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Optimization of repair limit replacement
policies for one unit systems

OPTIMIZATION OF
REPAIR LIMIT REPLACEMENT POLICIES
FOR ONE UNIT SYSTEMS (*)

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Abstract. − *A one unit system is considered which is repaired at failure or at age T. If a repair is not completed within a certain repair time limit S the unit is replaced. The introduced expected cost rate function simplifies the construction and analysis of the objective function. Due to it insight is much enlarged, as an economic interpretation can be given to the results.*

Keywords: Repair limit replacement policy, expected cost rate function, discounting, optimization.

Résumé. − *On considère un système à une unité avec réparation en cas de panne ou à l'âge T. Si une réparation n'est pas effectuée dans une certaine limite de temps S, l'unité est remplacée. La fonction d'espérance du coût, introducée dans cet article, simplifie la construction et l'analyse de la fonction objective. Comme une interprétation dans une façon économique est possible, les résultats peuvent être mieux compris.*

Mots clés : Réparation et remplacement préventif, fonction d'espérance du coût, coût escomptable, optimisation.

1. INTRODUCTION

Investigating a replacement policy one usually takes two steps. During the first step the objective function is obtained, while during the second step properties of the objective function are derived, e. g. concerning the existence and uniqueness of an optimal policy. With respect to a large class of replacement policies of one unit systems we have shown in Zijlstra [4] that the so-called expected cost rate function may be a helpful tool during both steps. Due to it the construction and analysis of the objective function are simplified and moreover insight is enlarged as an economic interpretation can be given to the results.

In this paper we show that the expected cost rate function is useful in a similar way when studying replacement situations with a repair phase. That kind of replacement problems have been studied earlier by Nakagawa and Osaki [1], and Nakagawa [2]. Our investigation generalizes and simplifies their work.

In section 2 the replacement with repair model is described and the objective function is obtained, while some of its properties are treated in section 3. The used optimization criterion is minimization of costs per time unit over an infinite timespan. In section 4 we explain that the given results need only a minor adjustment if one is minimizing the expected total discounted expenses.

2. MODEL AND OBJECTIVE FUNCTION

Model

We consider a one unit system which is repaired at failure or at age T whichever occurs first. A failure gives rise to an additional cost. Furthermore a repair is stopped if it is not completed within the repair time limit S, and the unit is then replaced by a new identical unit. It is assumed that a repaired unit is like new. So two alternating phases can be distinguished for the system: an operating phase 1 and a repair phase 2.

Objective function

Let $A(t, T, S)$ denote the total expenses spent during the time interval $[0, t)$, assuming that at time $t=0$ the first unit has been installed. The stochastic process $\{A(t, T, S); t \geq 0\}$ has a regenerative structure, that is it consists of successive cycles at the beginning of which the process restarts itself in a probabilistic sense. A new cycle begins every time when a new or repaired unit starts operating. In order to make all cycles, including the first one, identical in the sense of costs we say that there is a replacement cost c_2 at the beginning of *every* cycle, while a negative cost $-c_2$ occurs at the end of it if the repair is completed within the time limit S. Now we choose as objective function, $C(T, S)$, the total costs per unit time over an infinite timespan i.e.:

$$C(T, S) = \lim_{t \to \infty} \frac{A(t, T, S)}{t} = \frac{EX_1 + EX_2}{EZ_1 + EZ_2}, \qquad (2.1)$$

where the random variables: X_1, total cost during phase 1 of a cycle including c_2 and a possible additional cost due to a failure; X_2, total cost during repair phase 2 of a cycle including earnings in case of a successful repair; Z_i, length of phase i of a cycle, $i=1, 2$.

The latter identity in (2.1) is a well-known result from the theory about regenerative processes, see Ross [3].

Expected cost rate function

Elaborating (2.1) the concept of expected cost rate function is a useful tool. In order to introduce it we investigate the cost X_i during an arbitrary cycle. For convenience we assume that phase i of that cycle starts at time $u = 0$.

Let now $Y_i(u)$ denote the costs caused by *that particular* cycle in phase i during the time interval $[0, u)$, $u > 0$. By definition $Y_1(0) \equiv c_2$ and $Y_2(0) \equiv 0$. Notice that there may be costs during $[0, u)$ caused during the other phase or by following cycles but they are not contained in $Y_i(u)$. With respect to $Y_i(u)$ we make the important assumption that the following limit exists for all $u \geq 0$:

$$h_i(u) = \lim_{\delta \downarrow 0} \frac{E(Y_i(u+\delta) - Y_i(u) \mid Z_i > u)}{\delta} \qquad (2.2)$$

The conditional expectation is defined to be zero if $P(Z_i > u) = 0$. We call $h_i(u)$ the *expected cost rate function* for phase i of a cycle. Now we can state the important

THEOREM 2.1: *If the limit in (2.2) exists for all $u \geq 0$, then the expected cost during phase i of a cycle is given by:*

$$EX_i = EY_i(0) + \int_0^\infty h_i(u) \, \overline{G}_i(u) \, du \; = EY_i(0) + E\left(\int_0^{Z_i} h_i(u) \, du \right), \quad (2.3)$$

where:

$$\overline{G}_i(u) = P(Z_i > u), \qquad i = 1, 2.$$

Proof: Take $\delta > 0$, then we have:

$$EX_i = EY_i(\infty) = EY_i(0)$$

$$+ \sum_{u=0, \delta, 2\delta, \ldots} [E(Y_i(u+\delta) - Y_i(u) \mid Z_i > u) \, P(Z_i > u)$$

$$+ E(Y_i(u+\delta) - Y_i(u) \mid Z_i \leq u) \, P(Z_i \leq u)].$$

A phase does not cause costs after it has ended, i.e. if $Z_i \leq u$ then $Y_i(v) - Y_i(u) = 0$ for $v > u$ with probability 1. Hence:

$$EX_i = EY_i(0) + \sum_{u=0, \delta, 2\delta, \ldots} [E(Y_i(u+\delta) - Y_i(u) \mid Z_i > u) \, \overline{G}_i(u)].$$

Taking $\delta \downarrow 0$ we obtain in the usual way:

$$EX_i = EY_i(0) + \int_0^\infty h_i(u)\,\overline{G}_i(u)\,du.$$

From this the second identity in (2.3) can be found by a simple calculation. $\otimes$

Before applying this result to (2.1) we give some additional notation:

c_2, replacement cost (price of a new unit including installation):

$c_1 - c_2$, additional cost in case of a failure. We choose this notation in order to keep the results comparable with age replacement without repair and a cost c_1 for an unplanned replacement;

$F_i(u) = P(W_i \leq u), i = 1, 2,$ with W_1 and W_2, respectively, the lifelength and the repair time of a unit;

$\overline{F}_i(u) = 1 - F_i(u),\ i = 1, 2;$

$r_1(u) = f_1(u)/\overline{F}_1(u),$ the failure rate function;

$r_2(u) = f_2(u)/\overline{F}_2(u),$ the repair rate function;

It is assumed that $f_i(u) = dF_i(u)/du$ exists.

$k_1(u)$, operating cost intensity of a unit at age u; one may think of maintenance cost, but also of returns. In that case $k_1(u)$ is negative.

$k_2(u)$, repair cost intensity at repair age u, possibly including non-availability costs.

It is easily seen that:

$$G_1(u) = \begin{cases} F_1(u) & \text{if} \quad 0 \leq u < T, \\ 1 & \text{if} \quad u \geq T \end{cases}$$

and:

$$G_2(u) = \begin{cases} F_2(u) & \text{if} \quad 0 \leq u < S, \\ 1 & \text{if} \quad u \geq S. \end{cases}$$

Using the theorem given above and knowing that for a nonnegative random variable Z with distribution function G one has $EZ = \int_0^\infty [1 - G(u)]\,du$ we

rewrite (2.1) as:

$$C(T, S) = \frac{c_2 + \displaystyle\int_0^T h_1(u)\,\overline{F}_1(u)\,du + \int_0^S h_2(u)\,\overline{F}_2(u)\,du}{\displaystyle\int_0^T \overline{F}_1(u)\,du + \int_0^S \overline{F}_2(u)\,du} \qquad (2.4)$$

The expected cost rate functions in this basic expression for the objective function can be written down directly:

$$h_1(u) = k_1(u) + (c_1 - c_2)\,r_1(u)$$

and: $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (2.5)

$$h_2(u) = k_2(u) - c_2\,r_2(u).$$

3. ANALYSIS OF THE OBJECTIVE FUNCTION

In the analysis of the objective function an important role is played by a function having the following form:

$$C(v) = \frac{\gamma + \displaystyle\int_0^v a(u)\,du}{\delta + \displaystyle\int_0^v b(u)\,du}, \qquad (3.1)$$

with $v > 0$, $\delta > 0$ and $b(u) > 0$, $u \geq 0$.

Observe that $C(T, S)$ in (2.4) has this form as a function of T with S fixed and as a function of S with T fixed. In this section we state a few results concerning $C(v)$ as given in (3.1) and next we apply them to $C(T, S)$ in (2.4). Consider $C(v)$ on an open interval $I \subset (0, \infty)$ and suppose that a and b are continuous for $u \geq 0$. Furthermore let $D(v)$ be defined by:

$$D(v) = \frac{\delta a(v)}{b(v)} + \int_0^v \left\{ \frac{a(v)}{b(v)} b(u) - a(u) \right\} du - \gamma.$$

After some elementary calculations we find:

$$\frac{dC(v)}{dv} = \frac{b(v)\,D(v)}{\left[\delta + \displaystyle\int_0^v b(u)\,du\right]^2} = \frac{a(v) - C(v)\,b(v)}{\delta + \displaystyle\int_0^v b(u)\,du}. \qquad (3.2)$$

From (3.2) it follows that:

$$\frac{dC(v)}{dv} = 0 \quad \Leftrightarrow \quad D(v) = 0 \quad \Leftrightarrow \quad C(v) = \frac{a(v)}{b(v)}. \qquad (3.3)$$

•THEOREM 3.1: *If $a(u)/b(u)$ is increasing (decreasing) on an open interval I, then $C(v)$ has at most one extreme on I and, if it exists, the extreme is a minimum (maximum).*

Proof: For v', $v \in I$ with $v' > v$ we have:

$$D(v') - D(v) = \delta \left\{ \frac{a(v')}{b(v')} - \frac{a(v)}{b(v)} \right\}$$

$$+ \int_0^v \left\{ \frac{a(v')}{b(v')} - \frac{a(v)}{b(v)} \right\} b(u)\,du$$

$$+ \int_v^{v'} \left\{ \frac{a(v')}{b(v')} b(u) - a(u) \right\} du.$$

This expression shows that $D(v)$ is monotone on I if $a(u)/b(u)$ is monotone on I. Hence $D(v)$ will cross the v-axis on I [i. e. $D(v) = 0$] at most once and so $C(v)$ will have at most one extreme [i. e. $dC(v)/dv = 0$] because of (3.3). If $a(u)/b(u)$ is increasing (decreasing) the crossing will be from below (above), implying a minimum (maximum). $\otimes$

COROLLARY 3.2: *If $a(u)/b(u)$ is increasing on $[0, \infty)$ then:*

$$C(v') \leqq \frac{a(v')}{b(v')}$$

implies that $C(v)$ has no extreme for $v > v'$ and

$$C(v'') > \frac{a(v'')}{b(v'')}$$

implies that $C(v)$ has no extreme for $v \leqq v''$.

COROLLARY 3.3: *If $a(u)/b(u)$ is increasing on $[0, \infty)$ then $C(v)$ has no extreme on $(0, \infty)$ if at least one of the following conditions holds:*

(i) $$C(0) \leqq \frac{a(0)}{b(0)}, \qquad with \quad C(0) = \lim_{v \downarrow 0} C(v);$$

(ii) $$\lim_{v \to \infty} \left\{ C(v) - \frac{a(v)}{b(v)} \right\} > 0;$$

(iii) $$C(v) - \frac{a(v)}{b(v)} \downarrow 0 \quad for \quad v \to \infty.$$

If neither (i), *nor* (ii), *nor* (iii) *holds $C(v)$ has a unique minimum on $(0, \infty)$.*

In order to illustrate the given results a number of possible courses of C and increasing a/b have been drawn in the figure below.
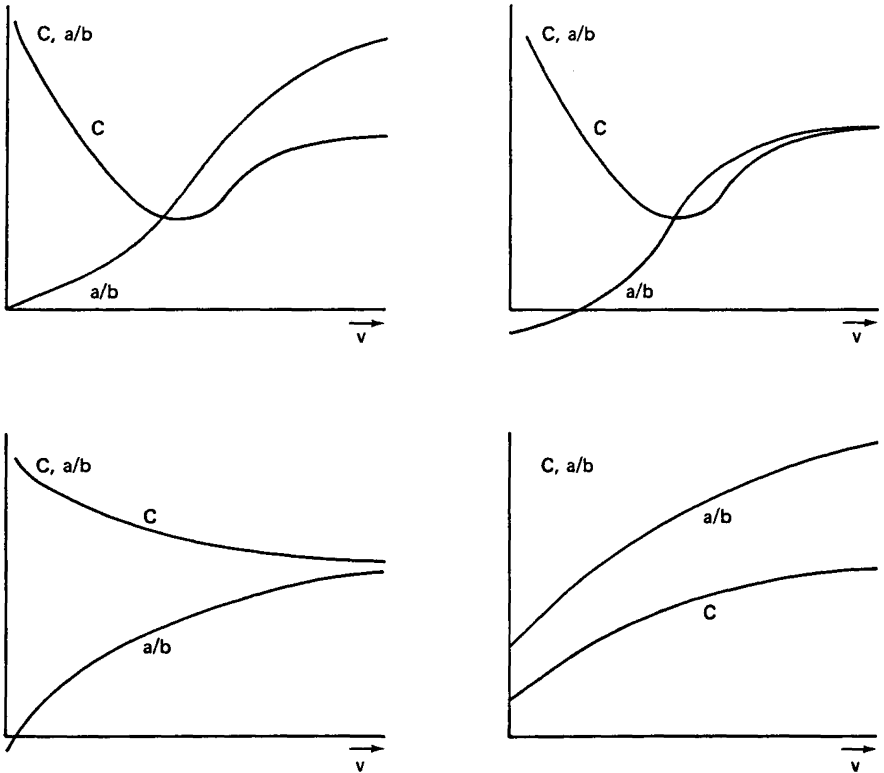
Possible courses of C with a/b increasing.

Now we apply the results given above to the objective function $C(T, S)$ as defined in (2.4), assuming that h_i and F_i, $i = 1, 2$ are continuous on $[0, \infty)$. If $C(T, S)$ has an extreme in a point of the set $W = \{(T, S) : 0 < T < \infty, 0 < S < \infty\}$, then its partial derivatives are equal to zero in that point. Together with (3.3) this consideration leads to the statement that:

$$C(T, S) = h_1(T) = h_2(S), \tag{3.4}$$

is a necessary condition for an extreme in point $(T, S) \in W$.

Economic interpretation

Relation (3.4) says that in the optimal situation (i.e. T and S are chosen in such a way that C is minimal) phase i should be stopped at the moment when its expected cost rate h_i becomes equal to the over-all cost per time unit, $i = 1, 2$. From theorem 3.1 it follows that such a time point will be reached at most once in the case of an increasing expected cost rate function. We observe that the expected cost rate functions have a meaning similar to that of marginal cost functions in Economics when one is concerned with non-failing, deteriorating equipment. This economic consideration makes many results concerning the behaviour of the objective function much more transparent.

In the search of an optimal policy the course of the expected cost rate function plays an essential role, as will be illustrated also by the next theorem.

THEOREM 3.4: (i) *If h_1 is increasing and h_2 is decreasing on $[0, \infty)$, then one has:*
- *no extremes on W;*
- *no maximum and at most one minimum on $\{(T, S) : 0 < T < \infty, S = 0\}$;*
- *no maximum and at most one minimum on $\{(T, S) : 0 < T < \infty, S = \infty\}$;*
- *no minimum and at most one maximum on $\{(T, S) : T = \infty, 0 < S < \infty\}$.*

(ii) *If h_1 and h_2 are increasing on $[0, \infty)$, then one has:*
- *at most one extreme, a minimum, on W;*
- *the minimum is assumed on $V = \{(T, S) : h_1(T) = h_2(S)\}$;*
- *a necessary and sufficient condition for such a minimum is that:*

$$C(T^-, S^-) \leqq h_1(T^-) \qquad and \qquad C(T^+, S^+) \geqq h_1(T^+),$$

with:

$$T^- = \inf_V(T) \qquad and \qquad T^+ = \sup_V(T)$$

and S^-, S^+ defined in a similar way.

REMARK 1: If $T^+ = \infty$ (or $S^+ = \infty$) and $h_1(T^+) = C(T^+, S^+)$ then $C(T^+, S^+) \geqq h_1(T^+)$ in the theorem above has to be replaced by $C(T, S) - h_1(T) \uparrow 0$ taking $T \to T^+$ along the line V, cf. corollary 3.3.

REMARK 2: Similar statements can be made about the behaviour of $C(T, S)$ in the case of h_1 decreasing and h_2 increasing or h_1 and h_2 both decreasing. If h_1 and h_2 are monotone only on certain intervals, statements as above can be restricted to those intervals, cf. theorem 3.1. We give no details. Theorem 3.4 illustrates sufficiently how to use theorem 3.1 in such specific situations.

The results of theorem 3.4 are direct consequences of theorem 3.1, its corollaries and (3.4). Therefore its proof is omitted. We give only some additional comment. In case (i) it is possible that (3.4) is fulfilled for a point of W. In that point C has a minimum as a function of T and a maximum as a function of S due to theorem 3.1 and (3.3), i.e. the point is a saddle-point. For the cases with $S = 0$, $S = \infty$, ... corollary 3.3 may be helpful in the search of an optimum.

As h_1 and h_2 are increasing in case (ii) extremes must be minima due to theorem 3.1. But C is continuous, so there can be at most one minimum.

Earlier work

Nakagawa and Osaki [1], and Nakagawa [2] consider the model given above for the special cases with no planned repair, that is T is assumed to be infinite, and with no planned replacement that is $S = \infty$. Their results, for instance those derived for some particular repair cost functions k_2 in the case $T = \infty$, are direct consequences of the theorems treated above. When using the concept of expected cost rate function these results can be found very easily and more insight is obtained into their meaning. In this context the following oversight in Nakagawa and Osaki [1], page 313 is detected directly: it is asserted that the optimum policy is no replacement $(S = \infty)$ in the case $T = \infty$, with $k_2(u) = \beta u^{\alpha}$, $\beta > 0$, $-1 < \alpha < 0$ and arbitrary F_2. Using (2.4), (2.5) and corollary 3.3 it is easy to choose F_2 and c_2 in such a way that for instance $S = 0$ is the optimal policy.

4. DISCOUNTING

Minimization of the expectation of the sum of discounted expenses over an infinite horizon is often used as optimization criterion. Applying this criterion to our repair and replacement model the expected cost rate function may again be a helpful tool. We explain this briefly.

Let $\alpha > 0$ be the discount rate and let $X^{(j)}$ denote the costs during the j-th cycle discounted to the beginning of that cycle, $j = 1, 2, \ldots$ Moreover let $Z^{(j)}$ be the length of the j-th cycle, then we have for the expected total expenses discounted to $t = 0$ (take $Z^{(0)} \equiv 0$):

$$C(T, S) = E\left[\sum_{j=1}^{\infty} X^{(j)} \exp\left(-\alpha(Z^{(1)} + \ldots + Z^{(j-1)})\right) \right]$$

$$= \frac{E\{X_1^{(1)} + X_2^{(1)} \exp(-\alpha Z_1)\}}{1 - E(\exp(-\alpha Z_1 - \alpha Z_2))}, \quad (4.1)$$

with $X_i^{(1)}$ the cost during phase i of the first cycle discounted to the beginning of that phase, $i = 1, 2$.

Let now $h_i(u)$ be the expected cost rate function as defined in (2.2) (not discounted), then one can find in a way similar to the one used in theorem 2.1 that:

and:

$$\left\{ \begin{array}{l} EX_1^{(1)} = c_2 + \displaystyle\int_0^\infty h_1(u) e^{-\alpha u} \overline{G}_1(u)\, du \\[4mm] EX_2^{(1)} = \displaystyle\int_0^\infty h_2(u) e^{-\alpha u} \overline{G}_2(u)\, du. \end{array} \right. \qquad (4.2)$$

Observing that:

$$E(\exp(-\alpha Z_i)) = 1 - \alpha \int_0^\infty e^{-\alpha u} \overline{G}_i(u)\, du, \qquad i = 1, 2$$

we get from (4.1) and (4.2):

$C(T, S)$

$$= \frac{c_2 + \displaystyle\int_0^T h_1(u) e^{-\alpha u} \overline{F}_1(u)\, du + E(\exp(-\alpha Z_1)) \int_0^S h_2(u) e^{-\alpha u} \overline{F}_2(u)\, du}{1 - E(\exp(-\alpha Z_1)) + \alpha E(\exp(-\alpha Z_1)) \displaystyle\int_0^S e^{-\alpha u} \overline{F}_2(u)\, du}$$

$$= \frac{c_2 + \displaystyle\int_0^T \left\{ h_1(u) - \alpha \int_0^S h_2(u) e^{-\alpha u} \overline{F}_2(u)\, du \right\} e^{-\alpha u} \overline{F}_1(u)\, du}{1 - E(\exp(-\alpha Z_2)) + E(\exp(-\alpha Z_2)) \displaystyle\int_0^T e^{-\alpha u} \overline{F}_1(u)\, du}$$

We see that $C(T, S)$ has the familiar form (3.1) as a function of S and as a function of T. So the results of section 3 are applicable. From (3.3) we obtain that for the optimal S^* and T^*:

$$h_2(S^*) = \frac{C(T^*, S^*)}{\alpha^{-1}}$$

and:

$$h_1(T^*) = \frac{C(T^*, S^*) E(\exp(-\alpha Z_2)) + \int_0^S h_2(u) e^{-\alpha u} \overline{F}_2(u) \, du}{\alpha^{-1}}$$

The first identity shows that a planned replacement should be carried out at a moment at which the expected cost rate becomes equal to the total expected discounted costs per discounted time unit. The second identity leads to a similar interpretation with respect to a planned repair, if one assumes that at time $t = 0$ one is starting with a repair phase.

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