optimal vertex of X_I.

Proof: Let $\bar{x}$ be an optimal vertex of X_I. If $\bar{x}$ is adjacent to x^1, then $\bar{x} \in C$. Otherwise, $\bar{x}$ can be expressed (Corollary 3.1) as

$$\begin{aligned}\bar{x} &= x^1 - \sum_{i=1}^{p} \sum_{j \in Q_{1i}} \bar{a}^j \\ &= x^1 + \sum_{i=1}^{p} (x^{1i} - x^1)\end{aligned}$$

where x^{1i}, $i = 1, \ldots, p$, are vertices of X_I adjacent to x^1. Then

$$0 < cx^1 - c\bar{x} = \sum_{i=1}^{p} \sum_{j \in Q_{1i}} c\bar{a}^j$$

Let $\{1, \ldots, p\} = P$, and let $P^+ = \{1, \ldots, k\}$. Since $c\bar{x} < cx^1$, $P^+ \neq 0$. From Corollary 3.2, the point

$$\begin{aligned}x^* &= x^1 - \sum_{i \in P^+} \sum_{j \in Q_{1i}} \bar{a}^j \\ &= x^1 + \sum_{i \in P^+} (x^{1i} - x^1)\end{aligned}$$

is a vertex of X_I, and from the definition of P^+,

$$\begin{aligned}cx^* &= cx^1 + \sum_{i \in P^+} c(x^{1i} - x^1) \\ &\leq cx^1 + \sum_{i=1}^{p} c(x^{1i} - x^1) = c\bar{x}.\end{aligned}$$

Thus, since $\bar{x}$ is optimal, so is x^*; and since the vertices x^{1i}, $i \in P^+$ are among those that generate C, clearly $x^* \in C$. *Q. E. D.*

The above results can be used to generate integer vertices of the feasible set X, adjacent to a given integer vertex x^1. Namely, by systematically generating composite columns of the form $\bar{a}^{j_*} = \sum_{j \in Q} \bar{a}^j$, where Q satisfies the requirements for $x^1 - \bar{a}^{j_*}$ to be a vertex of X_I adjacent to x^1, one can obtain all such vertices. The efficiency of a procedure based on these results will of course be highly dependent on the specific way in which they are used; and in view of the many options that are available, this topic requires further investigation.

REFERENCES

1. E. Balas and M. Padberg, On the Set Covering Problem. *Operations Research*, 20, 1972, p. 1152-1161.
2. E. Balas and M. Padberg, On the Set Covering Problem. II: An Algorithm for Set Partitioning. *Operations Research*, 23, 1975, p. 74-90.
3. J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions. I.* Springer, New York, 1970.