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CONDITIONS FOR LUMPING THE GRADES OF A HIERARCHICAL MANPOWER SYSTEM (*)

by VOLKER ABEL ([1])

Abstract. — *In manpower systems there is often a need to condense some grades (for example, organizational reasons or simplified forecasting). For hierarchical Markovian systems, conditions are derived when a grouping of neighboring grades is possible without losing the system's Markov property. The grades to be grouped must have the same leaving probabilities. For various policies of recruitment and promotion, all possible lumpings are given. It turns out that there are only a few types of them.*

Keywords : manpower system; hierarchical; Markovian; lumping.

Résumé. — *Dans les systèmes de gestion du personnel, il est souvent souhaitable de regrouper plusieurs niveaux hiérarchiques (par exemple pour des motifs d'organisation ou de simplification des prévisions). Pour les systèmes hiérarchiques markoviens, les conditions sont celles qui découlent du fait qu'un regroupement de niveaux voisins n'est possible que s'il ne fait pas perdre le caractère markovien du système. Les niveaux à regrouper doivent avoir des probabilités identiques de sortie. Pour différentes politiques de recrutement et de promotion, on indique tous les regroupements possibles. Il s'avère qu'il y en a peu.*

Mots clés : Système hiérarchique markovien de gestion du personnel; regroupements.

1. A MARKOVIAN MANPOWER SYSTEM

For a discrete-time manpower system with k grades (ranks), let $n_i(T)$ be the number of persons in grade i, $i=1, \ldots, k$, at time T, $T=0, 1, 2, \ldots$, and $\mathbf{n}(T)=(n_1(T), \ldots, n_k(T))$ the corresponding row vector. Then, $N(T)=\sum_{i=1}^{k} n_i(T)$ is the total number of persons in the system, and $M(T)=N(T)-N(T-1)$ is the number of newly created vacancies at time T. A person in grade i moves with probability $p(i, j)$ to grade j, and leaves with probability w_i the system. If a person is hired, he is recruited with probability r_i into grade i. Let $\mathbf{P}$ be the matrix of transition probabilities $p(i, j)$, $\mathbf{w}'$ the column vector of the w_i's, and $\mathbf{r}$ the row vector of the r_i's. We assume that all these probabilities only depend on the currently occupied rank, that they are constant over time, and that they are applied to a person independently of what happened to other persons. Under these Markovian assumptions we get the recursive relation:

$$\mathbf{n}(T+1)=\mathbf{n}(T)\cdot\mathbf{P}+\mathbf{n}(T)\cdot\mathbf{w}'\cdot\mathbf{r}+M(T+1)\cdot\mathbf{r} \tag{1}$$

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(*see* [1], chapter 4) where: $\mathbf{n}(T)\cdot\mathbf{P}$ represents normal internal movement; $\mathbf{n}(T)\cdot\mathbf{w}'\cdot\mathbf{r}$ represents recruits who replace leavers, and $M(T+1)\cdot\mathbf{r}$ represents recruits filling new vacancies.

This equation holds for expanding manpower systems ($M(T)>0$), for systems of constant size ($M(T)=0$), and for shrinking systems ($M(T)<0$) as long as $\mathbf{n}(T)\cdot\mathbf{w}'\geqq -M(T+1)$.

Putting $\mathbf{Q}=\mathbf{P}+\mathbf{w}'\cdot\mathbf{r}$, $\mathbf{Q}$ is a stochastic matrix, and the relation:

$$\mathbf{n}(T+1)=\mathbf{n}(T)\cdot(\mathbf{P}+\mathbf{w}'\cdot\mathbf{r})+M(T+1)\cdot\mathbf{r}=\mathbf{n}(T)\cdot\mathbf{Q}+M(T+1)\cdot\mathbf{r} \tag{2}$$

defines a Markov chain.

2. THE CONCEPT OF LUMPABILITY

When the number of states of a Markov chain is very large, we may be interested in condensing some states if this can be done without losing the Markov property. Consider a decomposition $\{S_i\}$, $i=1, \ldots, l$, of the state space S with pairwise disjoints sets S_i and $S_1 \cup S_2 \cup \ldots \cup S_l=S$. We call the Markov chain lumpable (groupable) with respect to the partition $\{S_i\}$, $i=1, \ldots, l$, if the resulting process with the l states $S_1, \ldots, S_l$ is also a Markov chain.

To simplify notation we denote the sets $S_1, \ldots, S_l$ by $\hat{1}, \ldots, \hat{l}$. If $\mathbf{Q}$ is the transition matrix underlying the Markov chain with state space S, we set $q(i, A)=\sum_{j\in A} q(i,j)$ for all $A\subset S$ and $i\in S$. Then the following criterion for lumpability holds (*see*, for example, [2], p. 166 ff):

A necessary and sufficient condition for a Markov chain to be lumpable with respect to a partitition $\{S_i\}$, $i=1, \ldots, l$, is that the probabilities $q(i, S_n)$ have the same value $\hat{q}(\hat{m}, \hat{n})$ for all states $i\in S_m$ for every pair of subsets S_m, S_n, $m\neq n$. The matrix $\hat{\mathbf{Q}}=(\hat{q}(\hat{m}, \hat{n}))$ is the transition matrix for the lumped Markov chain.

As an exemplary application of this concept we take the prediction of grade sizes in a manpower system by means of relation (2). Let the k grades be lumped with respect to a partition $S_1, \ldots, S_l$. Denote by $\mathbf{B}$ a $l\times k$ matrix such that its i-th row is a probability vector whose nonnull components are those corresponding to the states in S_i, and by $\mathbf{C}$ a $k\times l$ matrix such that the nonnull components of its i-th column are equal to 1 and also correspond to the states in S_i, $i=1, \ldots, l$. Then $\hat{\mathbf{Q}}=\mathbf{B}\cdot\mathbf{Q}\cdot\mathbf{C}$, and $\mathbf{C}\cdot\mathbf{B}\cdot\mathbf{Q}\cdot\mathbf{C}=\mathbf{Q}\cdot\mathbf{C}$ if and only if the Markov chain defined through $\mathbf{Q}$ is lumpable (*see* e. g. [2], p. 168). Note that matrix $\mathbf{C}$ controls the lumping. If we lump at time $\mathrm{T}+1$, we get:

$$n(\mathrm{T}+1)\cdot\mathbf{C}=(n(T)\cdot\mathbf{Q}+M(T+1)\cdot r)\cdot\mathbf{C}=n(T)\cdot\mathbf{Q}\cdot\mathbf{C}+M(T+1)\cdot r\cdot\mathbf{C}.$$

Lumping at time T, i. e. executing $\mathbf{n}(T)\cdot\mathbf{C}$, and then forecasting grade sizes for time $T+1$, we have:

$$\mathbf{n}(T)\cdot\mathbf{C}\cdot\hat{\mathbf{Q}}+M(T+1)\cdot\mathbf{r}\cdot\mathbf{C}=\mathbf{n}(T)\cdot\mathbf{C}\cdot\mathbf{B}\cdot\mathbf{Q}\cdot\mathbf{C}+M(T+1)\cdot\mathbf{r}\cdot\mathbf{C}$$
$$=\mathbf{n}(T)\cdot\mathbf{Q}\cdot\mathbf{C}+M(T+1)\cdot\mathbf{r}\cdot\mathbf{C}=\mathbf{n}(T+1)\cdot\mathbf{C}.$$

That is, we will get the same grade sizes if we first forecast and then lump, or if we first lump and then forecast. Of course, the latter procedure is more economical. Other reasons why one might wish to lump are given in [3], p. 175.

3. LUMPABILITY IN HIERARCHICAL MANPOWER SYSTEMS

Let the k grades be ordered such that 1 is the lowest, 2 the second-lowest, etc., and k the highest rank. A manpower system is called hierarchical if demotions are excluded (i. e. $p(i, j)=0$ for $i>j$), and if promotion is possible to the next rank only (i. e. $p(i, j)=0$ for $j>i+1$).

Because of the nature of hierarchical systems, we confine our investigation to the lumping of neighboring ranks. First we analyze the simplest mode of hiring, namely recruitment to the bottom rank only. Such a policy is hardly found in actual manpower systems as a whole, but is accurate for segments of them. Later, in proposition 3, we treat the case of recruitment into several ranks.

PROPOSITION 1: *Assume a hierarchical manpower system with* $\mathbf{r}=(1, 0, \ldots, 0)$. *If* $p(i, i+1)>0$ *for all* $i=1, \ldots, k-1$, *and if* h *is the lowest rank such that* $w_h=w_{h+1}=\ldots=w_k$, *then the only possible lumpings are of the form* $S_1=\{1\}, \ldots, S_{l-1}=\{l-1\}, S_l=\{l, l+1, \ldots, k\}$ *for* $l\geqq h$.

Proof: We have to show that the probabilities $q(i, S_n)$ have the same value for all $i\in S_m$ for every pair S_m, S_n, $m\neq n$. The only nontrivial cases are the ordered pairs S_l, S_n, $n=1, \ldots, l-1$. We get $q(i, S_n)=w_i$ for $n=1$, and $q(i, S_n)=0$ for $n>1$. Since w_i has the same value for all $i\in S_l$, lumpability is proved. A lumping such that $S_l=\{l, \ldots, k\}$ with $l<h$ is impossible as $q(i, S_1)$ is not constant in $i\in S_l$. Suppose a grouping of some neighboring singletons into a subset $S_*=\{i, \ldots, j\}$ with $1\leqq i<j\leqq l-1$ is possible. Then $q(s, S_{j+1})=0$ for $s<j$ and $q(s, S_{j+1})=p(j, j+1)>0$ for $s=j$, thus this lumping is also impossible. ◇

This proposition generalizes a theorem of Thomas and Barr [3] who assumed a rather simple manpower model without recruitment and wastage, and where the system merges into top rank.

As a complement to proposition 1, we give:

PROPOSITION 2: *Assume a hierarchical manpower system with* $\mathbf{r}=(1, 0, \ldots, 0)$. *Let the lowest rank with* $p(j, j+1)=0$ *be* j_1, *the second-lowest rank with* $p(j, j+1)=0$ *be* j_2, *etc., and the highest rank with* $p(j, j+1)=0$ *be* j_l. *The system is lumpable with respect to* $S_1=\{1, \ldots, j_1\}$, $S_2=\{j_1+1, \ldots, j_2\}, \ldots, S_l=\{j_{l-1}+1, \ldots, j_l\}, S_{l+1}=\{j_l+1, \ldots, k\}$ *if and only if constants* c_q, $q=2, 3, \ldots, l+1$, *exist with* $w_i=c_q$ *for all* $i \in S_q$, $q=2, 3, \ldots, l+1$.

No other lumpings are possible except lumpings that divide at least one S_q, $q=1, \ldots, l+1$ *into singletons, or lumpings that originate from unions of some neighboring* S_q*'s.*

Proof: Taking two sets, say S_m and S_n, we have: $q(i, S_n)=0$ for all $i \in S_m$ if $m<n$, or if $m>n>1$, and $q(i, S_n)=w_i$ for all $i \in S_m$ if $m>n=1$.

Thus we see that the transition probabilities $q(i, S_n)$ are independent of the choice of $i \in S_m$ if and only if constants c_m, $m=2, 3, \ldots, l+1$ exist such that $w_i=c_m$ for all $i \in S_m$.

Obviously, we may divide any S_q into its singletons and relax the requirement $w_i=c_q$ for $i \in S_q$ without violating lumpability. Other partitions of any S_q, and, as a consequence, any overlapping of some S_q's are prohibited because of the different promotion probabilities $p(i, i+1)$. We may unite two or more neighboring S_q's if w_i has the same value for all ranks in the union. ◇

Since the conditions of these two propositions are complementary, we have listed all possible lumpings in hierarchical manpower systems with recruitment to bottom rank only. Finally we discuss the case where recruitment into several ranks is possible as it is in systems which enlist new members in various ranks according to their educational background.

PROPOSITION 3: *Assume we have ranks* $1=j_1 \leqq j_2 \leqq \ldots \leqq j_l \leqq k$ *with* $r_{j_i}>0$ *for* $i=1, \ldots, l$ *and* $r_j=0$ *otherwise. A hierarchical manpower system is lumpable with respect to* $S_1=\{1, \ldots, j_2-1\}$, $S_2=\{j_2, \ldots, j_3-1\}, \ldots, S_l=\{j_l, \ldots, k\}$ *if and only if there are constants* c_q, $q=1, \ldots, l$ *such that:*

$$w_{j_q}=w_{j_q+1}=\ldots=w_{j_{q+1}-1}=c_q, \qquad q=1, \ldots, l-1,$$
$$w_{j_l}=w_{j_l+1}=\ldots=w_k=c_l.$$

Moreover, $p(j_q-1, j_q)=0$ *for* $q=2, 3, \ldots, l$ *is necessary for lumping.*

Proof: Lumpability with respect to this partition means that for every pair S_m, S_n, $m \neq n$:

$$q(i, S_n)=\sum_{j \in S_n}(p(i, j)+w_i \cdot r_j)=\sum_{j \in S_n} p(i, j)+w_i \cdot r_{j_n}$$

has the same value for all $i \in S_m$.

For $|m-n|>1$ and for $m=n+1$, this is equivalent to the requirement that the probabilities of leaving w_i are the same for all $i \in S_m$, since then $p(i, j)=0$ for all $i \in S_m$ and all $j \in S_n$. For $m=n-1$, and $i \neq j_n - 1$, $q(i, S_n)=w_i \cdot r_{j_n}$ is constant, since the w_i's are constant, say $w_i = c_{n-1}$ for all $i \in S_m$, by the statement above. For $m=n-1$, and $i=j_n-1$ we get $q(i, S_n)=p(j_n-1, j_n)+w_{j_n-1} \cdot r_{j_n}$ which by definition of lumpability must be equal to $c_{n-1} \cdot r_{j_n}$. That is, $p(j_n-1, j_n)=r_{j_n} \cdot (c_{n-1}-w_{j_n-1})$. As w_{j_n-1} equals the constant c_{n-1}, lumpability implies $p(j_n-1, j_n)=0$.

We may now summarize our findings. In all hierarchical manpower systems neighboring grades can be lumped if and only if the leaving probabilities of the states being grouped, are equal. In systems with recruitment to bottom rank only and with strictly positive promotion probabilities, solely the lumping of a bunch of upper ranks is possible, always including top rank. If some promotion probabilities vanish, or if recruitment into several ranks takes place, no other grouping than the segmentation of the system is allowed, i. e. the system falls apart into several separate systems. Thus, the possibilities of grouping without losing the Markov property are very limited.

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