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Adjacent vertices of the all 0-1 programming polytope


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ADJACENT VERTICES
OF THE ALL 0-1 PROGRAMMING POLYTOPE (*) (1)

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Abstract. — We give a constructive characterization of adjacency relations between vertices of the convex hull of feasible 0-1 points of an all 0-1 program. This characterization can be used, for instance, to generate all vertices of the convex hull, adjacent to a given vertex. As a by-product, we establish a strong bound on the diameter of the convex hull of feasible 0-1 points.

Any linear 0-1 programming problem can be brought (by using binary expansion on the slack variables, when necessary, or other devices) to the form of an equality-constrained all 0-1 program:

(P) \[ \min \{ cx \mid x \in X, x \text{ integer} \} \]

where \[ X = \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \} \]

and where \( A \) is \( m \times n \), and \( Ax = b \) implies \( x_j \leq 1, \forall j \in N = \{ 1, \ldots, n \} \).

We will assume, without loss of generality, that \( A \) has no identical columns or zero columns, and is of full row rank. The \( j \)-th column of \( A \) will be denoted \( a_j \).

Let \((P')\) denote the linear program associated with \((P)\), i. e.,

\[(P')\] \[ \max \{ cx \mid x \in X \} \]

Further, let \( X_I \) be the convex hull of the feasible 0-1 points, i. e.,

\[ X_I = \text{conv} \{ x \in X \mid x \text{ integer} \} \]

and let \( \text{vert } X (\text{vert } X_I) \) denote the set of vertices of \( X \) (of \( X_I \)). \( X_I \) is the all 0-1 programming polytope referred to in the title.

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It is a well-known property of 0-1 programs that every feasible integer solution is basic. Hence, \( \text{vert } X_I \subseteq \text{vert } X \). Further, a solution associated with a feasible basis \( B \), whose columns are indexed by \( I \), is integer if and only if \( \sum_{i \in Q} a_i = b \) for some \( Q \subseteq I \), and \( Q \) is unique whenever it exists.

Finally, if \( B \) is a feasible basis, \( I \) and \( J \) are the associated basic and nonbasic index sets, and \( \bar{a}_j = B^{-1} a_j \). To simplify notation, we assume the components of \( x \) to have been ordered so that \( I = \{ 1, \ldots, m \} \); thus the components of \( \bar{a}_j \) are \( \bar{a}_{ij} \), \( i = 1, \ldots, m \). Observe that \( \bar{a}_{ij} > 0 \) for at least one \( i \in I \) and every \( j \in J \), since \( X \) is bounded. Further, we denote

\[
\bar{a}^j = \left( \bar{a}_j - e_j \right)
\]

where \( e_j \) is the \((n - m)\)-dimensional unit vector whose \( j \)-th component is 1; i.e., the \( n \)-vector \( \bar{a}^j \) is the \( j \)-th column of the Tucker-tableau. The \( k \)-th component of \( \bar{a}^j \) is denoted by \( \bar{a}^j_k \).

Given a linear program, two bases are called adjacent if they differ in exactly one column. Two basic feasible solutions are called adjacent if they are adjacent vertices of the feasible set (i.e., distinct vertices contained in an edge, or 1-dimensional face). Two adjacent bases may be associated with the same solution; while two adjacent basic feasible solutions may be associated with two bases that are not adjacent to each other.

The results of this paper were first shown in [1], [2] to hold for the set partitioning problem, a special case of the problem considered here. Most of the proofs given in [2] carry over to the general case with only minor changes, but the main result (the sufficiency part of Theorem 3) requires a different approach. For the sake of completeness, we give all the proofs.

**Theorem 1**: Let \( x^1 \) and \( x^2 \) be two feasible integer solutions to \((P')\). Let \( B \) be a basis matrix associated with \( x^1 \), let \( I \) and \( J \) be the index sets for the basic and nonbasic variables respectively, and for \( i = 1, 2 \), let

\[
Q_i = \{ j \in N \mid x^i_j = 1 \}, \quad \bar{Q}_i = N - Q_i.
\]

Then, denoting \( \bar{a}_j = B^{-1} a_j \), \( j \in J \),

\[
\sum_{j \in J \cap Q_2} \bar{a}_{kj} = \begin{cases} 1 & k \in Q_1 \cap \bar{Q}_2 \\ -1 & k \in Q_2 \cap \bar{Q}_1 \cap I \\ 0 & k \in (Q_1 \cap Q_2) \cup (\bar{Q}_1 \cap \bar{Q}_2 \cap I) \end{cases}
\]

Proof: From the definition of \( Q_i \), \( i = 1, 2 \), we have

\[
\sum_{k \in Q_1} a_k = \sum_{k \in Q_2} a_k,
\]
which implies

\[
\sum_{j \in J_1 \cap Q_2} a_j = \sum_{k \in Q_2} a_k - \sum_{k \in Q_2 \cap I} a_k \\
= \sum_{k \in Q_1} a_k - \sum_{k \in Q_2 \cap I} a_k \\
= \sum_{k \in Q_1 \cap Q_2} a_k - \sum_{k \in Q_2 \cap I} a_k
\]

(by subtracting and adding \(\sum_{k \in Q_1 \cap Q_2} a_k\)).

Premultiplying the last equation by \(B^{-1}\) then produces (2), since the vectors \(a_k, k \in Q_1\) and \(k \in I\), are columns of \(B\). \(Q. E. D.\)

Next we state the converse of Theorem 1.

**Theorem 2:** Let \(x^1\) be a feasible integer solution to \((P')\), let \(B, I, J, Q_1\) and \(\bar{a}_j, j \in J\), be defined as in Theorem 1. Further, let the index set \(Q \subseteq J\) satisfy

\[
\sum_{j \in Q} \bar{a}_{kj} = \begin{cases} 0 & k \in Q_1 \\ 1 & k \in Q_1 \cap J_1 \\ -1 & k \in I_1 \cap \bar{Q}_1 \end{cases}
\]

Then \(x^2\) defined by

\[
x^2 = x^1 - \sum_{j \in Q} \bar{a}_j
\]

is a basic feasible solution to \((P')\), and

\[
x^2_j = \begin{cases} 1 & j \in Q_2 = Q \cup S \\ 0 & \text{otherwise} \end{cases}
\]

where

\[
S = \{ k \in Q_1 | \sum_{j \in Q} \bar{a}_{kj} = 0 \} \cup \{ k \in Q_1 \cap I | \sum_{j \in Q} \bar{a}_{kj} = -1 \}.
\]

**Proof:** Consider the problem \((\bar{P}')\) in \((n + 1)\)-space, obtained from \((P')\) by augmenting \(A\) with the composite column \(\bar{a}_{j_*} = \sum_{j \in Q} a_j\). The transformed column \(\bar{a}_{j_*} = B^{-1}a_{j_*}\) has an entry \(\bar{a}_{k_{j_*}} = 1\) for some \(k \in Q_1\), for otherwise (3) implies \(\bar{a}_{k_{j_*}} \leq 0, \forall k \in I\), which is impossible in view of the boundedness of the solution set. Pivoting on \(\bar{a}_{k_{j_*}} = 1\) yields a feasible solution \(\bar{x}^2\) to \((\bar{P}')\), defined by

\[
\bar{x}^2_j = \begin{cases} 1 & j \in \{j_*\} \cup S \\ 0 & \text{otherwise} \end{cases}
\]

Since

\[
\sum_{j \in S} a_j + a_{j_*} = \sum_{j \in S \cup Q} a_j = \epsilon
\]

it follows that \(x^2\) as defined by (5) is feasible for \((P')\). Since \(x^2\) is integer, it is also basic. From Theorem 1, relation (4) follows with \(Q = J_1 \cap Q_2\). \(Q. E. D.\)

A set \(Q \subseteq J\) for which (3) holds will be called *decomposable* if it can be
partitioned into two subsets, $Q^*$ and $Q^{**}$, such that (3) remains true when $Q$

is replaced by $Q^*$ and $Q^{**}$ respectively.

We now give a necessary and sufficient condition for two integer vertices of $X$ to be adjacent on $X_I$.

**Theorem 3**: Let $x^1$ and $x^2$ be two feasible integer solutions to $(P')$, with $B$, $I$, $J$, $Q_1$ and $a_j$, $j \in J$, defined as in Theorem 1, and $Q_2 = \{ j \in N \mid x_j^2 = 1 \}$. Then $x^2$ is adjacent to $x^1$ on $X_I$ (i.e., $x^1$ and $x^2$ lie on an edge of $X_I$) if and only if $Q = J \cap Q_2$ is not decomposable.

**Proof**: (i) Necessity. Suppose $Q$ is decomposable into $Q^*$ and $Q^{**}$. Then the vectors

$$
\pi x^i = \pi x^2 = \pi x^1 - \pi \left( \sum_{j \in Q^*} a^i_j \right) = \pi x^1 - \pi \left( \sum_{j \in Q^{**}} a^i_j \right) = \pi x^1 - \pi \left( \sum_{j \in Q^{**}} a^i_j \right) = \pi x^1
$$

where $S_2 = Q$, $S_3 = Q^*$, $S_4 = Q^{**}$, and $a^i_j$ is defined by (1), are all feasible integer solutions to $(P')$, hence vertices of $X_I$. Let $\pi x = \pi_0$ be a supporting hyperplane for $X_I$, such that $\pi x^i = \pi_0$ for $i = 1, 2$ and $\pi x \leq \pi_0$, $\forall x \in X_I$. (If no such hyperplane exists, then $x^1$ and $x^2$ are not adjacent on $X_I$, and the statement is proved.) Then from (6)

$$
\pi x^1 = \pi x^2
$$

$$
= \pi x^1 - \pi \left( \sum_{j \in Q^*} a^i_j \right) = \pi_0
$$

or

$$
\pi \left( \sum_{j \in Q^*} a^i_j \right) = 0,
$$

(7)

whereas

$$
\pi x^3 = \pi x^1 - \pi \left( \sum_{j \in Q^*} a^i_j \right) \leq \pi_0 = \pi x^1
$$

$$
\pi x^4 = \pi x^1 - \pi \left( \sum_{j \in Q^{**}} a^i_j \right) \leq \pi_0 = \pi x^1
$$

or

$$
\pi \left( \sum_{j \in Q^*} a^i_j \right) \geq 0, \quad \pi \left( \sum_{j \in Q^{**}} a^i_j \right) \geq 0.
$$

(8)

Then from (7) and (8) we have

$$
\pi \left( \sum_{j \in Q^*} a^i_j \right) = 0, \quad \pi \left( \sum_{j \in Q^{**}} a^i_j \right) = 0
$$

or $\pi x^3 = \pi x^4 = \pi_0$. Hence any supporting hyperplane for $X_I$ that contains $x^1$ and $x^2$, also contains $x^3$ and $x^4$; i.e., $x^1$ and $x^2$ cannot lie on an edge of, or be adjacent on, $X_I$.

(ii) Sufficiency. Suppose $x^1$ and $x^2$ are not adjacent on $X_I$. Let $F$ be the face of minimal dimension of $X_I$, which contains both $x^1$ and $x^2$ ($F$ is clearly
unique), and let $x_1^{i_1}, \ldots, x_1^{i_p}$ be the vertices of $F$ adjacent to $x_1$ on $F$. The (translated) convex polyhedral cone

$$C(x_1) = \{ x | x = x_1 + (x_1^{i_1} - x_1^{i_1})\lambda_i, \lambda_i \geq 0, i = 1, \ldots, p \}$$

is known (see for instance [3]) to be the intersection of those halfspaces $H_i^+, i = 1, \ldots, p$, such that $x^1 = \bigcap_{i=1}^p H_i$, where $H_i = bd H_i^+$. Since $\{ H_i^+ \}_{i=1}^p$ is a subset of the set of halfspaces whose intersection is $X_I$, clearly $X_I \subseteq C(x_1)$, and therefore every vertex $x$ of $F$ can be expressed as

$$x = x_1 + \sum_{i=1}^p (x_1^{i_1} - x_1^{i_1})\lambda_i, \quad \lambda_i \geq 0, \quad i = 1, \ldots, p.$$  \hspace{1cm} (9)

Since $x^2$ is not adjacent to $x_1$, $p \geq 2$. From Theorems 1 and 2,

$$x_1^{i_i} = x_1 - \sum_{j \in Q_{11}} a^j, \quad i = 1, \ldots, p$$  \hspace{1cm} (10)

and

$$x^2 = x_1 - \sum_{j \in Q} a^j$$  \hspace{1cm} (11)

where $Q_{11} \subseteq J, i = 1, \ldots, p$, and $Q \subseteq J$.

Since $F$ is the lowest-dimensional face of $X_I$ containing both $x_1$ and $x_2$, there exist $\lambda_i > 0$ for $i = 1, \ldots, p$, such that (9) holds with $x = x^2$. For if not, then $x_2$ is contained in a face $F'$ of $C(x_1^1)$ such that

$$\dim F' < \dim C(x_1) = \dim F.$$  

But $F'' = \text{aff} F' \cap X_I$ is a face of $X_I$ that contains both $x_1$ and $x_2$ and

$$\dim F'' = \dim F' < \dim F,$$  

which contradicts the assumption that $F$ is the lowest-dimensional face of $X_I$ containing both $x_1$ and $x_2$. Using (10) and the fact that (9) holds with $x = x^2$ for some $\lambda_i > 0, i = 1, \ldots, p$, we have

$$x^2 = x_1 + \sum_{i=1}^p (x_1^{i_1} - x_1^{i_1})\lambda_i$$

$$= x_1 - \sum_{i=1}^p (\sum_{j \in Q_{11}} a^j)\lambda_i$$

and from (11)

$$\sum_{j \in Q} a^j = \sum_{i=1}^p (\sum_{j \in Q_{11}} a^j)\lambda_i, \quad \text{with} \quad \lambda_i > 0, \quad i = 1, \ldots, p,$$

which implies $Q = \bigcup_{i=1}^p Q_{11}$.
We now partition \( Q \) into two subsets \( Q^* = Q_{11} \) and \( Q^{**} = Q - Q^* \). To complete the proof, we will show that (3) holds when \( Q \) is replaced by \( Q^{**} \) (for \( Q^* \) this follows from Theorem 1). This will be done by showing that \( x^{**} \) is a feasible integer solution to \((P')\), where

\[
x^{**} = x^1 - \sum_{j \in Q^{**}} \tilde{a}^j
= x^2 + \sum_{j \in Q^*} \tilde{a}^j
\tag{12}
\]

Theorem 1 then implies that (3) holds with \( Q \) replaced by \( Q^{**} \).

First, from Theorem 1 and the definition of \( Q^* \), \( x^{**} \) is integer. Next we show by contradiction that \( x^{**} \geq 0 \). Suppose \( x^{**} < 0 \). Then from (12), \( x^2 = 0 \) and \( \sum_{j \in Q_{11}} \tilde{a}^j = -1 \) (since \( Q^* = Q_{11} \)). But from (10), this implies (for \( i = 1 \)) \( x^1_k = 0 \), and hence

\[
\sum_{j \in Q_{11}} \tilde{a}^j \leq 0, \quad \forall i \in \{1, \ldots, p\}
\tag{13}
\]

But

\[
x^2_k = x^1_k - \frac{p}{i=1} \left( \sum_{j \in Q_{11}} \tilde{a}^j \right) \lambda_i, \quad \lambda_i > 0, \quad i = 1, \ldots, p;
\tag{14}
\]

hence \( x^2_k > 0 \), contradicting our earlier finding that \( x^2_k = 0 \). Hence, \( x^{**} \geq 0 \).

Suppose on the other hand that \( x^{**} > 1 \). By (12), \( x^2 = 1 \) and \( \sum_{j \in Q_{11}} \tilde{a}^j = 1 \) (since \( Q_{11} = Q^* \)). But from (10), this implies (for \( i = 1 \)) that \( x^1_k = 1 \), and hence that (13) holds with reversed inequality. Again from (14) we conclude that \( x^2_k < 1 \), contradicting our earlier finding that \( x^2_k = 1 \). Consequently, \( 0 \leq x^{**}_k \leq 1 \) for all \( k \in N \). Finally, \( Ax^{**} = b \), since

\[
A \tilde{a}^j = (B, R) \left( B^{-1} a_j - e_j \right)
= a_j - a_j = 0, \quad \forall j \in J
\]

where \( R \) is the submatrix of \( A \) consisting of the columns \( a_j, j \in J \). Hence \( x^{**} \) is a feasible 0-1 point. Q. E. D.

**Corollary 3.1**: Let \( x^1 \) and \( x^2 \) be two vertices of \( X_I \), and let \( B, I, J \) and \( \tilde{a}^j, j \in J \), be defined as above. Then \( x^2 \) is not adjacent to \( x^1 \) on \( X_I \), if and only if there exists a family of \( p \) sets \( Q_{1i} \subseteq J, i = 1, \ldots, p \), such that

(i) \( p \geq 2 \);
(ii) \( Q_{1i} \cap Q_{1k} = \emptyset, \forall i \neq k \);
(iii) the points

\[
x^{1i} = x^1 - \sum_{j \in Q_{1i}} \tilde{a}^j, \quad i = 1, \ldots, p
\]
are vertices of $X_I$, adjacent to $x^1$; and

$$(iv) \quad x^2 = x^1 - \sum_{i=1}^{p} \sum_{j \in Q_{1i}} \bar{a}^i$$

$$= x^1 + \sum_{i=1}^{p} (x^{1i} - x^1).$$

**Proof**: (a) Necessity. If $x^1$ and $x^2$ are not adjacent on $X_I$, then by Theorem 3 $Q = J \cap Q_2$ can be partitioned into two subsets $Q^*$ and $Q^{**}$ such that (3) holds with $Q$ replaced by $Q^*$ and $Q^{**}$. If

$$x^* = x^1 - \sum_{j \in Q^*} \bar{a}^j$$

and

$$x^{**} = x^1 - \sum_{j \in Q^{**}} \bar{a}^j$$

are both adjacent to $x^1$, the statement is proved; otherwise the reasoning can be applied to $Q^*$ and/or $Q^{**}$, and can be repeated as many times as needed to obtain pairwise disjoint sets $Q_{1i}, i = 1, \ldots, p$, with $p \geq 2$, which are not decomposable.

(b) Sufficiency. If the condition holds, then $Q = \bigcup_{i=1}^{p} Q_{1i} = J \cap Q_2$. Furthermore, (7) is satisfied when $Q$ is replaced by $Q_{1i}$ for $i = 1, \ldots, p$.

From (iii) it follows that the vectors $\sum_{j \in Q_{1i}} \bar{a}^j$ and $\sum_{j \in Q_{1h}} \bar{a}^j$ are mutually orthogonal for all $i \neq h$, $i, h \in \{1, \ldots, p\}$. Consequently, (3) also holds when $Q$ is replaced by $\bigcup_{i=2}^{p} Q_{1i}$. Thus $Q$ is decomposable into $Q_{11}$ and $\bigcup_{i=2}^{p} Q_{1i}$, hence $x^1$ and $x^2$ are not adjacent.

**Corollary 3.2**: If $x^1$ and $x^2$ are two non-adjacent vertices of $X_I$ related to each other by (iv), then for any subset $H$ of $\{1, \ldots, p\}$,

$$x^* = x^1 - \sum_{i \in H} \sum_{j \in Q_{1i}} \bar{a}^j$$

$$= x^1 + \sum_{i \in H} (x^{1i} - x^1)$$

is a vertex of $X_I$.

**Proof**: From (iii), the vectors $\sum_{j \in Q_{1i}} \bar{a}^i$ and $\sum_{j \in Q_{1h}} \bar{a}^i$ are pairwise orthogonal for all $i, h \in \{1, \ldots, p\}$, $i \neq h$; hence if (3) holds for $Q = \bigcup_{i=1}^{p} Q_{1i}$, then it also holds when $Q$ is replaced by $\bigcup_{i \in H} Q_{1i}$.

Q. E. D.
Corollary 3.2 can be given the following geometric interpretation. A path on $X_I$ between two vertices $x, y$ is a sequence of vertices $(x^1, x^2, \ldots, x^k)$, with $x^1 = x, x^k = y$, such that every pair of vertices $x^i, x^{i+1}, i = 1, \ldots, k-1$, is connected by an edge of $X_I$; the length of the path being $k - 1$. The edge-distance $d(x, y)$ between $x$ and $y$ is the length of a shortest path between $x$ and $y$. The diameter $\delta(X_I)$ of $X_I$ is the longest edge-distance between any two vertices of $X_I$.

Let $[a]$ denote the largest integer less than or equal to the real number $a$. For the next corollary, we assume that the matrix $A$ defining $X_I$ has no identical columns.

**Corollary 3.3**: $\delta(X_I) \leq \left\lceil \frac{z^*}{2} \right\rceil$ where

$$z^* = \max \left\{ \sum_{j=1}^{n} x_j \mid x \in X_I \right\}.$$ 

**Proof**: Let $x^1, x^2$ be a pair of vertices of $X_I$ which are at maximal edge-distance from each other, i.e., for which

$$d(x^1, x^2) = \delta(X_I).$$

Further, let $B$ be a basis associated with $x^1$; let $I, J, Q, \bar{a}^i, j \in J$, be defined as above.

From Corollary 3.1,

$$x^2 = x^1 - \sum_{i=1}^{p} \sum_{j \in Q_i} \bar{a}^i$$

and from Corollary 3.2, (15) holds with $p \geq \delta(X_I)$, since the sequence of vertices \(\{x^{10}, x^{11}, \ldots, x^{1p}\}\), of $X_I$, where $x^{10} = x^1$ and $x^{1p} = x^2$, with

$$x^{1k} = x^1 - \sum_{i=1}^{k} \sum_{j \in Q_i} \bar{a}^i \quad k = 1, \ldots, p,$$

defines a path of length $p$ between $x^1$ and $x^2$.

Now let $P = \{1, \ldots, p\}$, and let

$$P_1 = \{ i \in P \mid \sum_{j \in Q_i} \bar{a}^i = 1 \text{ for exactly one } k \in N \}.$$

If $P_1 = \emptyset$, then from (15) and the definition of $z^*$,

$$p \leq \left\lceil \frac{|Q_1|}{2} \right\rceil \leq \left\lceil \frac{z^*}{2} \right\rceil$$

which, together with $\delta(X_I) \leq p$, proves the corollary. Suppose now that $P_1 \neq \emptyset$. Then for each $i \in P_1$, the vector $\sum_{j \in Q_i} \bar{a}^i$ has at least two negative components.
For otherwise \( Q_{11} \) is a singleton, say \( Q_{11} = \{ h \} \), and \( \tilde{a}^h \) is of the form
\[
\tilde{a}^h = \left( \frac{e_i^m}{e_i^m - e_h^m} \right)
\]
(where \( e_j^k \) is the \( k \)-dimensional unit vector whose \( j \)-th entry is 1); which implies that the nonbasic column \( a_h \) of \( A \) is identical to a basic column, contrary to our assumption. Now let
\[
\begin{align*}
x^3 &= x^1 - \sum_{i \in P_1} \sum_{j \in Q_{11}} \tilde{a}^j, \\
x^4 &= x^1 - \sum_{i \in P - P_1} \sum_{j \in Q_{11}} \tilde{a}^j,
\end{align*}
\]
where both \( x^3 \) and \( x^4 \) are vertices of \( X_I \) (Corollary 3.2). Then
\[
x^4 = x^3 - \sum_{i \in P_1} \left( - \sum_{j \in Q_{11}} \tilde{a}^j \right) - \sum_{i \in P - P_1} \sum_{j \in Q_{11}} \tilde{a}^j;
\]
but in view of
\[
\sum_{j \in Q_{11}} \tilde{a}_j^i \neq 0 \Rightarrow \sum_{j \in Q_{11}} \tilde{a}_j^i = 0, \quad \forall k \in N, \forall i, h \in P, \quad i \neq h,
\]
(16) implies that \( p \leq \left[ \frac{1}{Q_3} \right] \), where \( Q_3 = \{ j \in N \mid x_j^3 = 1 \} \). Hence, in view of \( \delta(X_I) \leq p \) and \( |Q_3| \leq z^* \), the corollary follows. 

**Remark:** If in the definition of \( X_I \), \( A = (A_G, I_m) \) and \( b = (e^m) \), where \( I_m \) is the identity matrix of order \( m \), \( e^m = (1, \ldots, 1) \in R^m \), and \( A_G \) is the \( m \times \binom{m}{2} \) incidence matrix of the complete undirected graph with \( m \) vertices, then
\[
\delta(X_I) = \left[ \frac{z^m}{1} \right],
\]
since \( \delta(X_I) \) is achieved by the minimum distance between the empty matching and any maximum matching on the matching polytope. In this sense the upper bound on \( \delta(X_I) \) given in the above Corollary is a strongest possible one.

The property stated in the next Theorem, which does not hold for arbitrary integer programs, has some interesting algorithmic implications.

**Theorem 4:** Let \( x^1 \) be a non-optimal vertex of \( X_I \), let \( x_1^{i1}, i = 1, \ldots, k \), be those vertices of \( X_I \) adjacent to \( x^1 \), and such that \( c x_1^{i1} < c x^1, i = 1, \ldots, k \). Then the convex polyhedral cone
\[
C = \{ x \mid x = x^1 + \sum_{i=1}^{k} (x_1^{i1} - x^1) \lambda_i, \lambda_i \geq 0, i = 1, \ldots, n \}
\]
contains an optimal vertex of \( X_I \).

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Proof: Let $\bar{x}$ be an optimal vertex of $X_I$. If $\bar{x}$ is adjacent to $x^1$, then $\bar{x} \in C$. Otherwise, $\bar{x}$ can be expressed (Corollary 3.1) as

$$\bar{x} = x^1 - \sum_{i=1}^{p} \sum_{j \in Q_{ii}} \bar{a}^j,$$

$$= x^1 + \sum_{i=1}^{p} (x^{1i} - x^1)$$

where $x^{1i}, i = 1, \ldots, p,$ are vertices of $X_I$ adjacent to $x^1$. Then

$$0 < cx^1 - c\bar{x} = \sum_{i=1}^{p} \sum_{j \in Q_{ii}} c\bar{a}^i$$

Let $\{1, \ldots, p\} = P$, and let $P^+ = \{1, \ldots, k\}$. Since $c\bar{x} < cx^1$, $P^+ \neq 0$. From Corollary 3.2, the point

$$x^* = x^1 - \sum_{i \in P^+} \sum_{j \in Q_{ii}} \bar{a}^i$$

$$= x^1 + \sum_{i \in P^+} (x^{1i} - x^1)$$

is a vertex of $X_I$, and from the definition of $P^+$,

$$cx^* = cx^1 + \sum_{i \in P^+} c(x^{1i} - x^1)$$

$$\leq cx^1 + \sum_{i=1}^{p} c(x^{1i} - x^1) = c\bar{x}.$$  

Thus, since $\bar{x}$ is optimal, so is $x^*$; and since the vertices $x^{1i}, i \in P^+$ are among those that generate $C$, clearly $x^* \in C$. Q. E. D.

The above results can be used to generate integer vertices of the feasible set $X$, adjacent to a given integer vertex $x^1$. Namely, by systematically generating composite columns of the form $\bar{a}^h = \sum_{j \in Q} \bar{a}^j$, where $Q$ satisfies the requirements for $x^1 - \bar{a}^h$ to be a vertex of $X_I$ adjacent to $x^1$, one can obtain all such vertices. The efficiency of a procedure based on these results will of course be highly dependent on the specific way in which they are used; and in view of the many options that are available, this topic requires further investigation.

REFERENCES