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A note on minimum-dummy-activities PERT networks


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A NOTE ON MINIMUM-DUMMY-ACTIVITIES 
PERT NETWORKS (*)

by Marian Mrozek (1)

Abstract. — In the paper we present a polynomial-time method of verification if solutions to the minimum-dummy-activities problem in PERT networks produced by some suboptimal algorithms are optimal.

Keywords: Network construction, network analysis, PERT networks, arc-dual digraph.

Résumé. — Dans cet article, nous présentons une méthode à temps polynomial pour vérifier si les solutions au problème d'activités fictives minimum dans les réseaux de Pert données par des algorithmes suboptimaux sont en réalité suboptimales.

1. INTRODUCTION

The problem of the construction of an event-node PERT network which minimizes the number of vertices and dummy activities has been studied by many authors (the detailed bibliography can be found in [7]). The complete solution to the minimum-vertices problem was given by Cantor and Dimsdale [2] in 1969. In the same year Hayes [8] observed that the number of vertices and the number of dummy arcs cannot be minimized simultaneously in general. In 1979 Krishnamoorthy and Deo [4] proved that the minimum-dummy-activities problem is NP-complete. According to their result Syso suggested searching for a polynomial approximate algorithm and presented one in [7].

In the paper we consider the problem of the construction of a minimum-dummy-activities event-node PERT network in the class of all minimum-event

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networks. We prove that the problem has exactly one solution in a certain subclass of solutions and that the solution may be found in polynomial time on the base of algorithms presented by Cantor and Dimsdale [2], Sterboul and Wertheimer [6] and Mrozek [5]. Relatively often the above solution is also optimal in the general case, which may be verified in polynomial time too.

2. NOTATION

Let \((G, S)\) be a directed finite graph (or simply a digraph), where \(G\) is the set of its vertices and a relation \(S \subseteq G \times G\) is the set of its arcs. For an arc \(s \in S\) its initial and terminal vertices will be denoted \(s^-\) and \(s^+\) respectively, i.e. \(s = (s^-, s^+)\). Let \(a, b \in G\) and let \(\{s_i\}_{i=1}^n, n \geq 1,\) be a sequence of arcs in \(S\) such that \(a = s_1^-, s_n^+ = b\). If \(s_i^+ = s_{i+1}^-\) for \(i = 1, 2, \ldots, n-1\) then the sequence will be called a path in \(S\) from \(a\) to \(b\). We will say that the path is trivial if \(\{s_i^-\} \cup \{s_i^+\} \subseteq \{a, b\}\). The digraph \((G, S)\) will be called acircuit if for any vertex \(a \in G\) there is no path from \(a\) to \(a\).

The relation:

\[
tc S := \{(a, b) \in G^2 : \text{there exists a path in } S \text{ from } a \text{ to } b\},
\]

will be called the transitive closure of \(S\).

The relation:

\[
tr S := \{(a, b) \in S : \text{every path in } S \text{ from } a \text{ to } b \text{ is trivial}\},
\]

will be called the transitive reduction of \(S\). We will also use the notation:

\[
tc_0 S := tc S \cup \{(a, a) : a \in G\}.
\]

REMARK 2.1: The operations \(tr\) and \(tc\) satisfy the following conditions

\[
\begin{align*}
(2.1) \quad & tr S \subseteq S \subseteq tc S, \\
(2.2) \quad & \text{for any } S' \subseteq S \quad tc (tr S \cup S') = tc S, \\
(2.3) \quad & tr (tc S) = tr S.
\end{align*}
\]

For any set \(A\), its cardinality will be denoted by \(\text{card } A\).

3. THE PROBLEM

Let \((H, T)\) be an acircuit digraph. We recall (see [2]) that the triple \((G, S, k)\) is called an arc-dual digraph of \((H, T)\) if \((G, S)\) is an acircuit digraph and
$k : H \to S$ is a mapping such that:

$$\forall h_1, h_2 \in H (h_1, h_2) \in tc T \iff (k(h_1))^+, (k(h_2))^- \in t_c o S.$$  

Denote $S' := \{s \in S : \exists h \in H : k(h) = s\}$, $S^f := S \setminus S'$. Notice that in application to PERT networks $(H, T)$ may be considered as an activity network and $(G, S)$ as the corresponding event network, in which $S'$ and $S^f$ represent real and dummy activities respectively.

Let $AD(H, T)$ denote the class of all arc-dual digraphs of $(H, T)$. The following remark follows immediately from remark 2.1.

**Remark 3.1:** If $(G, S, k) \in AD(H, T)$ then $(G, tcS, k) \in AD(H, T)$ and $(G, trS \cup S', k) \in AD(H, T)$.

**Definition 3.1:** Digraphs $(G_i, S_i, k_i) \in AD(H, T)$, $(i = 1, 2)$ are said to be weakly isomorphic if there exists a bijection $f : G_1 \to G_2$ such that the following diagram

$$\begin{array}{ccc}
G_1 \times G_1 & \xrightarrow{f \times f} & G_2 \times G_2 \\
\downarrow H & & \downarrow \\
G_1 \times G_2 & & G_2 \times G_2
\end{array}$$

is commutative, i.e. for every $h \in H$:

$$f(k_1(h)^-) = k_2(h)^- \quad \text{and} \quad f(k_1(h)^+) = k_2(h)^+.$$

We distinguish the following two subclasses of $AD(H, T)$:

$$AD_0(H, T) := \{(G, S, k) \in AD(H, T) : G = G^+ \cup G^-\},$$

$$AD_1(H, T) := \{(G, S, k) \in AD_0(H, T) : \forall s \in S^f s^- \in G^+, s^+ \in G^-\}.$$  

where:

$$G^+ := \{g \in G : \exists h \in H g = k(h)^+\},$$

$$G^- := \{g \in G : \exists h \in H g = k(h)^-\}.$$  

**Definition 3.2:** An arc-dual digraph $(G, S, k) \in AD(H, T)$ will be called vertex minimal if for every $(G', S', k') \in AD(H, T)$:

$$\text{card } G \leq \text{card } G'.$$
A vertex minimal digraph \((G, S, k) \in AD(H, T)\) (or \(AD_1(H, T)\)) will be called arc-vertex minimal in \(AD(H, T)\) (or in \(AD_1(H, T)\)) if for any other vertex minimal digraph \((G', S', k') \in AD(H, T)\) (or \(AD_1(H, T)\)):

\[
\text{card } S \leq \text{card } S'.
\]

The following remark is obvious:

**Remark 3.2:** Any vertex minimal digraph \((G, S, k) \in AD(H, T)\) belongs to \(AD_0(H, T)\).

The following two theorems will be basic in the sequel. Since they are implicitly proved in [2], [5] and [6], we omit their proofs.

**Theorem 3.1:** Any two vertex minimal digraphs belonging to \(AD(H, T)\) are weakly isomorphic.

**Theorem 3.2:** There exists a vertex minimal digraph belonging to \(AD_1(H, T)\), in particular \(AD_1(H, T) \neq \emptyset\).

According to theorem 3.1, further on we may assume that all vertex minimal graphs in \(AD(H, T)\) have the same set of vertices, which we will denote by \(G_{H,T}\). Obviously they have also the same mapping \(k : H \to S\) and consequently the same set of real arcs \(S'\). Thus we may simply write \(S\) instead of \((G_{H,T}, S, k)\) in case of a vertex minimal digraph in \(AD(H, T)\).

The following remark is an immediate consequence of the definition of \(AD(H, T)\):

**Remark 3.3:** Let \(S_1, S_2\) be two vertex minimal digraphs in \(AD(H, T)\). Assume \((a, b) \in G^+_{H,T} \times G^-_{H,T}\). Then:

\[
(a, b) \in tc S_1 \iff (a, b) \in tc S_2.
\]

**4. MAIN RESULT**

**Lemma 4.1:** If \(S \in AD(H, T)\) and \(S' \subset AD_1(H, T)\) are vertex minimal then:

\[
(4.1) \quad tc S \supset tc S_1.
\]

**Proof:** To prove (4.1) it is enough to show that \(S_1 \subset tc S\). Let \(s \in S_1\). Since \(S' = S_1'\), we may assume that \(s \in S_1'\). By the definition of \(AD_1(H, T)\), there exist \(h_1, h_2 \in H\) such that \(k(h_1)^+ = s^-\) and \(k(h_2)^- = s^+\). Hence \((h_1, h_2) \in tc T\) and consequently \((k(h_1)^+, k(h_2)^-) = s \in tc_0 S\). Since \(s^- \neq s^+\), we obtain \(s \in tc S\). \(\square\)
Lemma 4.2: If $S_1, S_2 \in AD_1(H, T)$ are vertex minimal then:

(4.2) $tc S_1 = tc S_2$,

(4.3) $tr S_1 = tr S_2$.

Proof: (4.2) follows immediately from lemma 4.1. To prove (4.3) observe that by (2.3) and (4.2):

$$tr S_1 = tr (tc S_1) = tr (tc S_2) = tr S_2.$$ 

The following theorem follows immediately from the above lemma and remark 3.1.

Theorem 4.1: There exists exactly one arc-vertex minimal digraph in $AD_1(H, T)$. For any vertex minimal digraph $S \in AD_1(H, T)$ it equals $tr S \cup S'$.

Further on we will denote this unique in $AD_1(H, T)$ arc-vertex minimal digraph in $AD_1(H, T)$ by $S_{H, T}$.

The following relation in $G_{H, T}^2$ is important in the study of arc-vertex minimal digraphs in $AD(H, T)$:

$$A_{H, T} := \{(a, b) \in G_{H, T}^2 : \forall (a', b') \in G_{H, T}^+, (a', a) \in tc_0 S_{H, T}, (b, b') \in tc_0 S_{H, T} \Rightarrow (a', b') \in tc_0 S_{H, T}\}.$$

Its importance explains the following:

Theorem 4.2: For any vertex minimal digraph $S$ in $AD(H, T)$:

$$S \subset A_{H, T}.$$

Proof: Let $s \in S$. First assume that $s^- \in G_{H, T}^+$. If $s^+ \in G_{H, T}^-$ then it follows from remark 3.3 that $s \in tc_0 S_{H, T}$ and consequently $s \in A_{H, T}$. Assume $s^+ \in G_{H, T}^+$ and let $b' \in G_{H, T}^-$, $(s^+, b') \in tc_0 S_{H, T}$. Again by remark 3.3 $(s^-, b') \in tc_0 S_{H, T}$, thus $(s^-, b') \in tc_0 S$. Consequently $(s^-, b') \in tc_0 S_{H, T}$ and $s \in A_{H, T}$. The remaining cases can be proved in a similar way. □

Theorem 4.3: If $A_{H, T} \subset tc S_{H, T}$ then $S_{H, T}$ is the unique arc-vertex minimal digraph in $AD(H, T)$.

Proof: Assume $S$ is an arc-vertex minimal digraph in $AD(H, T)$. By theorem 4.2 $S \subset A_{H, T} \subset tc S_{H, T}$. Hence $S \in AD_1(H, T)$. It follows from theorem 4.1 that $S = S_{H, T}$. □

Theorem 4.4: The verification of the assumptions of theorem 4.3 can be done in polynomial time.

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Proof: From the construction presented in [5] and [6] it follows that at least one vertex minimal digraph in $AD_1 (H, T)$ can be found in polynomial time. In order to compute $S_{H, T}$ it is enough to construct the transitive reduction of any vertex minimal graph in $AD_1 (H, T)$, which also may be done in polynomial time (see [1]). To verify the assumptions of theorem 4.3 it is now necessary to construct $tc S_{H, T}$ and $A_{H, T}$. It is well known that the transitive closure may be found in polynomial time. What concerns $A_{H, T}$ one can easily construct an algorithm, which analyses all possible quadruplets $(a', a, b, b') \in G_{H, T}^4$, i.e. needs $O \left(n^4\right)$ time, where $n$ stands for the number of vertices in $G_{H, T}$. The verification of the inclusion $A_{H, T} \subseteq tc S_{H, T}$ can be obviously done in polynomial time. \[\square\]

For a subset $B \subseteq A_{H, T} \setminus tc S_{H, T}$ define:

$$rd B := \{s \in S_{H, T}^f : \text{there exists a non-trivial path in } B \cup S_{H, T} \text{ from } s^- \text{ to } s^+\}.$$ 

Let $A_0 := \{s \in A_{H, T} \setminus tc S_{H, T} : \text{rd } s \neq \emptyset\}$. 

**Theorem 4.5:** Assume that:

$$A_0 \subseteq tr \left( A_0 \cup S_{H, T} \right)$$

and

$$\forall s, s' \in A_0 \quad s \neq s' \quad \Rightarrow \quad \text{rd } s \cap \text{rd } s' = \emptyset.$$

Then $tr \left( A_0 \cup S_{H, T} \right)$ is an arc-vertex minimal digraph in $AD (H, T)$. If $A_0 = \emptyset$ or there is only one element in $A_0$ then the assumptions (4.4) and (4.5) are obviously satisfied. Additionally $A_0 = \emptyset$ is a necessary and sufficient condition for $S_{H, T}$ to be the unique arc-vertex minimal digraph in $AD (H, T)$. The verification of the assumptions (4.4) and (4.5) as well as the construction of $tr \left( A_0 \cup S_{H, T} \right)$ can be done in polynomial time.

Since the proof of the above theorem is mainly technical, we omit it.

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