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A DISCRETE TIME QUEUEING SYSTEM WITH DEPARTURES HAVING RANDOM MEMORY (*)

by U. K. Gupta (1)

Abstract. — *This paper deals with a discrete time, first-come-first-served, single channel queueing system in which arrivals have zero-step memory while departures have sometimes zero-step memory and sometimes one-step memory. Generating functions for the steady state queue length probabilities have been obtained explicitly for two models. Finally, some particular cases have been discussed.*

Résumé. — *Cet article traite d'un système de file d'attente à temps discret, à canal unique, où le premier arrivé est le premier servi. Les arrivées sont statistiquement indépendantes des arrivées et départs précédents, tandis que les départs sont correlés. On obtient explicitement les fonctions génératrices des probabilités de la longueur de la file d'attente dans le cas stationnaire.*
Quelques cas particuliers sont examinés pour terminer.

INTRODUCTION

Cox and Miller [1] consider a discrete time, limited space, single channel, first-come-first-served queueing system in which arrivals and departures are statistically independent of the previous arrivals and departures (i. e. arrivals and departures with zero-step memory — to be defined later) and obtain equilibrium probability distribution of the queue size. Chaudhry [2] considers the steady state behaviour of a discrete time, single channel, first-come-first-served queueing problem in which it is assumed that the arrivals at two consecutive time marks are statistically independent, whereas the departures are correlated (i. e. departures with one-step memory — to be defined later). However, it is not uncommon in practice to find situations where departures have sometimes one-step memory and sometimes zero-step memory. For example, let there be two clerks X and Y in an office. There arrive certain confidential files sealed and written on the cover "A" or "B" and certain general files. File "A" is to be kept by X and file "B" is to be handed over to Y by X for necessary action. Out of the general files, some are kept by X himself and the remaining are allocated to Y by X for disposal. If we now identify respectively the departure and no departure

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at a service channel with the keeping of a file by X to be disposed of by himself and allocating a file to Y for disposal, then:

(1) if a file coming to X is found confidential:

(*a*) he has to keep it with himself irrespective of the fact that he had/ had not kept a file with himself at the previous time mark if "A" is written on the cover of the file,

(*b*) he has to allocate it to Y irrespective of the fact that he had/had not kept a file with himself at the previous time mark if "B" is written on the cover of the file;

(2) if a file coming to X is a general one, X keeps it with himself or allocates it to Y depends on whether he (X) had/had not kept a file with himself at the previous time mark.

Case (1) corresponds to departures with zero-step memory while case (2) corresponds to departures with one-step memory.

In this paper we shall study the steady state behaviour of a discrete time, single channel, first-come-first-served queueing problem in which arrivals have zero-step memory but departures have sometimes zero-step memory and sometimes one-step memory. Two models (A) and (B) are considered. If just before a transition mark t_r the system is empty and an arrival takes place at t_r, then: (i) in model (A), it can not leave the system at the same time mark i. e. the probability of its departure and no departure at the same time mark t_r is zero and one respectively; (ii) in model (B), it has got equal chances of departing and not departing at the same time mark t_r i. e. the probability in each case is 1/2. In each model, probability generating function of the queue length is determined. Also queue length probabilities are determined explicitly. Mean queue length, variance etc. have been found out in these models. Some particular cases have also been added.

In a problem like this the whole time axis is divided into a number of intervals separated by transition marks $t_0, t_1, t_2 \ldots$ The arrivals/ departures can occur only at these transition time marks. If arrivals(departures) at a transition mark are independent of whether there were arrivals(departures) or not at the previous transition marks, they are called arrivals(departures) with zero-step memory or to have zero-step memory. If arrivals(departures) at a transition mark t_{r+1} depend on whether there were arrivals(departures) or not at the previous transition mark t_r only, they are called arrivals(departures) with one-step memory or to have one-step memory.

In this paper, arrivals have zero-step memory while departures have sometimes zero-step memory and sometimes one-step memory i. e. departures are associated with a random variable X, call it memory random variable, such that it assumes values 0 or 1. The probability that X takes the value 1 is p and that it takes the value 0 is q such that $p+q=1$. Within one transition duration X can have only one value either 0 or 1. The value of X from 0 to 1 or from 1 to 0 can change only just after a transition time mark and it will remain the same upto the next transition time mark. When X assumes the value 0, it means departures have zero-step memory and when X assumes the value 1, it means that departures have one-step memory.

MODEL *A*

The queueing model investigated in this paper involves the following assumptions:

(1) The probability of an arrival and no arrival at a transition mark is λ_1 and λ_0 respectively such that $\lambda_0+\lambda_1=1$.

(2) The probability of more than one arrival or more than one departure at a transition mark is assumed to be zero.

(3) When the system is empty just before a transition time mark t_r and a unit arrives at t_r, the probability of its departure at t_r is zero and that of no departure is one.

(4) When just before a transition mark t_{r+1} the queue length is $n>0$ and $X=1$, then:

(i) if there is a departure at t_r:

$$\text{Prob. (departure at } t_{r+1})=a_{11},$$

$$\text{Prob. (no departure at } t_{r+1})=a_{10};$$

(ii) if there is no departure at t_r:

$$\text{Prob. (departure at } t_{r+1})=a_{01},$$

$$\text{Prob. (no departure at } t_{r+1})=a_{00}.$$

Thus the Transition Probability Matrix (T.P.M.) for departures and no departures at two consecutive time marks t_r and t_{r+1} is given by:

$$\begin{array}{ll} & \overbrace{\quad\text{no departure}\qquad\text{departure}\quad}^{\text{To } t_{r+1}} \\ \text{From } t_r \begin{cases}\text{no departure at } t_r \\ \text{departure at } t_r\end{cases} & \begin{bmatrix} a_{00} & \qquad\quad a_{01} \\ a_{10} & \qquad\quad a_{11}\end{bmatrix}\end{array}$$

where $a_{00}+a_{01}=a_{10}+a_{11}=1$.

(5) When just before a transition mark t_{r+1} the queue length is $n>0$ and $X=0$, then:

$$\text{Prob. (departure at } t_{r+1}) = b_1,$$

$$\text{Prob. (no departure at } t_{r+1}) = b_0,$$

such that $b_1 + b_0 = 1$.

(6) The system follows the "First-Come-First-Served" queue discipline. Define:

$P_{n,0}^{(1)}(t_r+0) \equiv$ Probability that just after the transition mark t_r the queue length (including the one being served) is n, $X=1$ and there was no departure at the transition mark t_r.

$P_{n,1}^{(1)}(t_r+0) \equiv$ Probability that just after the transition mark t_r the queue length is n, $X=1$ and there was a departure at the transition mark t_r.

$P_n^{(0)}(t_r+0) \equiv$ Probability that just after the transition mark t_r the queue length is n and $X=0$.

$P_n(t_r+0) \equiv$ Probability that just after the transition mark t_r the queue length is n.

The queueing system described above is governed by the following equations:

$$P_{n,0}^{(1)}(t_r+0) = p\,[\lambda_0 a_{00} P_{n,0}^{(1)}(t_r-0) + \lambda_0 a_{10} P_{n,1}^{(1)}(t_r-0) + \lambda_0 b_0 P_n^{(0)}(t_r-0) + \lambda_1 a_{00} P^{(1)}{}_{n-1,0}(t_r-0) + \lambda_1 a_{10} P_{n-1,1}^{(1)}(t_r-0) + \lambda_1 b_0 P_{n-1}^{(0)}(t_r-0)] \quad \text{for} \quad n \geqq 2, \tag{1}$$

$$P_{1,0}^{(1)}(t_r+0) = p\,[\lambda_0 a_{00} P_{1,0}^{(1)}(t_r-0) + \lambda_0 a_{10} P_{1,1}^{(1)}(t_r-0) + \lambda_0 b_0 P_1^{(0)}(t_r-0) + \lambda_1 P_{0,0}^{(1)}(t_r-0) + \lambda_1 P_{0,1}^{(1)}(t_r-0) + \lambda_1 P_0^{(0)}(t_r-0)], \tag{2}$$

$$P_{0,0}^{(1)}(t_r+0) = p\,[\lambda_0 P_{0,0}^{(1)}(t_r-0) + \lambda_0 P_{0,1}^{(1)}(t_r-0) + \lambda_0 P_0^{(0)}(t_r-0)], \tag{3}$$

$$P_{n,1}^{(1)}(t_r+0) = p\,[\lambda_0 a_{01} P_{n+1,0}^{(1)}(t_r-0) + \lambda_0 a_{11} P_{n+1,1}^{(1)}(t_r-0) + \lambda_0 b_1 P_{n+1}^{(0)}(t_r-0) + \lambda_1 a_{01} P_{n,0}^{(1)}(t_r-0) + \lambda_1 a_{11} P_{n,1}^{(1)}(t_r-0) + \lambda_1 b_1 P_n^{(0)}(t_r-0)] \quad \text{for} \quad n \geqq 1, \tag{4}$$

$$P_{0,1}^{(1)}(t_r+0) = p\,[\lambda_0 a_{01} P_{1,0}^{(1)}(t_r-0) + \lambda_0 a_{11} P_{1,1}^{(1)}(t_r-0) + \lambda_0 b_1 P_1^{(0)}(t_r-0)], \tag{5}$$

$$P_n^{(0)}(t_r+0) = q\,[\lambda_0 a_{00} P_{n,0}^{(1)}(t_r-0) + \lambda_0 a_{10} P_{n,1}^{(1)}(t_r-0) + \lambda_0 b_0 P_n^{(0)}(t_r-0) + \lambda_0 a_{01} P_{n+1,0}^{(1)}(t_r-0) + \lambda_0 a_{11} P_{n+1,1}^{(1)}(t_r-0) + \lambda_0 b_1 P_{n+1}^{(0)}(t_r-0) + \lambda_1 a_{00} P_{n-1,0}^{(1)}(t_r-0) + \lambda_1 a_{10} P_{n-1,1}^{(1)}(t_r-0) + \lambda_1 b_0 P_{n-1}^{(0)}(t_r-0) + \lambda_1 a_{01} P_{n,0}^{(1)}(t_r-0) + \lambda_1 a_{11} P_{n,1}^{(1)}(t_r-0) + \lambda_1 b_1 P_n^{(0)}(t_r-0)] \quad \text{for} \quad n \geqq 2, \tag{6}$$

$$P_1^{(0)}(t_r+0)=q\,[\lambda_0\,a_{00}\,P_{1,0}^{(1)}(t_r-0)+\lambda_0\,a_{10}\,P_{1,1}^{(1)}(t_r-0)+\lambda_0\,b_0\,P_1^{(0)}(t_r-0)$$
$$+\lambda_0\,a_{01}\,P_{2,0}^{(1)}(t_r-0)+\lambda_0\,a_{11}\,P_{2,1}^{(1)}(t_r-0)+\lambda_0\,b_1\,P_2^{(0)}(t_r-0)$$
$$+\lambda_1\,P_{0,0}^{(1)}(t_r-0)+\lambda_1\,P_{0,1}^{(1)}(t_r-0)+\lambda_1\,P_0^{(0)}(t_r-0)$$
$$+\lambda_1\,a_{01}\,P_{1,0}^{(1)}(t_r-0)+\lambda_1\,a_{11}\,P_{1,1}^{(1)}(t_r-0)+\lambda_1\,b_1\,P_1^{(0)}(t_r-0)], \quad (7)$$

$$P_0^{(0)}(t_r+0)=q\,[\lambda_0\,P_{0,0}^{(1)}(t_r-0)+\lambda_0\,P_{0,1}^{(1)}(t_r-0)+\lambda_0\,P_0^{(0)}(t_r-0)$$
$$+\lambda_0\,a_{01}\,P_{1,0}^{(1)}(t_r-0)+\lambda_0\,a_{11}\,P_{1,1}^{(1)}(t_r-0)+\lambda_0\,b_1\,P_1^{(0)}(t_r-0)], \quad (8)$$

$$P_n(t_r+0)=P_{n,0}^{(1)}(t_r+0)+P_{n,1}^{(1)}(t_r+0)+P_n^{(0)}(t_r+0) \qquad \text{for} \quad n\geqq 0. \quad (9)$$

For steady state, as $r \to \infty$:

$$P_{n,i}^{(1)}(t_r\pm 0) \to P_{n,i}^{(1)} \qquad (i=0,\,1),$$
$$P_n^{(0)}(t_r\pm 0) \to P_n^{(0)},$$

and

$$P_n(t_r\pm 0) \to P_n.$$

Define the generating functions for the steady state probabilities:

$$R(\alpha,\,1,\,i)=\sum_{n=0}^{\infty}\alpha^n\,P_{n,i}^{(1)} \qquad (i=0,\,1),$$
$$R(\alpha,\,0)=\sum_{n=0}^{\infty}\alpha^n\,P_n^{(0)},$$
$$R(\alpha)=\sum_{n=0}^{\infty}\alpha^n\,P_n.$$

Letting $r \to \infty$ in equations (1)-(3), multiplying by appropriate powers of α, summing over n from 0 to ∞ and making use of the generating functions, we get:

$$[pa_{00}-h(\alpha)]\,R(\alpha,\,1,\,0)+pa_{10}\,R(\alpha,\,1,\,1)+pb_0\,R(\alpha,\,0)+p\,K=0. \quad (10)$$

Similarly dealing with equations (4)-(5) and (6)-(8):

$$pa_{01}\,R(\alpha,\,1,\,0)+[pa_{11}-\alpha\,h(\alpha)]\,R(\alpha,\,1,\,1)+pb_1\,R(\alpha,\,0)-p\,K=0, \quad (11)$$

$$q\,(a_{01}+\alpha\,a_{00})\,R(\alpha,\,1,\,0)+q\,(a_{11}+\alpha\,a_{10})\,R(\alpha,\,1,\,1)$$
$$+[q\,(b_1+\alpha\,b_0)-\alpha\,h(\alpha)]\,R(\alpha,\,0)-q\,(1-\alpha)\,K=0, \quad (12)$$

where

$$h(\alpha)=\frac{1}{\lambda_0+\lambda_1\,\alpha}$$

and

$$K=a_{01}P^{(1)}_{0,0}+a_{11}P^{(1)}_{0,1}+b_1P^{(0)}_0.$$

Writing (10), (11) and (12) in the Matrix form and making some elementary row transformations, we have:

$$\begin{bmatrix} p-h(\alpha) & p-\alpha h(\alpha) & p \\ pa_{01} & pa_{11}-\alpha h(\alpha) & pb_1 \\ q(a_{01}+\alpha a_{00}) & q(a_{11}+\alpha a_{10}) & q(b_1+\alpha b_0)-\alpha h(\alpha) \end{bmatrix} \times \begin{bmatrix} R(\alpha,1,0) \\ R(\alpha,1,1) \\ R(\alpha,0) \end{bmatrix} = \begin{bmatrix} 0 \\ pK \\ q(1-\alpha)K \end{bmatrix} \tag{13}$$

Solving (13):

$$R(\alpha,1,0)=\frac{Kp\alpha h(\alpha)[1-\alpha h(\alpha)]}{A(\alpha)},$$

$$R(\alpha,1,1)=\frac{Kp\alpha h(\alpha)[h(\alpha)-1]}{A(\alpha)},$$

$$R(\alpha,0)=\frac{Kq\alpha h^2(\alpha)[1-\alpha]}{A(\alpha)},$$

where

$$A(\alpha)=\alpha h(\alpha)[p(a_{01}-a_{11})+\{pa_{11}+qb_1+\alpha(pa_{00}+qb_0)\}h(\alpha)-\alpha h^2(\alpha)],$$

so that

$$R(\alpha)=R(\alpha,1,0)+R(\alpha,1,1)+R(\alpha,0)$$

$$=\frac{K(1-\alpha)g(\alpha)}{p(a_{01}-a_{11})g^2(\alpha)+[pa_{11}+qb_1+\alpha(pa_{00}+qb_0)]g(\alpha)-\alpha}$$

[where $g(\alpha)=1/h(\alpha)=\lambda_0+\lambda_1\alpha$]

$$=\frac{K(\lambda_0+\lambda_1\alpha)}{(\lambda_0+\lambda_1\alpha)(p\lambda_0a_{01}+p\lambda_1a_{11}+qb_1)-\lambda_1\alpha}. \tag{14}$$

Using the normalising condition $\underset{\alpha\to 1-0}{\mathrm{Lt}} R(\alpha)=1$, we get:

$$K=p\lambda_0a_{01}+p\lambda_1a_{11}+qb_1-\lambda_1. \tag{15}$$

Therefore:

$$R(\alpha)=\frac{(\lambda_0+\lambda_1\alpha)(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1-\lambda_1)}{(\lambda_0+\lambda_1\alpha)(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)-\lambda_1\alpha}. \tag{16}$$

Applying Leibnitz differentiation formula to (16), we obtain:

$$P_n=\frac{1}{n!}\left[\frac{d^n}{d\alpha^n}R(\alpha)\right]_{\alpha=0}$$

$$=\left\{\begin{array}{ll}\dfrac{\lambda_0\lambda_1(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1-\lambda_1)}{XY}\left(\dfrac{Y}{X}\right)^n & \text{for } n\geqq 1\\[2ex] \dfrac{\lambda_0(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1-\lambda_1)}{X} & \text{for } n=0\end{array}\right\} \tag{17}$$

where

and

$$\left.\begin{array}{l}X=\lambda_0(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)\\[1ex] Y=\lambda_1[1-(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)]\end{array}\right\} \tag{18}$$

It is obvious that the steady state queue size distribution exists if

$$Y/X<1 \quad \text{i. e. if } \lambda_1<\frac{pa_{01}+qb_1}{1+p(a_{01}-a_{11})}.$$

The left side of this inequality is obviously the mean number of arrivals at a time mark and we prove below that the right side is the mean number of departures at a time mark.

Just after a time mark t_r, the queueing system can be only in three states E_0, E_1 and E_2 — states E_0, E_1, E_2 respectively mean the system has zero memory, the system has memory one and there was no departure at t_r, the system has memory one and there was a departure at t_r. Now the stochastic matrix P can be written as

$$\begin{array}{c} \\ E_0\\ E_1\\ E_2\end{array}\begin{array}{c}\begin{array}{ccc}E_0 & E_1 & E_2\end{array}\\ \begin{bmatrix} q & pb_0 & pb_1\\ q & pa_{00} & pa_{01}\\ q & pa_{10} & pa_{11}\end{bmatrix}\end{array}$$

Let z_1, z_2, z_3 stand for the limiting probabilities of the states E_0, E_1, E_2 respectively. Then the vector $Z=(z_1, z_2, z_3)$ is given by:

$$ZP=Z. \tag{19}$$

Solving (19):

$$z_1 = q,$$

$$z_2 = \frac{p(qb_0 + pa_{10})}{1 + p(a_{01} - a_{11})},$$

$$z_3 = \frac{p(qb_1 + pa_{01})}{1 + p(a_{01} - a_{11})}.$$

Thus

mean number of departures at a time mark

$$= z_1 b_1 + z_2 a_{01} + z_3 a_{11}$$

$$= \frac{pa_{01} + qb_1}{1 + p(a_{01} - a_{11})}.$$

Mean Queue Length

Differentiating (16) w. r. t. α at $\alpha = 1$, the mean queue length (i. e. the expected number in the system), L, is given by:

$$L = \frac{\lambda_0 \lambda_1}{p \lambda_0 a_{01} + p \lambda_1 a_{11} + qb_1 - \lambda_1}. \tag{20}$$

Variance

L, being an expected value, fluctuations in the number waiting can occur, which can be best seen by calculating the variance, V:

$$V = R''(1) + R'(1) - [R'(1)]^2$$

$$= \frac{\lambda_0 \lambda_1 [(\lambda_0 - \lambda_1)(p \lambda_0 a_{01} + p \lambda_1 a_{11} + qb_1) + \lambda_1^2]}{(p \lambda_0 a_{01} + p \lambda_1 a_{11} + qb_1 - \lambda_1)^2}. \tag{21}$$

Expected Number in the waiting line

L_q, the expected number in the waiting line (excluding the one being served) can be determined as:

$$L_q = \sum_{n=1}^{\infty} (n-1) P_n$$

$$= \frac{\lambda_1^2 [1 - (p \lambda_0 a_{01} + p \lambda_1 a_{11} + qb_1)]}{(p \lambda_0 a_{01} + p \lambda_1 a_{11} + qb_1)(p \lambda_0 a_{01} + p \lambda_1 a_{11} + qb_1 - \lambda_1)}. \tag{22}$$

The probability that not less than a given number is in the system

The probability that the number in the system is greater than or equal to $j\,(\geqq 1)$ is given by:

$$\sum_{n=j}^{\infty} P_n = \frac{\lambda_0 \lambda_1}{Y} \left(\frac{Y}{X}\right)^j \tag{23}$$

[X and Y being given by (18)].

If we put $j=1$, we get $\lambda_1/(p\lambda_0 a_{01} + p\lambda_1 a_{11} + qb_1)$, the probability that an arrival must wait.

PARTICULAR CASES

(1) If we put $p=0$ and $q=1$ in (16), we obtain the generating function for the queue length for the queueing system in which departures have simply zero-step memory:

$$R(\alpha) = \frac{(\lambda_0 + \lambda_1 \alpha)(b_1 - \lambda_1)}{b_1 \lambda_0 - b_0 \lambda_1 \alpha}, \tag{24}$$

which is the same as (21) of Chaudhry, except for notations.

(2) If we put $p=1$ and $q=0$ in (16), we obtain the generating function for the queue length for the queueing system in which departures have only one-step memory:

$$R(\alpha) = \frac{(\lambda_0 + \lambda_1 \alpha)(\lambda_0 a_{01} - \lambda_1 a_{10})}{(\lambda_0 + \lambda_1 \alpha)(\lambda_0 a_{01} + \lambda_1 a_{11}) - \lambda_1 \alpha}. \tag{25}$$

For $a_{00} = a_{11}$ and $a_{01} = a_{10}$ we have the result (12) of Chaudhry, except for notations.

(3) If we put $p=q=1/2$ in (16), we obtain the generating function for the queue length for the queueing system in which departures have equal chances for having one-step memory and zero-step memory:

$$R(\alpha) = \frac{(\lambda_0 + \lambda_1 \alpha)(\lambda_0 a_{01} + \lambda_1 a_{11} + b_1 - 2\lambda_1)}{(\lambda_0 + \lambda_1 \alpha)(\lambda_0 a_{01} + \lambda_1 a_{11} + b_1) - 2\lambda_1 \alpha}. \tag{26}$$

MODEL *B*

In this model the assumption (3) of model (A) is modified as follows:

When the system is empty just before a transition mark t_r and a unit arrives at t_r, then its departure and no departure have equal chances to happen at t_r.

That is, if just before a transition mark t_r, the system is empty and an arrival takes place at t_r, then

$$\text{Prob.(departure at } t_r) = \frac{1}{2},$$

$$\text{Prob.(no departure at } t_r) = \frac{1}{2}.$$

Proceeding as in model (A), we see that the system is governed by the following steady state equations:

$$P_{n,0}^{(1)} = p\,[\lambda_0 a_{00} P_{n,0}^{(1)} + \lambda_0 a_{10} P_{n,1}^{(1)} + \lambda_0 b_0 P_n^{(0)} + \lambda_1 a_{00} P_{n-1,0}^{(1)} + \lambda_1 a_{10} P_{n-1,1}^{(1)} + \lambda_1 b_0 P_{n-1}^{(0)}] \quad \text{for} \quad n \geqq 2, \tag{27}$$

$$P_{1,0}^{(1)} = p\left[\lambda_0 a_{00} P_{1,0}^{(1)} + \lambda_0 a_{10} P_{1,1}^{(1)} + \lambda_0 b_0 P_1^{(0)} + \lambda_1 \frac{1}{2} P_{0,0}^{(1)} + \lambda_1 \frac{1}{2} P_{0,1}^{(1)} + \lambda_1 \frac{1}{2} P_0^{(0)}\right], \tag{28}$$

$$P_{0,0}^{(1)} = p\,[\lambda_0 P_{0,0}^{(1)} + \lambda_0 P_{0,1}^{(1)} + \lambda_0 P_0^{(0)}], \tag{29}$$

$$P_{n,1}^{(1)} = p\,[\lambda_0 a_{01} P_{n+1,0}^{(1)} + \lambda_0 a_{11} P_{n+1,1}^{(1)} + \lambda_0 b_1 P_{n+1}^{(0)} + \lambda_1 a_{01} P_{n,0}^{(1)} + \lambda_1 a_{11} P_{n,1}^{(1)} + \lambda_1 b_1 P_n^{(0)}] \quad \text{for} \quad n \geqq 1, \tag{30}$$

$$P_{0,1}^{(1)} = p\left[\lambda_0 a_{01} P_{1,0}^{(1)} + \lambda_0 a_{11} P_{1,1}^{(1)} + \lambda_0 b_1 P_1^{(0)} + \lambda_1 \frac{1}{2} P_{0,0}^{(1)} + \lambda_1 \frac{1}{2} P_{0,1}^{(1)} + \lambda_1 \frac{1}{2} P_0^{(0)}\right], \tag{31}$$

$$P_n^{(0)} = q\,[\lambda_0 a_{00} P_{n,0}^{(1)} + \lambda_0 a_{10} P_{n,1}^{(1)} + \lambda_0 b_0 P_n^{(0)} + \lambda_0 a_{01} P_{n+1,0}^{(1)} + \lambda_0 a_{11} P_{n+1,1}^{(1)} + \lambda_0 b_1 P_{n+1}^{(0)} + \lambda_1 a_{00} P_{n-1,0}^{(1)} + \lambda_1 a_{10} P_{n-1,1}^{(1)} + \lambda_1 b_0 P_{n-1}^{(0)} + \lambda_1 a_{01} P_{n,0}^{(1)} + \lambda_1 a_{11} P_{n,1}^{(1)} + \lambda_1 b_1 P_n^{(0)}] \quad \text{for} \quad n \geqq 2, \tag{32}$$

$$P_1^{(0)} = q\left[\lambda_0 a_{00} P_{1,0}^{(1)} + \lambda_0 a_{10} P_{1,1}^{(1)} + \lambda_0 b_0 P_1^{(0)} + \lambda_0 a_{01} P_{2,0}^{(1)} + \lambda_0 a_{11} P_{2,1}^{(1)} + \lambda_0 b_1 P_2^{(0)} + \lambda_1 \frac{1}{2} P_{0,0}^{(1)} + \lambda_1 \frac{1}{2} P_{0,1}^{(1)} + \lambda_1 \frac{1}{2} P_0^{(0)} + \lambda_1 a_{01} P_{1,0}^{(1)} + \lambda_1 a_{11} P_{1,1}^{(1)} + \lambda_1 b_1 P_1^{(0)}\right], \tag{33}$$

$$P_0^{(0)}=q\Big[\lambda_0 P_{0,0}^{(1)}+\lambda_0 P_{0,1}^{(1)}+\lambda_0 P_0^{(0)}+\lambda_0 a_{01} P_{1,0}^{(1)}+\lambda_0 a_{11} P_{1,1}^{(1)}$$

$$+\lambda_0 b_1 P_1^{(0)}+\lambda_1\frac{1}{2} P_{0,0}^{(1)}+\lambda_1\frac{1}{2} P_{0,1}^{(1)}+\lambda_1\frac{1}{2} P_0^{(0)}\Big], \quad (34)$$

$$P_n=P_{n,0}^{(1)}+P_{n,1}^{(1)}+P_n^{(0)} \qquad \text{for} \quad n\geqq 0. \quad (35)$$

Proceeding as in model (A), we get:

$$\begin{bmatrix} p-h(\alpha) & p-\alpha h(\alpha) & p \\ pa_{01} & pa_{11}-\alpha h(\alpha) & pb_1 \\ q(a_{01}+\alpha a_{00}) & q(a_{11}+\alpha a_{10}) & q(b_1+\alpha b_0)-\alpha h(\alpha) \end{bmatrix}$$

$$\times\begin{bmatrix} R(\alpha, 1, 0) \\ R(\alpha, 1, 1) \\ R(\alpha, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ pK(\alpha) \\ q(1-\alpha)K(\alpha) \end{bmatrix}, \quad (36)$$

where

$$h(\alpha)=\frac{1}{\lambda_0+\lambda_1\alpha}$$

and

$$K(\alpha)=\Big[(a_{01} P_{0,0}^{(1)}+a_{11} P_{0,1}^{(1)}+b_1 P_0^{(0)}) - \frac{\alpha\lambda_1}{2(\lambda_0+\lambda_1\alpha)}(P_{0,0}^{(1)}+P_{0,1}^{(1)}+P_0^{(0)})\Big].$$

Solving (36):

$$R(\alpha)=\frac{K(\alpha)(\lambda_0+\lambda_1\alpha)}{(\lambda_0+\lambda_1\alpha)(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)-\lambda_1\alpha}. \quad (37)$$

The normalising condition $\operatorname{Lt}_{\alpha\to 1-0} R(\alpha)=1$ gives:

$$\left(a_{01}-\frac{1}{2}\lambda_1\right)P_{0,0}^{(1)}+\left(a_{11}-\frac{1}{2}\lambda_1\right)P_{0,1}^{(1)}+\left(b_1-\frac{1}{2}\lambda_1\right)P_0^{(0)}$$

$$=p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1-\lambda_1. \quad (38)$$

Equation (29) gives:

$$(p\lambda_0-1)P_{0,0}^{(1)}+p\lambda_0 P_{0,1}^{(1)}+p\lambda_0 P_0^{(0)}=0. \quad (39)$$

Solving (31) and (34):

$$pq\lambda_0 P_{0,0}^{(1)} + (pq\lambda_0 + q) P_{0,1}^{(1)} + (pq\lambda_0 - p) P_0^{(0)} = 0. \tag{40}$$

Writing (38), (39) and (40) in the Matrix form and making some elementary row transformations, we obtain:

$$\begin{bmatrix} a_{01} - \frac{1}{2}\lambda_1 & a_{11} - \frac{1}{2}\lambda_1 & b_1 - \frac{1}{2}\lambda_1 \\ p\lambda_0 - 1 & p\lambda_0 & p\lambda_0 \\ q & q & -p \end{bmatrix} \times \begin{bmatrix} P_{0,0}^{(1)} \\ P_{0,1}^{(1)} \\ P_0^{(0)} \end{bmatrix} = \begin{bmatrix} p\lambda_0 a_{01} + p\lambda_1 a_{11} + qb_1 - \lambda_1 \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

Solving (41):

$$P_{0,0}^{(1)} = \frac{p\lambda_0 A}{D},$$

$$P_{0,1}^{(1)} = \frac{p\lambda_1 A}{D},$$

$$P_0^{(0)} = \frac{qA}{D},$$

where

$$A = p\lambda_0 a_{01} + p\lambda_1 a_{11} + qb_1 - \lambda_1$$

and

$$D = p\lambda_0 a_{01} + p\lambda_1 a_{11} + qb_1 - \frac{1}{2}\lambda_1$$

so that

$$K(\alpha) = \frac{\left\{ \begin{array}{l} [2(\lambda_0 + \lambda_1 \alpha)(p\lambda_0 a_{01} + p\lambda_1 a_{11} + qb_1) \\ \quad - \lambda_1 \alpha] [p\lambda_0 a_{01} + p\lambda_1 a_{11} + qb_1 - \lambda_1] \end{array} \right\}}{(\lambda_0 + \lambda_1 \alpha)[2(p\lambda_0 a_{01} + p\lambda_1 a_{11} + qb_1) - \lambda_1]}. \tag{42}$$

Therefore:

$$R(\alpha)=\frac{\left\{\begin{array}{l}[2(\lambda_0+\lambda_1\alpha)(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)\\ \quad -\lambda_1\alpha][p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1-\lambda_1]\end{array}\right\}}{\left\{\begin{array}{l}[(\lambda_0+\lambda_1\alpha)(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)\\ \quad -\lambda_1\alpha][2(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)-\lambda_1]\end{array}\right\}}. \tag{43}$$

Proceeding as in model (A) we can find out the queue length probabilities explicitly which are given by:

$$P_n=\left\{\begin{array}{ll}\dfrac{\lambda_1 K}{(2K+\lambda_1)Y}\left(\dfrac{Y}{X}\right)^n & \text{for} \quad n\geqq 1\\ \dfrac{2K}{2K+\lambda_1} & \text{for} \quad n=0\end{array}\right\}, \tag{44}$$

where K, X, Y are given by (15) and (18).

Again, we see that steady state exists if

$$\lambda_1<\frac{pa_{01}+qb_1}{1+p(a_{01}-a_{11})}.$$

Mean queue length

The mean queue length, L, is given by:

$$L=\frac{\lambda_0\lambda_1(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)}{[2(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)-\lambda_1][p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1-\lambda_1]}. \tag{45}$$

Variance

The variance, V, is given by:

$$V=\frac{\lambda_0\lambda_1 B[2(\lambda_0-\lambda_1)B^2+3\lambda_1^2 B-\lambda_1^2]}{(2B-\lambda_1)^2(B-\lambda_1)^2}, \tag{46}$$

where

$$B=p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1.$$

Expected number in the waiting line

L_q, the expected number in the waiting line (excluding the one being served) is given by:

$$L_q=\frac{\lambda_1^2[1-(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)]}{(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1-\lambda_1)[2(p\lambda_0 a_{01}+p\lambda_1 a_{11}+qb_1)-\lambda_1]}. \tag{47}$$

The probability that not less than a given number is in the system

The probability that the number in the system is greater than or equal to $j\,(\geqq 1)$ is given by:

$$\sum_{n=j}^{\infty} P_n = \frac{\lambda_0 \lambda_1}{2X - \lambda_0 \lambda_1} \left(\frac{Y}{X}\right)^{j-1}. \tag{48}$$

The probability that an arrival must wait is given by (putting $j=1$):

$$\lambda_1/[2(p\lambda_0 a_{01} + p\lambda_1 a_{11} + qb_1) - \lambda_1]. \tag{49}$$

PARTICULAR CASE

If we put $p=1$ and $q=0$ in (43), we obtain the probability generating function for the queue length for the queueing model in which the departures have only one-step memory:

$$R(\alpha) = \frac{(\lambda_0 a_{01} - \lambda_1 a_{10})\,[2(\lambda_0 + \lambda_1 \alpha)(\lambda_0 a_{01} + \lambda_1 a_{11}) - \lambda_1 \alpha]}{[2(\lambda_0 a_{01} + \lambda_1 a_{11}) - \lambda_1]\,[(\lambda_0 + \lambda_1 \alpha)(\lambda_0 a_{01} + \lambda_1 a_{11}) - \lambda_1 \alpha]} \tag{50}$$

For $a_{00} = a_{11}$ and $a_{01} = a_{10}$ we have the result (33) of Chaudhry, except for notations.

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