Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle

R. L. Garg

S. K. Khanna

Queue dependent additional server queueing problem with batch arrivals

Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle, tome 17, nº 4 (1983), p. 391-398.

<http://www.numdam.org/item?id=RO_1983__17_4_391_0>

R.A.I.R.O. Recherche opérationnelle/Operations Research
(vol. 17, n° 4, nov. 1983, p. 391 à 398)

QUEUE DEPENDENT ADDITIONAL SERVER QUEUEING PROBLEM WITH BATCH ARRIVALS (*)

by R. L. GARG and S. K. KHANNA (1)

Abstract. — *We consider in this paper the steady state behaviour of a Queueing System with queue length dependent additional server facility wherein arrivals occur in batches of variable size. Whenever the queue length in front of the first server reaches a certain lenght, the system adds another server. Steady state probabilities and expected queue lengths in Single Server System and Additional Server System are calculated explicitly. Associating the costs with the opening of new server and the waiting of the customers, a criterion to obtain the Decision Point at which the application of additional server will be profitable is discussed.*

Keywords: Queues with several channels; Bulk queues; Design and control of queues.

Résumé. — *Nous considérons dans cet article le comportement asymptotique d'un système disposant d'un serveur supplémentaire si la longueur de la file d'attente le nécessite. On suppose que les arrivées se font en grappes. Chaque fois que la file d'attente au premier guichet atteint une certaine longueur, le système ouvre un autre guichet. On calcule explicitement les probabilités asymptotiques et les moyennes de ces longueurs. Considérons les coûts de l'ouverture d'un nouveau guichet et de l'attente des clients, on obtient un critère pour obtenir le point et l'introduction d'un nouveau guichet est profitable.*

INTRODUCTION

A large number of queueing problems where the service is through parallel channels have been solved by various authors. In particular Kendall [1] considers the steady state behaviour of many server queueing system. Saaty [2] studies the transient behaviour of the system M/M/C. Romani [3], Phillips [4] and Murari [5] obtain the steady state probabilities of queueing problems with variable number of service channels. More recently Singh [6] discusses a Markovian queue wherein the system starts another server at some cost whenever the queue length in front of the first server reaches a certain length.

(*) Received in february 1982.
(1) Department of Mathematics, Kurukshetra University, Kurukshetra 132 119, India.

R.A.I.R.O. Recherche opérationnelle/Operations Research, 0399-0559/1983/391/$ 5.00

In this paper an attempt is made to modify the results due to Singh so as to bring them more close to many practical situations, where arrivals occur in batches of variable size, which may exist in banks, election booths, etc.

Thus the queueing system studied in this paper can be described as follows:

1. The arrivals follow Poisson distribution and occur in batches of variable size with maximum size T; $\lambda\, C_i\, dt$ is the first order probability that a batch of arrivals consists of exactly $i\,(1 \leqq i \leqq T)$ units; where $\sum_{i=1}^{T} C_i = 1$.

2. The queue discipline is first come first served.

3. The service facility consists of one regular and one additional service channel. The regular service channel is always at the disposal of customers, irrespective of the queue length (the number of customers waiting including those being served). The additional service channel operates instantaneously on an arrival when there are $N\,(>2\,T)$ customers in the queue and stops operating when the queue length again reduces to N.

4. The service time distribution for each server is negative exponential with parameter μ.

A relationship is developed among costs, the traffic intensity ρ, the maximum size of the arriving batch T, probabilities $C_i\,(1 \leqq i \leqq T)$ and the maximum allowable size N in front of the first server. For particular values of T and C_i's, the ratios of the costs is given in a table for different values of N and ρ. The graphs for ratios of the costs against ρ for these values of N are plotted. It is demonstrated with an example that how from the graphs the decision point for N at which the application of second server is profitable can be calculated. Results of Singh are obtained as particular cases.

STEADY STATE SOLUTION

Let:

$P_n \equiv$ Steady state probability that there are n customers in the system at any time.

Elementary probability reasoning leads to the following difference equations governing the system:

$$P_1 - \rho\, P_0 = 0,$$

$$P_n - (1+\rho)\, P_{n-1} + \rho \sum_{i=1}^{T} C_i P_{n-1-i} = 0; \qquad 2 \leqq n \leqq N,$$

$$P_{N+1}-\frac{1}{2}(1+\rho)\,P_N+\frac{\rho}{2}\sum_{i=1}^{T} C_i P_{N-i}=0,$$

$$P_n-\left(1+\frac{\rho}{2}\right)P_{n-1}+\frac{\rho}{2}\sum_{i=1}^{T} C_i P_{n-1-i}=0; \qquad N+2\leqq n,$$

where:

$$\rho=\frac{\lambda}{\mu} \qquad \text{and} \qquad P_j=0 \qquad \text{for} \quad j<0.$$

Solutions of these difference equations are:

$$\begin{aligned}
P_n=\sum_{i=1}^{T} a_i\,\alpha_i^n\,P_0&=\sum_{i=1}^{T} b_i\,\alpha_i^n\,P_0; \qquad 1\leqq n\leqq T+1\\
&=\sum_{i=1}^{T} b_i\,\alpha_i^n\,P_0; \qquad T+2\leqq n\leqq N-T\\
&=\sum_{i=1}^{T} b_i\,\alpha_i^n\,P_0=\sum_{i=1}^{T} d_i\,\beta_i^n\,P_0; \qquad N-T+1\leqq n\leqq N\\
&=\sum_{i=1}^{T} d_i\,\beta_i^n\,P_0; \qquad N+2\leqq n.
\end{aligned}$$

and:

$$P_{N+1}=\frac{1}{2}\sum_{i=1}^{T} b_i\,\alpha_1^{N+1}\,P_0=\sum_{i=1}^{T} d_i\,\beta_i^{N+1}\,P_0$$

where:

$$\left.\begin{aligned}
a_i&=\frac{\alpha_i^{T-1}}{\prod\limits_{\substack{r=1\\ r\neq i}}^{T}(\alpha_i-\alpha_r)}\\
b_i&=\frac{\sum\limits_{g=1}^{T} a_g\left[\rho\sum\limits_{j=1}^{T}\sum\limits_{k=0}^{j-1} C_{T-k}\,\alpha_i^{k-j}\,\alpha_g^{j}-\alpha_g^{T+1}\right]}{\alpha_i(1-\alpha_i)\prod\limits_{\substack{r=1\\ r\neq i}}^{T}(\alpha_i-\alpha_r)}\\
d_i&=\frac{\sum\limits_{g=1}^{T} b_g\left[\rho\sum\limits_{j=1}^{T}\sum\limits_{k=0}^{j-1} C_{T-k}\,\beta_i^{k-j}\,\alpha_g^{N-T+j}-\alpha_g^{N+1}\right]}{2\,\beta_i^{N-T+1}(1-\beta_i)\prod\limits_{\substack{r=1\\ r\neq i}}^{T}(\beta_i-\beta_r)}
\end{aligned}\right\}\ 1\leqq i\leqq T$$

and α_i, $\beta_i (1 \leqq i \leqq T)$ are respectively the non-unity roots of the algebraic equations:

$$Z^{T+1}-(1+\rho) Z^T+\rho (C_1 Z^{T-1}+C_2 Z^{T-2}+\ldots+C_T)=0,$$

$$Z^{T+1}-\left(1+\frac{\rho}{2}\right) Z^T+\frac{\rho}{2}(C_1 Z^{T-1}+C_2 Z^{T-2}+\ldots+C_T)=0.$$

Since $\sum_{n=0}^{\infty} P_n=1$, each β_i must lie within the unit circle and by Rouche's theorem it can be easily shown that this condition holds provided $\rho<2/T$.

Now assuming that this condition is satisfied and using normal condition that $\sum_{n=0}^{\infty} P_n=1$, we find that:

$$P_0=\left[1+\sum_{i=1}^{T} \frac{b_i \alpha_i (1-\alpha_i^N}{1-\alpha_i}+\frac{d_i \beta_i^{N+1}}{1-\beta_i}\right]^{-1}.$$

MEAN QUEUE LENGTH

Let L_2 denote the mean queue length:

$$L_2=\sum_{n=0}^{\infty} n P_n=\sum_{n=1}^{N} n P_n+\sum_{n=N+1}^{\infty} n P_n.$$

Substituting the value of P_n already obtained we have

$$L_2=\sum_{i=1}^{T}\left[\frac{b_i \alpha_i \{1-(N+1) \alpha_i^N+N \alpha_i^{N+1}\}}{(1-\alpha_i)^2}\right.$$

$$\left.+\frac{d_i \beta_i^{N+1} (N+1-N \beta_i)}{(1-\beta_i)^2}\right] P_0.$$

SINGLE SERVER SYSTEM

When additional server does not operate α_i must lie in a unit circle, for which $\rho<1/T$. The mean queue length L_1 for the system can be obtained by making $N \to \infty$ in the expression for L_2:

$$L_1=\sum_{i=1}^{T}\frac{b_i\,\alpha_i}{(1-\alpha_i)^2}\,Q_0$$

where:

$$Q_0=\left[1+\sum_{i=1}^{T}\frac{b_i\,\alpha_i}{1-\alpha_i}\right]^{-1}.$$

PARTICULAR CASE

When $T=1$, $C_1=1$, $C_i=0\,(2\leqq i\leqq T)$, i. e. when the arrival occur singly the situation corresponds to Queues Dependent Servers by V. P. Singh [6].

In this case we have:

$$\alpha_1=\rho,\qquad a_1=1,\qquad b_1=1,\qquad \beta_1=\frac{\rho}{2},\qquad d_1=2^N$$

and:

$$P_n=\rho^n\,P_0;\qquad 0\leqq n\leqq N=\frac{\rho^n}{2^{n-N}}\,P_0;\qquad N<n.$$

where:

$$P_0=\frac{(1-\rho)(2-\rho)}{2-\rho-\rho^{N+1}}$$

and:

$$L_2=\frac{\rho(2-\rho)}{(1-\rho)(2-\rho-\rho^{N+1})}-\frac{(2-\rho)(N-\rho\,N+1)\,\rho^{N+1}}{(1-\rho)(2-\rho-\rho^{N+1})}+\frac{(1-\rho)(2N-\rho\,N+2)\,\rho^{N+1}}{(2-\rho)(2-\rho-\rho^{N+1})}$$

These results are in confirmity with that of V. P. Singh [6].

PROFITABILITY CRITERION

Let R_1 be the unit profit associated with L_1-L_2 and R_2 is the cost to the system of providing the additional server. It is profitable for the system to have additional server only if:

$$R_1 (L_1 - L_2) > R_2 (\text{Prob. } n > N),$$

$$\text{Prob. } (n > N) = \sum_{n=N+1}^{\infty} P_n = \sum_{i=1}^{T} \frac{d_i \beta_i^{N+1}}{1-\beta_i} P_0,$$

$$L_1 - L_2 = \sum_{i=1}^{T} \frac{b_i \alpha_i}{(1-\alpha_i)^2} (Q_0 - P_0)$$

$$+ \sum_{i=1}^{T} \left[\frac{b_i \alpha_i^{N+1} \{ (N+1) - N \alpha_i \}}{(1-\alpha_i)^2} - \frac{d_i \beta_i^{N+1} (N+1-N\beta_i)}{(1-\beta_i)^2} \right] P_0.$$

$$\frac{R_2}{R_1} < \left[\frac{\left\{ \sum_{i=1}^{T} b_i \alpha_i/(1-\alpha_i)^2 \right\} \left\{ \sum_{j=1}^{T} (d_j \beta_j^{N+1}/1-\beta_j) - (b_j \alpha_j^{N+1}/1-\alpha_j) \right\}}{1 + \sum_{j=1}^{T} b_j \alpha_j/1-\alpha_j} \right.$$

$$\left. + \sum_{i=1}^{T} \frac{b_i \alpha_i^{N+1} \{ (N+1) - N \alpha_i \}}{(1-\alpha_i)^2} - \frac{d_i \beta_i^{N+1} \{ (N+1) - N \beta_i \}}{(1-\beta_i)^2} \right]$$

$$\times \left[\sum_{i=1}^{T} \frac{d_i \beta_i^{N+1}}{1-\beta_i} \right]^{-1}.$$

It may be noted that the above relation depends only on T, N, ρ, R_2/R_1 and $C_i (1 \leqq i \leqq T)$. Thus in terms of any four value fifth can be calculated.

For $T=5$:

$$C_{i+1} = \binom{4}{i} q^{4-i} p^i; \qquad i = 0, 1, 2, 3, 4$$

where:

$$p + q = 1, \qquad p = \frac{1}{2}, \qquad q = \frac{1}{2}.$$

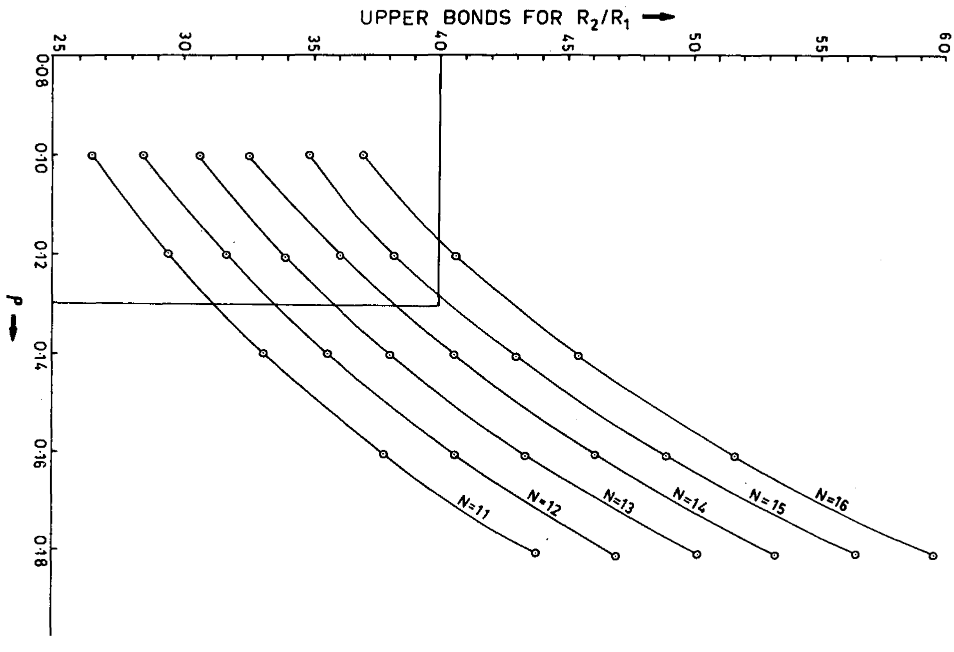
The upper bounds of R_2/R_1 for various values of N and ρ are given in the Table.

TABLE

Values of the upper bounds of R_2/R_1 for various ρ and N

ρ. . . . . N	.10	.12	.14	.16	.18
11.	26.45	29.47	33.17	37.86	43.94
12.	28.47	31.69	35.62	40.64	47.1
13.	30.61	33.88	38.09	43.42	50.29
14.	32.45	36.11	40.61	46.22	53.42
15.	34.84	38.22	43.00	48.99	56.59
16.	37.06	40.65	45.49	51.70	59.75

From the tabular values the graphs for the upper bounds of R_2/R_1 against ρ, for various values of N, are plotted.



From the graphs the decision point for N at which the application of second server is profitable can be calculated for the given values of R_2/R_1 and ρ, for example, if $\rho = .13$, $R_2/R_1 = 40$; the point $(.13, 40)$ lies between the graphs for $N = 14$ and $N = 15$. Therefore, in this case, it will be profitable for the system to provide the additional server when $N \geqq 15$.

ACKNOWLEDGEMENTS

The authors are grateful to Professor C. Mohan, Head of the Department of Mathematics, Kurukshetra University, Kurukshetra for his keen interest in the work and Dr. N. K. Jain, Post-Doctoral Fellow, for useful discussions during the preparation of this paper.

REFERENCES

1. D. G. Kendall, *Stochastic Processes Occurring in the Theory of Queues and their Analysis by the Method of Imbedded Markov Chain*, Ann. Math. Statist., Vol. 24, 1953, pp. 338-354.
2. T. L. Saaty, *Elements of Queueing Theory with Applications*, McGraw Hill Book Co. Inc. New York, 1961.
3. J. Romani, *A Queueing Model with Variable Number of Channels*, Trabajos de Estadistica, Vol. 8, No. 3, 1957, pp. 175-189.
4. Jr. R. Cecil Phillips, *A Variable Channels Queueing Model with Limited Number of Channels*. Thesis, Georgia Institute of Technology, 1960.
5. K. Murari, *Some C-Additional Service Channels Limited Space Queueing Problems*, Zamm., Vol. 51, 1971, pp. 157-162.
6. V. P. Singh. *Queue Dependent Servers*, J. Eng. Math., Vol. 7, No. 2, 1973, pp. 123-126.