Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle

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Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle, tome 14, nº 2 (1980), p. 109-113.

<http://www.numdam.org/item?id=RO_1980__14_2_109_0>

R.A.I.R.O. Recherche opérationnelle/Operations Research
(vol. 14, n° 2, mai 1980, p. 109 à 113)

QUEUES WITH INFINITELY MANY SERVERS

by Lajos TAKÁCS

Abstract. — *In the time interval* $(0, \infty)$ *calls arrive in a telephone exchange in accordance with a recurrent process. There are an unlimited number of available lines. The holding times are mutually independent and identically distributed positive random variables and are independent of the arrival times. In this paper we determine the limit distribution of the number of the busy lines at the arrival of the n-th call as* $n \to \infty$.

Résumé. — *Dans l'intervalle* $(0, \infty)$ *des appels téléphoniques sont arrivés à la centrale selon un processus récurrent. Il y a des guichets disponibles en nombre infini. Les durées de service sont des variables aléatoires indépendantes positives avec la même fonction de distribution et indépendantes aussi des instants d'arrivée. Dans cet article nous déterminons la distribution-limite du nombre des guichets occupés à l'arrivée du n-ième appel quand* $n \to \infty$.

1. INTRODUCTION

In a telephone exchange there are an unlimited number of available lines and a connection is made immediately after the arrival of each call. Denote by τ_n the arrival time and by χ_n the holding time of the n-th call arriving in the time interval $(0, \infty)$. We assume that $\tau_n - \tau_{n-1}$ $(n=1, 2, \ldots, \tau_0=0)$ and χ_n $(n=1, 2, \ldots)$ are independent sequences of mutually independent and identically distributed positive random variables.

Denote by ξ_n the number of the busy lines at the arrival of the n-th call. In this paper we shall determine the limit distribution of ξ_n as $n \to \infty$. The analogous problem of finding the limit distribution of $\xi(t)$, the number of the busy lines at time t, as $t \to \infty$ has already been solved by the author [1].

We shall use the notations

$$\mathbf{P}\{\tau_n - \tau_{n-1} \leqq x\} = F(x), \qquad \mathbf{P}\{\tau_n \leqq x\} = F_n(x)$$

and

$$\mathbf{P}\{\chi_n \leqq x\} = H(x) \qquad \text{for} \quad n=1, 2, \ldots .$$

(*) Received August 1979.
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Furthermore,

$$\varepsilon_j = \begin{cases} 1 & \text{if} \quad \tau_j \leqq \chi_j, \\ 0 & \text{if} \quad \tau_j > \chi_j, \end{cases} \tag{1}$$

for $j=1, 2, \ldots$

The expected number of calls arriving in the time interval $(0, x]$ is

$$m(x) = \sum_{n=1}^{\infty} F_n(x) \tag{2}$$

for $x \geqq 0$.

2. THE DISTRIBUTION OF ξ_n

We observe that if $n > 1$, then the random variable ξ_n has the same distribution as $\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{n-1}$. Obviously, ξ_n is equal to the number of subscripts $i=1, 2, \ldots, n-1$ for which $\tau_i + \chi_i \geqq \tau_n$. If we replace $\chi_1, \chi_2, \ldots, \chi_{n-1}$ by $\chi_{n-1}, \chi_{n-2}, \ldots, \chi_1$ respectively, and $\tau_1 - \tau_0, \tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}$ by $\tau_n - \tau_{n-1}, \tau_{n-1} - \tau_{n-2}, \ldots, \tau_1 - \tau_0$ respectively, then we can conclude that ξ_n has the same distribution as the number of subscripts $i=1, 2, \ldots, n-1$ for which $\chi_i \geqq \tau_i$. Accordingly,

$$\mathbf{P}\{\xi_n = k\} = \mathbf{P}\{\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{n-1} = k\} \tag{3}$$

for $k=0, 1, \ldots, n-1$.

The r-th binomial moment of ξ_n is

$$\mathbf{E}\left\{\binom{\xi_n}{r}\right\} = \sum_{1 \leqq i_1 < i_2 < \ldots < i_r \leqq n-1} \mathbf{E}\{\varepsilon_{i_1} \varepsilon_{i_2} \ldots \varepsilon_{i_r}\} \tag{4}$$

for $r=1, 2, \ldots, n-1$, and 1 for $r=0$. If $r=1, 2, \ldots, n-1$, then by (1) we obtain that

$$\mathbf{E}\{\varepsilon_{i_1} \varepsilon_{i_2} \ldots \varepsilon_{i_r}\} = \int_0^\infty \int_0^\infty \ldots \int_0^\infty [1-H(x_1)]\,[1-H(x_1+x_2)] \ldots \times [1-H(x_1+\ldots+x_r)]\, dF_{i_1}(x_1)\, dF_{i_2}(x_2) \ldots dF_{i_r}(x_r). \tag{5}$$

The distribution of ξ_n is uniquely determined by its binomial moments. See reference [2]. We have

$$\mathbf{P}\{\xi_n = k\} = \sum_{r=k}^{n-1} (-1)^{r-k} \binom{r}{k} \mathbf{E}\left\{\binom{\xi_n}{r}\right\} \tag{6}$$

for $k=0, 1, \ldots, n-1$.

3. THE LIMIT DISTRIBUTION OF ξ_n

Now we shall prove the following theorem.

THEOREM 1: *If*

$$a=\int_0^{\infty} x\,dH(x)<\infty, \tag{7}$$

then the limit distribution

$$\lim_{n\to\infty} \mathbf{P}\{\xi_n=k\}=P_k \tag{8}$$

exists, and

$$P_k=\sum_{r=k}^{\infty}(-1)^{r-k}\binom{r}{k}B_r \tag{9}$$

for $k=0, 1, 2, \ldots$, *where* $B_0=1$ *and*

$$B_r=\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}[1-H(x_1)]\,[1-H(x_1+x_2)]\ldots \times [1-H(x_1+\ldots+x_r)]\,dm(x_1)\,dm(x_2)\ldots dm(x_r) \tag{10}$$

for $r=1, 2, \ldots$

Proof: By (3) we have

$$\mathbf{P}\{\xi_n\leqq k\}=\mathbf{P}\{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_{n-1}\leqq k\} \tag{11}$$

for $k\geqq 0$ and $n\geqq 1$ where $\varepsilon_1+\ldots+\varepsilon_{n-1}$ is 0 for $n=1$. For $n=1, 2, \ldots$ the events $\{\varepsilon_1+\ldots+\varepsilon_{n-1}\leqq k\}$ form a decreasing sequence. Thus by the continuity theorem for probabilities we obtain that

$$\lim_{n\to\infty}\mathbf{P}\{\xi_n\leqq k\}=\mathbf{P}\left\{\sum_{j=1}^{\infty}\varepsilon_j\leqq k\right\} \tag{12}$$

for $k\geqq 0$. Hence the limit (8) exists and

$$P_k=\mathbf{P}\left\{\sum_{j=1}^{\infty}\varepsilon_j=k\right\} \tag{13}$$

for $k=0, 1, 2, \ldots$

The random variables $\varepsilon_1+\ldots+\varepsilon_{n-1}$ $(n=1, 2, \ldots)$ form a nondecreasing sequence and if $a<\infty$, then

$$\mathbf{P}\left\{\sum_{j=1}^{\infty}\varepsilon_j<\infty\right\}=1. \tag{14}$$

Consequently, $B_0=1$ and

$$B_r=\lim_{n\to\infty} \mathbf{E}\left\{\binom{\varepsilon_1+\ldots+\varepsilon_{n-1}}{r}\right\} \tag{15}$$

for $r=1, 2, \ldots$ By (4), (5) and (15) we get (10).

If $a<\infty$, then there exists a positive constant C such that

$$B_r \leqq C^r/r! \tag{16}$$

for $r \geqq 0$. If $r=0$, then (16) is trivial. Let $r \geqq 1$.

Since

$$m(t+h)-m(t) \leqq 1+m(h) \tag{17}$$

for $t \geqq 0$ and $h \geqq 0$, by (10) we get the inequality

$$B_r \leqq \left(\frac{1+m(h)}{h}\right)^r \int_0^\infty \int_0^\infty \ldots \int_0^\infty [1-H(x_1-h)] \times [1-H(x_1+x_2-h)] \ldots [1-H(x_1+\ldots+x_r-h)] \times dx_1\, dx_2 \ldots dx_r \leqq \left(\frac{1+m(h)}{h}\right)^r \frac{(a+h)^r}{r!} \tag{18}$$

for $r=1, 2, \ldots$ and any $h>0$. This proves (16).

By (16) and by the results of reference [2] we can conclude that (9) is valid for any $k=0, 1, 2, \ldots$

4. EXAMPLES

(i) Let

$$F(x)=1-e^{-\lambda x} \tag{19}$$

for $x \geqq 0$ and

$$a=\int_0^\infty x\, dH(x)<\infty. \tag{20}$$

Then

$$m(x)=\lambda x \tag{21}$$

for $x \geqq 0$,

$$B_r=(\lambda a)^r/r! \tag{22}$$

for $r \geqq 0$ and

$$P_k=e^{-\lambda a}(\lambda a)^k/k! \tag{23}$$

for $k \geqq 0$.

(ii) Let

$$H(x)=1-e^{-\mu x} \tag{24}$$

for $x \geqq 0$ and

$$\varphi(s)=\int_0^\infty e^{-sx}\,dF(x) \tag{25}$$

for $s \geqq 0$.

Then $a=1/\mu$ is finite,

$$B_r=\prod_{i=1}^{r}\left(\frac{\varphi(i\mu)}{1-\varphi(i\mu)}\right)$$

for $r \geqq 1$, $B_0=1$ and P_k is given by (9) for $k \geqq 0$.

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