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Dror Zuckerman

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OPTIMAL REPLACEMENT RULE-DISCOUNTED COST CRITERION (*)

by Dror ZUCKERMAN [1]

Abstract. — *A replacement model under additive damage is considered. When the system fails it must be replaced and a failure cost is incurred. If the system is replaced before failure a smaller cost is incurred. We consider the problem of specifying a replacement rule which minimizes the expected total discounted cost.*

1. INTRODUCTION

We examine a production system which operates for random time until it fails. The system is subject to a sequence of random shocks. Each shock causes a random amount of damage and these damages accumulate additively. The successive shock magnitudes $Y_1, Y_2, \ldots$ are positive, independent and identically distributed random variables, having a known distribution function $B(y)$. If the accumulated damage in the system is x, then the elapsed time until the next shock is a random variable characterized by the exponential density function $f_x(t) = \lambda(x)\, e^{-\lambda(x)t}$. A failure can occur only at the occurrence of a shock and the probability of such a failure is a function of the accumulated damage in the system. More explicity, if at time t the cumulative damage is x and a shock of magnitude y occurs, then the system fails with known probability $1 - r(x+y)$. We refer to $r(.)$ as the survival function.

Upon failure the system is replaced by a new one having the same properties and the replacement cycles are repeated indefinitely. Each replacement costs C dollars and each failure adds a cost of K dollars, thus providing an incentive for a controller to attempt to replace before failure occurs. We allow a controller to replace the system at any stopping time T before failure time.

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[1] The Hebrew University of Jerusalem.

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Throughout, we assume that the following two conditions are satisfied:

(a) the survival function $r(z)$ is monotonically nonincreasing function of z;

(b) the shock rate, $\lambda(x)$, is monotonically nondecreasing over the state space of the damage process.

The problem is to find an optimal replacement rule that balances the cost of replacement with the cost of failure and results in a minimum expected total discounted cost.

Assuming that the cumulative damage is observable by the controller, and that his decisions may be based on its current value, we show that an optimal replacement policy is a control limit policy. The term "control limit policy" refers to a policy in which we replace the system either upon failure or when the accumulated damage first exceeds a fixed critical value ξ^*, whichever occurs first.

Barlow and Proschan [1] assumed that the age of the system in current service was maintained as a control variable, but no further information on the state of the production system was available. Esary Marshall and Proschan [4] investigated the property of a breakdown model for which the instants at which damage to the system occurs are Poisson distributed over time and the magnitude of damage caused by each shock (disturbance) equals one.

Taylor [5] derives an optimal replacement rule which minimizes the total long run average cost for the breakdown model in which shocks occur to the system at a constant rate.

The following will be standard notation used throughout the paper:

$$E_x[\,.\,]=E[\,.\,|\,X(0)=x], \qquad P_x(\,.\,)=P(\,.\,|\,X(0)=x),$$

and reserve $E(P)$ without affixes for expectation (probability) conditional on $X(0)=0$. The notation $E[Y;\,A]$, where Y is a random variable and A is an event, refers to the expectation $E[I_A Y]=E[Y\,|\,I_A=1]\,P(A)$, where I_A is the set characteristic function of A.

2. PRELIMINARIES

Let $\{\hat{X}(t);\ 0\leqq t<\delta\}$ be the stochastic process representing the cumulative damage process up to the failure time, δ, of the system.

Let Δ be a distinct point not in R_+ and define

$$X(t)=\begin{cases}\hat{X}(t) & \text{if} \quad t<\delta,\\ \Delta & \text{if} \quad t\geqq\delta.\end{cases} \tag{1}$$

A key tool for us is the following formula:

$$E_x[e^{-\alpha T} f(X(T)); A] - f(x) = E_x\left[\int_0^T e^{-\alpha s}\{A_f(X(s)) - \alpha f(X(s))\}\, ds\right], \quad (2)$$

where

$$A_f(x) = \lim_{t \downarrow 0} t^{-1}\{E_x[f(X(t)); A] - f(x)\}. \quad (3)$$

Formula (2) is valid for any function f such that $f(x)$ and $A_f(x)$ are bounded and continuous (*see* [2], p. 376).

Let

$$w(x) = E_x[\delta].$$

We proceed with the following proposition.

PROPOSITION 1: *$w(x)$ is a bounded function of x.*

Proof: Let B be such that $r(B) = \alpha < 1$. Let N be the number of shocks until failure.

Define

$$K_n(x) = P_x\{N > n\}.$$

Recalling that $r(.)$ is nonincreasing we obtain

$$K_1(x) = \int r(x+y)\, dF(y) \leqq r(B) = \alpha \qquad \text{for} \quad x \geqq B,$$

and by induction it follows that

$$K_n(x) \leqq [P_x\{N > 1\}]^n \leqq \alpha^n \qquad \text{for} \quad x \geqq B.$$

Hence

$$E_x[N] = \sum_{n \geqq 0} K_n(x) \leqq \frac{1}{1-\alpha} \qquad \text{for} \quad x \geqq B.$$

Since the expected time between two successive shocks is bounded from above by $\lambda(0)$, we have

$$w(x) \leqq \frac{1}{\lambda(0)} E_x[N] \leqq \frac{1}{\lambda(0)(1-\alpha)} \qquad \text{for} \quad x \geqq B. \quad (4)$$

On the other hand, for $X<B$:

$$w(x)-w(B)$$

$$\leqq \frac{1}{\lambda(0)} \text{ [mean number of } Y_K\text{'s needed to achieve } Y_1+Y_2+\ldots+Y_n \geqq B]$$

$$\leqq \frac{1}{\lambda(0)} M(B), \quad (5)$$

where $M(z)$ is the renewal function associated with the distribution function B. i. e.,

$$M(z)=\sum_{n \geqq 0} P\{Y_1+Y_2+\ldots+Y_n \leqq z\}.$$

From inequalities (4) and (5) it follows that

$$w(x) \leqq \frac{1}{\lambda(0)}\left[\frac{1}{1-\alpha}+M(B)\right] \quad \text{for all} \quad x, \qquad (6)$$

as required. □ Using proposition 1 we obtain that for any permissible stopping time T, $E[T]$ is finite.

3. OPTIMAL PLANNED REPLACEMENT

We allow a controller to institute a *planned replacement* at any stopping time $T<\delta$. Upon failure the system must be replaced by a new identical one and the replacement cycles are repeated indefinitely.

Every replacement costs C dollars, and a *failure replacement*, the event $\{T=\delta\}$ invokes an additional cost of K dollars. We now attempt to minimize the expected total discounted cost. For a given stopping time T, the expected discounted cost from the first replacement cycle is

$$U_{T,\alpha}(1)=CE[e^{-\alpha T};\ T<\delta]+(C+K)E[e^{-\alpha T};\ T=\delta],$$

where α is the discount factor.

Generally, the expected discounted cost from the n-th replacement cycle is

$$U_{T,\alpha}(n)=U_{T,\alpha}(1)\{E[e^{-\alpha T}]\}^{n-1}.$$

Clearly we can restrict our attention to the following set of stopping times:

$$S=\{T;\ X(T)\neq 0\}.$$

For each stopping time $T \in S$, $E[T] \geqq 1/\lambda(0) > 0$. On the other hand, for any permissible stopping T, $E[T] < \infty$, therefore, the expected total discounted cost associated with a stopping time T, will be

$$U_{T,\alpha} = E\left[\lim_{n \to \infty} \sum_{i=1}^{n} \left\{\begin{array}{l}\text{discounted cost associated with the } i\text{-th replacement}\\ \text{cycle when a stopping time } T \text{ is employed}\end{array}\right\}\right].$$

By applying the dominated convergence theorem it follows

$$U_{T,\alpha} = \lim_{n \to \infty} \sum_{i=1}^{n} U_{T,\alpha}(i) = \frac{CE[e^{-\alpha T};\ T<\delta] + (C+K)E[e^{-\alpha T};\ T=\delta]}{1-E[e^{-\alpha T}]}. \tag{7}$$

Let $U_\alpha^* = \inf_T U_{T,\alpha}$ be the optimal discounted cost. Also let

$$R(x) = \int r(x+y)\,dB(y).$$

The optimal policy will be determined with the aid of the following theorem.

THEOREM 1: *A stopping time T^* is optimal if and only if it maximizes*

$$E\left[\int_0^T e^{-\alpha s} J(X(s))\,ds\right], \tag{8}$$

where

$$J(x) = \alpha(U_\alpha^* + C) - \lambda(x) K[1-R(x)]. \tag{9}$$

Proof: To begin the derivation, note that for every stopping time T:

$$U_\alpha^* \leqq \frac{CE[e^{-\alpha T};\ T<\delta] + (C+K)E[e^{-\alpha T};\ T=\delta]}{1-E[e^{-\alpha T}]}. \tag{10}$$

A stopping time T minimizes the total discounted cost if and only if T maximizes

$$\theta_T \equiv U_\alpha^* \{1-E[e^{-\alpha T}]\} - CE[e^{-\alpha T};\ T<\delta] - (C+K)E[e^{-\alpha T};\ T=\delta], \tag{11}$$

and the maximum value of θ_T is zero. θ_T can be rearranged to give

$$\theta_T = -C - \{E[(U_\alpha^* + C+K)e^{-\alpha T}] - (U_\alpha^* + C+K)\} + \{E[K e^{-\alpha T};\ T<\delta] - K\}. \tag{12}$$

Now, we may use formula (2) to express

$$\theta_T = -C - E\left[\int_0^T e^{-\alpha s}(g(X(s)) - \alpha(U_\alpha^* + C+K))\,ds\right] + E\left[\int_0 e^{-\alpha s}(h(X(s)) - \alpha K)\,ds\right], \tag{13}$$

where

$$g(x)=\lim_{t\downarrow 0} t^{-1}\{E_x[U_\alpha^* + C + K] - (U_\alpha^* + C + K)\} = 0 \tag{14}$$

and

$$\begin{aligned} h(x) &= \lim_{t\downarrow 0} t^{-1}\{E_x[K;\ t<\delta] - K\} \\ &= \lim_{t\downarrow 0} t^{-1}\left\{(1-\lambda(x)\,t)\,K + \lambda(x)\,t\,K\int r(x+y)\,dB(y) + o(t) - K\right\} \\ &= -\lambda(x)\,K\,[1-R(x)]. \end{aligned} \tag{15}$$

Using (14) and (15), (13) can be rearranged to give

$$\theta_T = -C + \left[\int_0^T e^{-\alpha a}\{\alpha(U_\alpha^* + C) - \lambda(X(s))\,K\,[1-R(X(s))]\}\,ds\right]. \tag{16}$$

This should make it clear that T^* is an optimal stopping time if and only if it maximizes (8), as required. □

The following concludes the proof of optimality.

THEOREM 2: *An optimal replacement policy T^* is a control limit policy. Furthermore, the optimal critical value is given by*

$$\xi_\alpha^* = \inf\{x \geqq 0;\ J(x) \leqq 0\}.$$

Proof: Let us consider the following stopping time

$$T^* = \min\{\inf\{t \geqq 0;\ X(t) \geqq \xi_\alpha^*\},\ \delta\}.$$

It can easily be seen that $J(x)$ is nonincreasing in x. Thus by definition of T^* we obtain for all $t<\delta$:

$$J(X(t)) > 0 \qquad \text{if and only if} \quad t < T^*.$$

For every stopping time T:

$$\begin{aligned} \theta_{T^*} - \theta_T &= E\left[\int_0^{T^*} J(X(s))\,ds\right] - E\left[\int_0^T J(X(s))\,ds\right] \\ &= E\left[\int_T^{T^*} J(X(s))\,ds;\ T<T^*\right] - E\left[\int_{T^*}^T J(X(s))\,ds;\ T \geqq T^*\right] \geqq 0 \end{aligned} \tag{17}$$

Thus, T^* maximizes θ_T, and this completes the proof of the optimality of T^*. □

Example: In order to illustrate computational procedures, let us consider the following model:

(i) the survival function is given by

$$r(x)=\begin{cases} 1 & \text{if} \quad 0 \leqq x < L, \\ 0 & \text{if} \quad x \geqq L. \end{cases}$$

In words, system failure occurs when the cumulative damage first exceeds a fixed threshold L;

(ii) the magnitude of damage associated with each shock equals one;

(iii) the shock rate as a function of the cumulative damage in the system is given by

$$\lambda(x)=a+x \qquad (a>0).$$

Without loss of generality we may assume that the threshold L is an integer. Furthermore, it suffices to consider stopping times of the form

$$T_\xi=\min\{\inf\{t \geqq 0;\ X(t) \geqq \xi\},\ \delta\},$$

for $\xi=1, 2, \ldots, L$.

Let t_i ($i=0, 1, \ldots, L-1$) be the time interval between the i-th shock and the $i+1$ shock.

$\{t_i\}_{i=0, 1, 2, \ldots, L-1}$ is a sequence of independent random variables exponentially distributed with parameters $\lambda(i)=a+i$, respectively. Since

$$\{T_\xi<\delta\}=0 \qquad \text{if} \quad \xi=L,$$

and

$$\{T_\xi=\delta\}=0 \qquad \text{if} \quad \xi=1, 2, \ldots, L-1,$$

the total discounted cost associated with a stopping time T_ξ is given by

$$U_{T_\xi, \alpha}=\begin{cases} \dfrac{(C+K)\prod_{i=0}^{L-1}((a+i)/((a+i)+\alpha))}{1-\prod_{i=0}^{L-1}((a+i)/((a+i)+\alpha))} & \text{if} \quad L=\xi, \\[2ex] \dfrac{C\prod_{i=0}^{\xi-1}((a+i)/((a+i)+\alpha))}{1-\prod_{i=0}^{\xi-1}((a+i)/((a+i)+\alpha))} & \text{if} \quad \xi=1, 2, \ldots, L-1. \end{cases}$$

In order to find the optimal policy, we have simply to minimize $U_{T_\xi, \alpha}$ for $\xi = 1, 2, \ldots, L$.

REMARK: The level of difficulty in expressing $U_{T_\xi, \alpha}$ explicitly, depends heavily on the structure of the survival function $r(.)$ and the distribution function B. In some cases simulation methods are needed in order to determine the optimal policy.

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