REVUE FRANÇAISE D'AUTOMATIQUE, D'INFORMATIQUE ET DE RECHERCHE OPÉRATIONNELLE. RECHERCHE OPÉRATIONNELLE

NAOTO KAIO

SHUNJI OSAKI

Optimum age replacement policy with two failure modes

Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle, tome 13, nº 2 (1979), p. 205-208.

<http://www.numdam.org/item?id=RO_1979__13_2_205_0>

R.A.I.R.O. Recherche opérationnelle/Operations Research
(vol. 13, n° 2, mai 1979, p. 205 à 209)

OPTIMUM AGE REPLACEMENT POLICY WITH TWO FAILURE MODES (*)

by NAOTO KAIO (1) and SHUNJI OSAKI (1)

Abstract. – *In this paper, we treat the optimum age replacement model with discounting, in which two types of failure modes are considered. Introducing an exchange, a replacement, and a salvage costs, we obtain the optimum policy which minimizes the expected total discounted cost. It is shown that, under certain conditions, there exists a finite and unique optimum policy. Furthermore, the relations to the earlier contributions are shown.*

Résumé. – *Dans cet article, nous traitons du modèle d'âge optimal de remplacement avec amortissement, dans lequel deux types de modes de panne sont considérés. En introduisant des coûts d'échange, de remplacement et de réparation, nous déterminons la politique optimale qui minimise l'espérance du coût total avec amortissement. Il est montré que sous certaines conditions il existe une politique optimale finie et unique. En outre, on montre les relations avec des recherches antérieures.*

1. INTRODUCTION

Many contributions have made to age replacement policies (e. g., Barlow and Proschan [1, p. 85], Fox [2], and Osaki and Nakagawa [3]). An age replacement model is easy to be applied to real systems for its simplicity.

In this paper, we treat the age replacement policy, especially, the age replacement model with discounting in which two types of failure modes and a salvage and some other costs are considered. In real systems, it can be considered that two types of failures, i. e., a catastrophic one and a degradation one, occur independently.

In particular, we use the continuous discount rate, and obtain the optimum policy to minimize the expected total discounted cost. We show that there exists a finite and unique optimum policy under certain conditions. Finally, we also show the relations between the results of this paper and the results obtained in the earlier contributions.

(*) Reçu juin 1978.
(1) Department of Industrial Engineering, Hiroshima University, Hiroshima 730, Japan.

R.A.I.R.O. Recherche opérationnelle/Operations Research, 0399-0842/1979/205/$ 4.00

2. MODEL AND ASSUMPTIONS

Consider a one-unit system, where each failed unit is scrapped. The planning horizon is infinite and a unit starts operating at time 0. If the unit does not fail up to a prespecified time instant $t_0 \in [0, \infty)$, the unit is exchanged by a new identical unit at that time instant t_0. If the unit fails before that time instant t_0, the failed unit is replaced by the new one at the failure time instant. Exchange (replacement) is made instantaneously, and exchanged (replaced) new unit takes over its operation immediately. The similar cycle repeats itself again and again.

Assume that the lifetime for each unit obeys an arbitrary mixture distribution $a_1 F_1(t)+a_2 F_2(t)$, where $a_1 \geqq 0$, $a_2 \geqq 0$, and $a_1+a_2=1$. $F_i(t)$ $(i=1, 2)$ has a density $f_i(t)$, i. e. it may be considered as that the index $i=1$ implies a catastrophic failure and $i=2$ implies a degradation one. The costs considered here are the following; a constant cost c_0 is suffered for the exchange of a nonfailed unit at the prespecified time instant t_0, a cost c_i is suffered for the replacement of a failed unit by a failure mode i $(i=1, 2)$ before the time instant t_0, and a cost k per unit time is suffered for the residual lifetime of the exchanged unit which is still able to operate. Furthermore, we introduce an exponential type discount rate α (> 0), where a unit of cost is discounted $e^{-\alpha t}$ after a time interval t, and assume that $c_i > c_0$ $(i=1, 2)$, and $c_0+k/\alpha > 0$. These assumptions seem to be reasonable.

Under the above assumptions, we define an interval from the beginning of the unit to exchange (replacement) as one cycle, and analyze this model noting that each exchange (replacement) time instant is a regeneration point.

3. ANALYSIS AND THEOREM

The expected total cost per one cycle is

$$\begin{aligned}\varphi_\alpha(t_0)=c_0 e^{-\alpha t_0}&[a_1\bar{F}_1(t_0)+a_2\bar{F}_2(t_0)]\\&+a_1 c_1\int_0^{t_0} e^{-\alpha t}dF_1(t)+a_2 c_2\int_0^{t_0} e^{-\alpha t}dF_2(t)\\&+k\int_{t_0}^{\infty} e^{-\alpha t}[a_1\bar{F}_1(t)+a_2\bar{F}_2(t)]\,dt,\end{aligned} \tag{1}$$

where $\bar{\psi}(.)\equiv 1-\psi(.)$ in general. Just after one cycle, a unit of cost is discounted as follows;

$$\delta_\alpha(t_0)=1-\alpha\int_0^{t_0} e^{-\alpha t}[a_1\bar{F}_1(t)+a_2\bar{F}_2(t)]\,dt. \tag{2}$$

Thus, when a unit starts operating at time 0, the expected total discounted cost for an infinite time span is

$$C_\alpha(t_0)=\varphi_\alpha(t_0)/[1-\delta_\alpha(t_0)] = \Big[c_0 e^{-\alpha t_0}\{a_1\bar{F}_1(t_0)+a_2\bar{F}_2(t_0)\}$$

$$+a_1 c_1\int_0^{t_0} e^{-\alpha t}dF_1(t)+a_2 c_2\int_0^{t_0} e^{-\alpha t}dF_2(t)+k\int_{t_0}^{\infty} e^{-\alpha t}$$

$$\times\{a_1\bar{F}_1(t)+a_2\bar{F}_2(t)\}dt\Big]\Big/\Big[\alpha\int_0^{t_0} e^{-\alpha t}\{a_1\bar{F}_1(t)+a_2\bar{F}_2(t)\}dt\Big] \quad (3)$$

(*see* Fox [2]), and

$$C_\alpha(\infty)=[a_1 c_1 F_1^*(\alpha)+a_2 c_2 F_2^*(\alpha)]/[a_1\bar{F}_1^*(\alpha)+a_2\bar{F}_2^*(\alpha)], \quad (4)$$

where $F_i^*(\alpha)\equiv\int_0^{\infty} e^{-\alpha t}dF_i(t)$ $(i=1, 2)$.

Differentiating the expected total discounted cost $C_\alpha(t_0)$ with respect to t_0, setting it equal to zero and arranging it, we have $q_\alpha(t_0)=0$, where

$$q_\alpha(t_0)=[H(t_0)/\alpha-(c_0+k/\alpha)]\,[1-\delta_\alpha(t_0)]-\varphi_\alpha(t_0), \quad (5)$$

where

$$H(t_0)=[(c_1-c_0)a_1 f_1(t_0)+(c_2-c_0)a_2 f_2(t_0)]/[a_1\bar{F}_1(t_0)+a_2\bar{F}_2(t_0)]. \quad (6)$$

It is further noted that

$$q_\alpha(\infty)=[-\alpha(c_0+k/\alpha)+H(\infty)]\,[\{a_1\bar{F}_1^*(\alpha)+a_2\bar{F}_2^*(\alpha)\}/\alpha] - [a_1 c_1 F_1^*(\alpha)+a_2 c_2 F_2^*(\alpha)]. \quad (7)$$

Here, we have the following theorem for the optimum exchange time t_0^* minimizing the expected total discounted cost $C_\alpha(t_0)$. The proof is given in Appendix.

THEOREM 1 : (1) *If $H(t_0)$ is strictly increasing and $q_\alpha(\infty)>0$, then there exists a finite and unique optimum exchange time t_0^* $(0<t_0^*<\infty)$ satisfying $q_\alpha(t_0)=0$ and the corresponding expected total discounted cost is*

$$C_\alpha(t_0^*)=H(t_0^*)/\alpha-(c_0+k/\alpha). \quad (8)$$

(2) *If $H(t_0)$ is strictly increasing and $q_\alpha(\infty)\leqq 0$, or if $H(t_0)$ is decreasing, then the optimum exchange time is $t_0^*\to\infty$, i. e. a unit continues to operate until it fails. The corresponding expected total discounted cost is given by the formula* (4).

4. REMARKS

We have treated the age replacement model with two failure modes, and have obtained a theorem on the optimum exchange policy minimizing the expected total discounted cost $C_\alpha(t_0)$. It has been shown that there exists a finite and unique optimum exchange policy under certain conditions.

In particular, if $F_1(t) \equiv F_2(t) \equiv F(t)$, $c_1 \equiv c_2$, and $k \equiv 0$, then this model coincides with the model discussed by Fox [2], and the expected total discounted cost is

$$C_\alpha(t_0) = \left[c_0 e^{-\alpha t_0} \bar{F}(t_0) + c_1 \int_0^{t_0} e^{-\alpha t} dF(t) \right] \Big/ \left[\alpha \int_0^{t_0} e^{-\alpha t} \bar{F}(t) dt \right]. \quad (9)$$

Moreover, we have

$$\lim_{\alpha \to 0} \alpha C_\alpha(t_0) = [c_1 F(t_0) + c_0 \bar{F}(t_0)] \Big/ \left[\int_0^{t_0} \bar{F}(t) dt \right], \quad (10)$$

which is equal to the expected cost per unit time in the steady-state for the age replacement model with no discounting (*see* Barlow and Proschan [1, p. 87]).

APPENDIX

THE PROOF OF THEOREM 1

Differentiating $C_\alpha(t_0)$ with respect to t_0 and setting it equal to zero imply the equation $q_\alpha(t_0) = 0$. Further,

$$q'_\alpha(t_0) = H'(t_0)\,[1 - \delta_\alpha(t_0)]/\alpha. \quad \text{(A-1)}$$

First, we assume the case that $H(t_0)$ is strictly increasing. Thus, we have that $q'_\alpha(t_0) > 0$, i. e., $q_\alpha(t_0)$ is strictly increasing.

If $q_\alpha(\infty) > 0$, then there exists a finite and unique t_0^* $(0 < t_0^* < \infty)$ which minimizes the expected total discounted cost $C_\alpha(t_0)$ as a finite and unique solution to $q_\alpha(t_0) = 0$, since $q_\alpha(0) < 0$ and $q_\alpha(t_0)$ is strictly increasing and continuous. Substituting the relation of $q_\alpha(t_0^*) = 0$ into $C_\alpha(t_0^*)$ in the formula (3) yields the formula (8).

If $q_\alpha(\infty) \leqq 0$, then $C'_\alpha(t_0) \leqq 0$ for any non-negative t_0 and thus $C_\alpha(t_0)$ is a strictly decreasing function. Thus, the optimum exchange time is $t_0^* \to \infty$.

Secondly, we assume the case that $H(t_0)$ is decreasing. Thus, we have that $q'_\alpha(t_0) \leqq 0$, i. e., $q_\alpha(t_0)$ is decreasing. We have that $q_\alpha(\infty) < 0$ since $q_\alpha(0) < 0$, i. e., $C_\alpha(t_0)$ is a strictly decreasing function. Thus, the optimum exchange time is $t_0^* \to \infty$.

Q.E.D.

REFERENCES

1. R. E. Barlow and F. Proschan, *Mathematical Theory of Reliability*, Wiley, New York, 1965.
2. B. Fox, Age Replacement with Discounting, *Oper. Res.*, Vol. 14, 1966, p. 533-537.
3. S. Osaki and T. Nakagawa, A Note on Age Replacement, *I.E.E.E. Trans. Rel.*, Vol. R-24, 1975, p. 92-94.