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## CAPACITY EXPANSION OF A DETERIORATING FACILITY UNDER UNCERTAINTY (\*)

by Charles S. TAPIERO <sup>(1)</sup>

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*Abstract. — An optimum single capacity expansion of a deteriorating facility is determined when future demand for the facility is given by an evolution of probability distributions. Several probabilistic deterioration processes are considered and the probability distribution of a facility's extinction is found. For practical applications, it is suggested that simulation be used in determining the costs and benefits of two or more expansion plans.*

### 1. INTRODUCTION <sup>(2)</sup>

Capacity expansions are strategic decisions involving the highest levels of management. They are designed to fulfill one or both of the following purposes:

- (1) replace deteriorating facilities or improve the efficiency of production processes by a newer technology;
- (2) meet anticipated demand growth for production services.

Capacity expansion involves present outlays measurable only in probabilistic terms with returns expected in the future. For this reason, problems of capacity expansion require a careful evaluation of the risk implications of a particular expansion.

Approaches to capacity expansion have ranged from non-linear programming to heuristic methods. Foremost among these studies is the dynamic programming approach (Manne [7], Manne and Veinott [9], Erlenkotter [2, 3, 4], Sengupta and Fox [11], Tapiero [13]) which provides an optimum sequence of capacity expansion when a future demand is known. These approaches, however, assume a demand growth and non-depreciating facilities. Over time, facilities have a deteriorating productive capacity while demand

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forecasts can at best be characterized by probability distributions. Manne [8], for example, suggested a Brownian motion model for describing a random evolution of demand for a product and obtained an optimum capacity expansion for a facility of the non-depreciating type.

The purpose of this paper is to consider a problem of capacity expansion of deteriorating facilities under uncertainty. The following assumptions are made:

- (1) the deterioration rate is described by a stochastic process;
- (2) a demand forecast is given by an evolution of probability distributions. Specifically, we assume as given the mean and variance evolutions of the forecasts;
- (3) only one plant is built at the initial planning time;
- (4) the riskless discount rate  $r$  is given.

An optimum capacity expansion is obtained when the planning time  $T$  (used in computing the costs and benefits of the expansion) is given. The cost criterion is assumed to consist of an expansion cost, an excess capacity and a capacity shortage cost. When a facility is to be evaluated over its lifetime, we compute the probability of facility's extinction and use it as a planning time. For simplicity, we have assumed that under such circumstances, the cost criterion consists of the expansion cost and a net return proportional to a facility's productive capacity at time  $t$ . If the facility capacity expansion is  $K$  and it has a deterioration rate  $\delta$ , we find that its expected life is approximately  $(\ln K)/\delta^2$ . Using this expected life as a planning time, an optimum capacity expansion is determined.

For practical purposes, when more realistic and complex assumptions lead to analytically intractable solutions, simulation can be used to compare the economic and risk implications to a firm of two or more expansions. In this case, the problem is not to solve for an optimum capacity expansion but to compare the economic and risk properties of two (or more) competing capacity expansion alternatives. For demonstration purposes, a simulation comparing two such expansions is included. Such a simulation approach can be used only after the theoretical probability structures of capacity deterioration processes has been found. In other words, simulation becomes an essential tool in comparing capacity expansions when the underlying probability processes of capacity and demand are established. The first part of the paper shows how such probability processes can be found and how they can be manipulated to yield an optimum capacity expansion when the cost function is a simple one.

## 2. THE CAPACITY OF A DETERIORATING FACILITY

Assume that at time  $t=0$  the present time, a capacity expansion  $K$  is implemented whose cost is  $G(K)$ . Let  $x$  be the capacity at a future time  $t$  and

assume that the facility deteriorates randomly where  $\delta$  is the deterioration rate. If  $P(x, t)$  is the probability of a capacity  $x$  at time  $t$ , and if the deterioration process is a stochastic process described by a random walk where  $\Delta t \delta x =$  probability of losing to deterioration one unit of capacity in a small time interval  $\Delta t$ , then, a probabilistic evolutions for the capacity  $x$  at time  $t$  is given by (3):

$$\left. \begin{aligned} \frac{dP(x, t)}{dt} &= P(x+1, t) \delta(x+1) - P(x, t) \delta(x), \\ P(x=K, t_0) &= 1. \end{aligned} \right\} \quad (2.1)$$

When  $\delta$  is a constant, (2.1) describes a simple death process whose solution is a binomial probability distribution

$$\left. \begin{aligned} P(x, t) &= \binom{K}{x} p(\delta, t)^x [1 - p(\delta, t)]^{K-x}, \\ p(\delta, t) &= e^{-\delta t}. \end{aligned} \right\} \quad (2.2)$$

The probability generating function of (2.1) is well known and given by (4):

$$Q(z, t) = \{1 + (z-1)e^{-\delta t}\}^K \quad (2.3)$$

with

$$E(x(t)) = K e^{-\delta t}, \quad \text{var}(x(t)) = K e^{-\delta t} (1 - e^{-\delta t}).$$

Therefore, if we expand a productive capacity by  $K$  units at time  $t=0$ , and if the deterioration rate  $\delta$  is constant, the probability distribution of having a capacity of  $x$  units at time  $t$  is given by the binomial distribution (2.2). When the deterioration rate  $\delta$  is probabilistic, it is necessary to randomize the random walk (2.1). However, since (2.1) has a solution given by (2.2), a randomized deteriorations process as in (2.1) has a probability given by the mixture distributions of (2.2).

(3) This is in fact the well known death-process in queueing theory, with initial condition given by the  $K$ -the expansion, and a death rate equalling the capacity deterioration rate.

(4) The probability generating function is defined by

$$Q(z, t) = \sum_{x=0}^K P(x, t) z^x$$

and for the binomial probability distribution (2.2) is well known.

Specifically, let  $f(e^{-\delta t})$  be the probability distribution of  $\exp(-\delta t)$ , then the probability distribution of the capacity  $x$  at time  $t$  is given by

$$P(x, t) = \binom{K}{x} \int_0^1 f(e^{-\delta t}) e^{-\delta t K} (1 - e^{-\delta t})^{Kx} d(e^{-\delta t}) \tag{2.4}$$

A general approximate solution for (2.4) is given by Hald [5]. When we assume that  $\exp(-\delta t)$  has a beta probability distribution, we can show, that the probability distribution of  $x$  at time  $t$  has a Polya–Eggenberger distribution [1, 10]. That is, if

$$f(e^{-\delta t}) = \left. \begin{aligned} & \frac{\Gamma(a(t) + b(t))}{\Gamma(a(t))\Gamma(b(t))} [e^{-\delta t}]^{a(t)-1} [1 - e^{-\delta t}]^{b(t)-1}, \\ & a(t) > 0, b(t) > 0. \end{aligned} \right\} \tag{2.5}$$

Then inserting (2.5) into (2.4) and solving for  $P(x, t)$  yields <sup>(6)</sup>:

$$P(x, t) = \frac{\Gamma(K + 1)\Gamma(x(t) + a(t))\Gamma(K - x(t) + b(t))}{\Gamma(x(t) + 1)\Gamma(K - x(t) + 1)\Gamma(K + a(t) + b(t))} \tag{2.6}$$

which has a mean given by:

$$E\{x(t)\} = K a(t) / [a(t) + b(t)], \tag{2.7}$$

and a variance given by:

$$\text{var}\{x(t)\} = \frac{K a(t) b(t)}{[a(t) + b(t)]^2} + \frac{K(K - 1) a(t) b(t)}{[a(t) + b(t)]^2 [a(t) + b(t) + 1]} \tag{2.8}$$

Thus, the capacity on hand at time  $t$  has a probability distribution given by (2.6) whose mean and variance is given by (2.7) and (2.8). When  $\delta$  is a constant, and when  $K$  is large ( $K > 30$ ), a normal approximation to (2.2) is accurate, and the probability distribution of a capacity  $x$  at time  $t$  is given by:

$$P(x, t) = \left. \begin{aligned} & \frac{1}{\sqrt{2\pi}\sigma(t)} \exp\left\{-\frac{1}{2} \frac{(x - \mu(t))^2}{\sigma^2(t)}\right\}, \\ & \mu(t) = K e^{-\delta t}, \\ & \sigma^2(t) = K e^{-\delta t} (1 - e^{-\delta t}). \end{aligned} \right\} \tag{2.9}$$

This approximation is useful in computing an optimum capacity expansion when the demand for products is also normally distributed.

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<sup>(5)</sup> That is, a probability distribution whose parameter is also given by a probability distribution.  
<sup>(6)</sup> Such computations are straightforward if we note that (2.5) and (2.4) yield a Beta integral. A solution to this problem is, however, pointed out in Johnson and Kotz [6].

When a facility's deterioration rate is also function of its age,  $\delta$  is a time variant and (2.1) is a non-homogeneous stochastic process, whose probability generating function can be shown to be

$$Q_x(z, t) = [1 + (z - 1) e^{\rho(t)}]^K, \tag{2.10}$$

where

$$\rho(t) = - \int_0^t \delta(\tau) d\tau,$$

which corresponds to a binomial probability distribution whose probability parameter is

$$p(t) = e^{\int_0^t \delta(\tau) d\tau} \tag{2.11}$$

TABLE I

Probability	P.G.F.	Mean $\mu(t)$	Variance $\sigma^2(t)$
Assumption about deterioration: $\delta = \text{Const.}$			
$\binom{K}{x} e^{-\delta t x} (1 - e^{-\delta t})^{K-x}$	$\{1 + (z - 1) e^{-\delta t}\}^K$	$K e^{-\delta t}$	$K e^{-\delta t} (1 - e^{-\delta t})$
Assumption about deterioration: $f(e^{-\delta t}) \sim \text{Beta}$			
$\frac{\left\{ \Gamma \cdot (K + 1) \Gamma(x + a(t)) \right\} \times \Gamma(K - x + b(t))}{\left\{ \Gamma(x + 1) \Gamma(K - x + 1) \right\} \times \Gamma(K + a + b)}$	-	$\frac{K a(t)}{a(t) + b(t)}$	$\frac{K a(t) b(t)}{[a(t) + b(t)]^2} + \frac{K(K - 1) a(t) b(t)}{[a(t) + b(t)]^2 (a(t) + b(t) + 1)}$
Assumption about deterioration: normal approximation (large K)			
$\frac{1}{\sqrt{2\pi} \sigma(t)} \times e^{-(1/2)(x - \mu(t)/\sigma(t))^2}$	-	$K e^{-\delta t}$	$K e^{-\delta t} (1 - e^{-\delta t})$
Assumption about deterioration: $\delta(\tau) = \text{time variant}$			
$\binom{K}{x} e^{-\int_0^t \delta(\tau) d\tau x} \times \left(1 - e^{-\int_0^t \delta(\tau) d\tau}\right)^{K-x}$	$[1 + (z - 1) e^{-\int_0^t \delta(\tau) d\tau}]^K$	$K e^{-\int_0^t \delta(\tau) d\tau}$	$K e^{-\int_0^t \delta(\tau) d\tau} \left(1 - e^{-\int_0^t \delta(\tau) d\tau}\right)$

when  $\delta(t)$  is time invariant, then (2.11) reduces to  $\exp(-\delta t)$  which is the case of (2.2). Equations (2.2), (2.5), (2.9) and (2.11) provide four stochastic models for capacity deterioration which may be used to compute the probability distributions of a facility's productive life. These distributions will be used to establish upper bounds on the economic planning time of a facility's expansion. Furthermore, when the future demand for a product is probabilistically given, probability distribution of excess and capacity shortage can be computed. To simplify our presentations the probability results are summarized in table 1.

**3. OPTIMUM CAPACITY EXPANSION OF A DETERIORATING FACILITY**

We shall first assume that the planning time used to compute the economic value of a capacity expansion is given by  $T$  (where  $T$  is smaller than the facility's life). The capacity expansion size is chosen to minimize the expected cost of

$$G(K) + \left\{ \begin{array}{ll} \sum_{t=0}^T c_1 (\tilde{x}(t) - \tilde{D}(t))(1+r)^{-t} & \text{if } \tilde{x}(t) - \tilde{D}(t) \geq 0, \\ \sum_{t=0}^T c_2 (\tilde{D}(t) - \tilde{x}(t))(1+r)^{-t} & \text{if } \tilde{x}(t) - \tilde{D}(t) < 0. \end{array} \right\} \quad (3.1)$$

where:  $G(K)$ , the capacity expansion cost;  $\tilde{x}(t)$ , the capacity at time  $t$  – a random variable;  $\tilde{D}(t)$ , the demand at time  $t$  – a random variable;  $r$ , the riskless discount rate;  $T$ , the planning time;  $c_1$ , cost per unit excess capacity;  $c_2$ , cost per unit capacity shortage.

This criterion will be simplified further when we consider a probabilistic planning time. The expected value of (3.1) is given by:

$$G(K) + \sum_{t=0}^T c_1 (1+r)^{-t} \int_0^\infty \tilde{z}(t) g(\tilde{z}(t)) d\tilde{z}(t) + \sum_{t=0}^T c_2 (1+r)^{-t} \int_{-\infty}^0 \tilde{z}(t) g(\tilde{z}(t)) d\tilde{z}(t),$$

where  $\tilde{z}(t) = \tilde{x}(t) - \tilde{D}(t)$  and  $g(\tilde{z}(t))$  is the probability distribution of  $\tilde{z}(t)$ . If we assume  $\delta$  constant and use the normal approximation to (2.2) and if the demand is normally distributed with mean  $\mu(t)$  and variance  $\sigma^2(t)$ , then  $\tilde{z}(t)$  has also a normal probability distribution. Specifically, the time variant mean and variance of this distribution is:

$$\left. \begin{array}{l} \mu_z(t) = E \{ \tilde{z}(t) \} = K e^{-\delta t} - \mu(t), \\ \sigma_z^2(t) = \text{var} \{ \tilde{z}(t) \} = K e^{-\delta t} (1 - e^{-\delta t}) + \sigma^2(t). \end{array} \right\} \quad (3.3)$$

Also, using the normal distribution equality:

$$\int_0^\infty \tilde{z}(t) g(\tilde{z}(t)) d\tilde{z}(t) = \mu_z(t) [1 - F(-\mu_z(t)/\sigma_z(t))] + \sigma_z(t) f(-\mu_z(t)/\sigma_z(t)), \quad (3.4)$$

where  $f$  and  $F$  are the standard and cumulative standard normal distribution. Since

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-((t-m)^2/2\sigma^2)} dt = \frac{1}{2} \left( 1 + \operatorname{erf} \left( \frac{\mu - m}{\sigma\sqrt{2}} \right) \right),$$

where

$$\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt.$$

But since  $\operatorname{erf}(-u) = -\operatorname{erf}(u)$ , we obtain, in equation (3.4):

$$\int_0^\infty \tilde{z}(t) \tilde{g}(z(t)) d\tilde{z}(t) = \mu_z \left[ \frac{1}{2} + \frac{1}{2} \operatorname{erf}(\mu_z(t)/\sqrt{2}\sigma_z(t)) \right] + \frac{\sigma_z(t)}{\sqrt{2\pi}} \exp \left\{ -\mu_z^2(t)/2\sigma_z^2(t) \right\}. \quad (3.5)$$

Equation (3.2) together with (3.5) can be written dropping the time subscript, as:

$$G(K) + \sum_{t=0}^T (c_1 + c_2)(1+r)^{-t} \left\{ \mu_z \left[ \frac{1}{2} + \frac{1}{2} \operatorname{erf}(\mu_z/\sqrt{2}\sigma_z) \right] - \frac{\sigma_z}{\sqrt{2\pi}} \exp(-\mu_z^2/2\sigma_z^2) \right\} + \sum_{t=0}^T c_2(1+r)^{-t} \mu_z. \quad (3.6)$$

A derivative of (3.6) with respect to  $K$  yields one equation in one unknown (3.7) which may be solved numerically for  $K$ , or by trial and error.

$$\begin{aligned} \frac{\partial G}{\partial K} + \sum_{t=0}^T (c_1 + c_2)(1+r)^{-t} \left\{ e^{-\delta t} \left[ \frac{1}{2} + \frac{1}{2} \operatorname{erf} \left( \frac{\mu_z}{\sqrt{2}\sigma_z} \right) \right] \right. \\ \left. + \frac{e^{-\delta t} e^{-\mu_z^2/2\sigma_z^2}}{\sigma_z\sqrt{2\pi}} \left[ K\sqrt{2} e^{-\delta t} \left( 1 - \frac{\mu_z(1-e^{-\delta t})}{\sigma_z^{1/2}} \right) \right. \right. \\ \left. \left. + \frac{(\mu_z^2 + 1)(1-e^{-\delta t}) - 2\mu_z\sigma_z^2}{2} \right] \right\} - \sum_{t=0}^T c_2(1+r)^{-t} e^{-\delta t} = 0. \quad (3.7) \end{aligned}$$

For example, assume a normally distributed demand with time variant mean and variance given by  $\mu(t) = 100 + 10t + 5 \sin(\pi t/3)$  and  $\sigma^2(t) = 10t$ , also  $G(K) = 10 K^{0.80}$ ,  $c_1 = 1.0$  and  $c_2 = 3.0$ ,  $r = 0.06$  and  $T = 30$ . An optimum capacity expansion is then given by  $K^*$  ( $\delta = 0.10$ ):

$$K^* = 300 \text{ units}$$

From further numerical computations (see table II) we note that large deterioration rates do not imply smaller expansions. The size of the expansion is as indicated in (3.6) a function of knowledge of the demand forecast [i. e. the magnitude of  $\sigma^2(t)$ ]. When the forecasted demand is not known,  $\sigma^2(t)$  is very large and a Taylor's series approximation to the exponentials in (3.6) is acceptable and yields:

TABLE II  
Optimum Capacity Expansions

$r$ $\delta$	0.00	0.02	0.04	0.06	0.08	0.10	0.12
0.01	370	340	300	270	230	210	200
0.02	430	370	330	280	250	220	210
0.03	480	410	350	290	260	230	210
0.04	520	440	360	300	260	240	210
0.05	590	460	360	310	270	240	210
0.06	—	460	370	310	270	240	210
0.07	—	450	370	320	270	230	210
0.08	590	450	370	310	270	230	200
0.09	560	450	370	310	260	230	200
0.10	540	440	370	300	250	220	200
0.11	530	430	360	290	250	220	200
0.12	520	420	350	280	240	220	200
0.13	500	400	340	270	240	220	200
0.14	480	380	330	260	240	210	200
0.15	460	370	320	260	230	210	200

$$\exp\left\{-\mu_z^2(t)/2\sigma_z^2(t)\right\} = 1 - \frac{\mu_z^2(t)}{2\sigma_z^2(t)}.$$

Also  $\text{erf}(\varepsilon) \approx 0$  for  $\varepsilon$  very small. Thus (3.6) can be written as:

$$G(K) + \sum_{t=0}^T (c_1 - c_2)(1+r)^{-t} \left\{ \frac{\mu_z}{2} \right\} + \sum_{t=0}^T (c_1 + c_2)(1+r)^{-t} \frac{\sigma_z}{\sqrt{2\pi}}. \quad (3.8)$$

Subtracting (3.8) from (3.6) we obtain:

$$\sum_{t=0}^T \left( \frac{c_1 + c_2}{2} \right) (1+r)^{-t} \left\{ \mu_z \text{erf}[\mu_z/\sqrt{2}\sigma_z] + \frac{\sigma_z}{\sqrt{2\pi}} [\exp(-\mu_z^2/\sigma_z^2) - 1] \right\}, \quad (3.9)$$

which is the value (in an expected cost sense and given the deterioration rate) of a demand's forecast. Since  $\exp(-\mu_z^2/\sigma_z^2) < 1$ , the larger the variance the larger the value of the demand forecast.

When the planning time  $T$  is not specifically given, we can compute the capacity's expansion cost over the facilities' life time. That is, the planning time  $T$  is a random variable expressing the probability that  $x$ , the capacity, equals zero.

For the four cases considered in this paper [equations (2.2), (2.6) and (2.9)] the probability that the capacity is equal to zero at time  $T$  is given below:

TABLE III  
Probabilities of capacity extinction

	Probabilities
$\delta = \text{Const.} \dots\dots\dots$	$\{1 - e^{-\delta T}\}^K$
$f(e^{-\delta T}) \sim \text{Beta} \dots\dots\dots$	$\frac{\Gamma(a(T))\Gamma(K+b(T))}{\Gamma(K+a(T)+b(T))} = B(a(T), K+b(T))$
Normal approximation . . . . .	$\frac{1}{\sqrt{2\pi}} (K e^{-\delta T} (1 - e^{-\delta T}))^{1/2} \exp\{- (1/2) K^2 e^{-2\delta T/K} e^{-\delta T} (1 - e^{-\delta T})\}$
$\delta(\tau)$ -time variant . . . . .	$\left\{ 1 - e^{-\int_0^T \delta(\tau) d\tau} \right\}^K$

Consider for simplicity the case of a constant deterioration rate and compute the Laplace transform of the probability of extinction. Then,

$$\left. \begin{aligned}
 G(s) &= \int_0^\infty e^{-sT} (1 - e^{-\delta T})^K dT \\
 G(s) &= (1/\delta) B(s/\delta, K + 1),
 \end{aligned} \right\} \tag{3.10}$$

where  $B(\alpha, \beta)$  is the Beta function

$$B(s/\delta, K + 1) = \frac{\Gamma(s/\delta)\Gamma(K + 1)}{\Gamma(s/\delta + K + 1)}. \tag{3.11}$$

The mean extinction time is clearly given by:

$$-\frac{\Gamma(K+1)}{\delta^2} \frac{d}{d(s/\delta)} \left[ \frac{\Gamma(s/\delta)}{\Gamma(s/\delta + K + 1)} \right] \text{ at } s=0. \quad (3.12)$$

The mean is therefore

$$\frac{1}{\delta^2} \left\{ \sum_{i=1}^K (1/i) \right\}. \quad (3.13)$$

For  $K$  large, an approximation to (3.13) is:

$$\frac{1}{\delta^2} \sum_{i=1}^K (1/i) \approx \int_1^K (1/\delta^2) j dj = \frac{1}{\delta^2} \ln K. \quad (3.14)$$

Using the mean extinction time as the facility's planning time, we consider a cost criterion given by

$$G(K) - \sum_{t=0}^{\tilde{T}} p(1+r)^{-t} \tilde{x}(t), \quad (3.15)$$

where  $p$  is a net return per unit capacity  $\tilde{x}(t)$  at time  $t$ . If we further approximate the problem by considering the continuous form of (3.15) we note that the expected value of the facility is given by:

$$G(K) - \int_0^\infty \int_0^\infty \int_0^{\tilde{T}} p x(t) e^{-rt} dt f(\tilde{x}(t)) f(\tilde{T}) d\tilde{x}(t) d\tilde{T}. \quad (3.16)$$

Replacing the expected value of capacity,  $Ke^{-\delta t}$ , and integrating yields:

$$G(K) + \int_0^\infty pK \frac{[e^{-(r+\delta)T} - 1]}{(r+\delta)} f(T) dT.$$

Since  $E\{e^{-(r+\delta)T}\}$  is the Laplace transform of  $T$ , we obtain an expected value given by:

$$G(K) - \frac{pK}{(r+\delta)} \left[ 1 - \frac{B(r+\delta/\delta, K+1)}{\delta} \right]. \quad (3.17)$$

The optimum capacity expansion is thus found by solving for  $K$  in

$$\frac{\partial G}{\partial K} - \frac{p}{r+\delta} \left[ 1 - \frac{B(r+\delta/\delta, K+1)}{\delta} \right] + \frac{pK}{r+\delta} \frac{\partial}{\partial K} \frac{B(r+\delta/\delta, K+1)}{\delta} = 0. \quad (3.18)$$

An analytical solution to this equation is very difficult, numerical results are however, fairly easy to obtain. Simplification can also be used in obtaining less

precise but analytically tractable results. Namely, using the mean extinction time as the planning horizon, we obtain an optimum capacity expansion given by a solution of:

$$\frac{\partial G}{\partial K} \left( \frac{1}{\delta^2} - \frac{1}{r+\delta} \right) p K^{-(\delta+r)/\delta^2} = \frac{p}{r+\delta}. \quad (3.19)$$

If the deterioration rate is small, (3.19) reduces to

$$\frac{\partial G(K)}{\partial K} = p/r. \quad (3.20)$$

The analytical problems encountered in solving the optimum capacity expansion problem are, as we note, difficult ones. For this reason, simplifying assumptions were made yielding an approximate capacity expansion. For practical application, however, the capacity probability distributions defined earlier allow the simulation of complex capacity expansion problems. Instead of solving a numerical problem to determine the optimum size of an expansion, we may instead be able to compare the returns and risks of two capacity expansion plans.

## 5. CONCLUSION

This paper has provided an approach to determining an optimum capacity expansion under uncertainty. We have considered, as in Smith [12], a single plant expansion but with a deteriorating capacity and a probabilistic future demand. Thus, the paper can be viewed as an extension of past studies on capacity expansion. Capacity expansions require a large commitment of resources by a firm in the expectation of future returns. The costs of such an expansion, the risks of expansion (or of not expanding) are very important in determining the size of capacity expansion. To reach a judicious decision concerning capacity expansion, it is necessary to quantify this risk and include it as a part of managerial decision criteria.

This paper has reduced the mathematical problem of determining an optimum capacity expansion to an analytically tractable form. When analytical solutions cannot be obtained, the capacity probability distributions can be used to compare through simulation alternative expansion plans.

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