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Ernest Koenigsberg

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QUEUE SYSTEMS WITH BALKING: A STOCHASTIC MODEL OF PRICE DISCRIMINATION (*)

by Ernest KŒNIGSBERG ([1])

Abstract. — *Queueing Systems with balking have been used to explain the role of uncertainty and capacity in behavior of a monopoly (De Vany, 1976) and a duopoly (Kœnigsberg, 1980). In both these models service is on a FIFO basis and there is no distinction between customers. In this paper customers are distinguished by their impatience, by their expectation of alternative service and by priority classifications which depend on their own utility functions and the loss of customers in each class is determined under simple behavior rules. The utility functions consider both price and the quality of service. The models, assuming profit maximizing behaviors by the firm, result in the usual conditions for price discrimination by the monopolist. While the models hold for a wide range of distributions of service time, only the $M/M/1$ queue is discussed in this paper.*

Keywords: Queue models; Balking; Price Discrimination.

Résumé. — *Les systèmes de files d'attente avec obstruction ont été utilisés pour expliquer le rôle de l'incertitude et de la capacité dans le comportement d'un monopole (De Vany, 1976) et d'un duopole (Kœnigsberg, 1980). Dans ces deux modèles le service est du type FIFO, et l'on ne fait aucune distinction entre les clients. Dans le présent article, les clients se différencient par leur impatience, par leur attente d'un autre service, et par des classifications de priorité qui dépendent de leur propres fonctions d'utilité; la perte de clients dans chaque classe est déterminée sous des règles de comportement simples. Les fonctions d'utilité considèrent à la fois le prix et la qualité de service. Ces modèles, supposant que la firme cherche à maximiser son profit, donnent les conditions usuelles pour la discrimination du prix par le monopoliste. Bien que les modèles soient valables pour un large éventail de lois de probabilité du temps de service, seule la file d'attente $M/M/1$ est examinée dans cet article.*

Queueing systems with balking have been used by De Vany (1976, 1977) as a possible model to explain the role of uncertainty and capacity in the behavior of a monopoly. In the De Vany model, service is on a first come-first served basis with no distinction between customers. Here we introduce classes of customers and several different queue disciplines and examine the resulting loss of customers by the monopolist.

Customers can be distinguished by classes because they value the service differently, because they have different alternatives to the offered service or because they have different cost for waiting time. Because of these factors

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([1]) Schools of Business Administration, University of California, Berkeley, CA.

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each customer class chooses a "balking" number or, perhaps, a bribe to pay to advance their position in the queue.

The priority ordering of customers can arise from price differences—i. e., a form of price discrimination by the monopolist. Customers who pay a higher price receive better service, i. e., have a shorter waiting time, than those who pay a lower price. We assume that higher priority custumers preempt lower priority customers in service and go to the head of the line when a high priority customer is in service. As De Vany pointed out in his paper, customers with a high value of waiting time are willing to pay a premium for better service.

I. MULTIPLE CUSTOMER CLASSES

We consider queueing systems with M classes of customers in which a customer with lower class index m is more impatient than all customers with higher index m' ($m=1, 2, \ldots, M$). The impatience of class i is expressed in a balking number B_i, with $B_{m'} > B_m$ or a reneging rate $k_i Y$ (with $k_m > k_{m'}$) or a bribe b_i (with $b_m > b_{m'}$). We will discuss the balking case in detail and only digress on the other forms of impatience.

Following De Vany (1976) we assume that a customer of priority class i chooses the balking number B_i so that his expected gain G_n^i by remaining in the monopolistic line when he is the n-th customer is greater than or equal to his expected gain G_a^i by seeking service from an alternative supplier:

$$G_n^i = R(p_i) - w_i n_i/\mu_i, \tag{1.1}$$

$$G_a^i = R^a(p_i^a) - c - w_i t - w_i (L^a + 1)/\mu^a, \tag{1.2}$$

where p^a, L^a and μ^a are the expected price, line length, and capacity at the alternate source, $R(p_i)$ is the return to the customer, w_i is the waiting cost per unit time, t is the time to change sources and c is the fixed cost of a change.

A customer could select B_i so that the expected gain when the balking number is B_i is just greater than the expected gain using an alternate source. If this is done the expected gain using the monopolist is always greater than the expected gain using the alternate source. For the customer i:

$$B_i \leqq (\mu_i/w_i)[R(p_i) - R(p^a) + c + w_i(t + (L^a + 1)/\mu^a)]. \tag{1.3}$$

Customers differ in their return functions R_i, their unit cost of waiting and the expected cost of changing to an alternate source and perhaps in the characteristics of their alternate source. We expect the following to hold:

$$\partial B_i/\partial p_i < 0, \qquad \partial B_i/\partial w_i < 0, \qquad \partial B_i/\partial p^a > 0 \qquad \text{and} \qquad \partial B_i/\partial (L^a/\mu^a) > 0.$$

Even when there is a single price (and a single priority class) customers will select a balking number (or reneging rate or bribe) according to their gain functions. If there are two groups of customers with distinctly different gain functions each will have a unique balking number B_i [as given for example, by Equation (1.3), and demand function $(\lambda_i(p_i))$].

A related system in which customer departures without service are probabilistic rather than deterministic has been reported by Kœnigsberg (1980). Within the problem framework, the probability that a customer will depart at time t given n in the queue at time zero is $(1-e^{-k(n-1)t})$. In the queueing literature this departure process is known as "reneging". (We note that the customer being served cannot leave the system by this process.)

We assume that customers of type i have a reneging rate k_i. The reneging rate can be examined in economic terms. We could write (for example):

$$k_i \propto \frac{w_i}{(p_a - p_i) + w_i L_a/\mu_a + c}, \tag{1.4}$$

where p_a is price, μ_a the service rate and L_a the expected line length at the alternate server for class i customers. w_i is the unit waiting time cost for class i customers. The exact form of k_i is not important, but the sign of the derivatives should be:

$$\partial k_i/\partial w_i > 0, \qquad \partial k_i/\partial p_i > 0, \qquad \partial k_i/\partial p_a < 0, \qquad \partial k_i/\partial (L_a/\mu_a) < 0.$$

We expect the reneging rate of the more impatient customer to be greater than that of the less impatient $(k_1 > k_2)$. This would arise if the unit waiting cost $w_1 > w_2$ or if $(p_a - p_1) < (p_a - p_2)$.

An equivalent class structure can be obtained if a class of entering customers can "buy" relative priority by means of a bribe (Kleinrock, 1957). If a portion α of customers pay a fixed bribe to achieve priority status [e. g. $\lambda_1 = \alpha\lambda$, $\lambda_2 = (1-\alpha)\lambda$] and each customer retains its balking numbers, the system will be identical to the ones mentioned previously [*see* Kleinrock (1976), p. 135ff].

II. CONSERVATION LAWS

Kleinrock (1976) pointed out that "so long as the queueing discipline selects customers in a way that is independent of their service time (or any measure of their service time) then the distribution of the number in the system will be invariant to the order of service: the same will also be shown to be true for the average waiting time of customers." For single server systems with Poisson arrivals and any general service time distribution ($M/G/1$ systems) the general conservation law yields, (without balking):

$$\sum_i \lambda_i W_i^q = \lambda W_0^q/(1-\rho). \tag{2.1}$$

Here W_i^q is the average waiting time for a customer of class i, $\lambda = \sum_i \lambda_i$ is the total arrival rate and

$$W_0^q = \lambda x^2/2,$$

where x^2 is the variance of the service time.

When balking occurs, the proportion of customers of type i "lost" depends on the queue discipline, but the unfinished work in the system during any busy period $U(t)$ is independent of the order of service, where:

$$E(U(t)) = U = N_0^q + \Sigma p_i W_{ii}^q. \tag{2.2}$$

Thus, given a set arrival rates λ_i and of balking numbers, B_i the total awaiting time (over all classes i) will be independent of the queue discipline.

The conservation law holds for a wide range of conservative systems (such as $G/G/1$, $M/M/c$, etc.). Here we concentrate on systems of the type ($M/M/1$) and priority on a pre-emptive non-resume basis.

III. CUSTOMER DECISIONS

In these models the queue discipline system is established by the firm when it selects the prices and the priority rules. Given the prices for each priority class the customers compute their balking numbers by using, say, Equation (1.3) for each priority class: the selected class for a customer group would be the highest class (with a positive B_i) for which the expected gain exceeds the gain from the alternate source.

By changing prices for the customer classes the monopolist changes the customers' balking numbers (or reneging rates) and, in effect, changes the arrival rates for the several classes.

Even when there is a single price (and a single priority class) customers will select a balking number (or reneging rate). If there are two groups of customers with distinct gain functions, each will have a distinct balking number B_i [as given, for example, by Equation (1.3)], and demand function. The customer decision is the value of B_i or k_i or the magnitude of the bribe paid to improve the position in the queue.

(a) Priority Systems

We consider a queueing system with M classes of customers where a customer with lower index m has priority for service over all customers with higher index m' ($m=1, 2, \ldots, M$). If a customer with lower index m arrives when a customer with a higher index is being served, the higher index customer leaves the service center without completing service and rejoins the queue. The "preempted" customer begins service *anew* after all higher priority customers who arrive before he reenters service are served. The priority system is thus "pre-emptive non-resume". Customers of class i have a balking number B_i.

Thus the system is a single server pre-emptive non-resume priority system with balking. Given the fact that balking occurs, the firm providing the service will lose some customers of each class and hence lose revenue. We are interested in determining for a given set of values (λ_m, μ_m, B_m) what numbers $\lambda'_m (<\lambda_m)$ of each class of customers will be served. Latter we will consider ways in which the firm may set its "control" variables λ_m and μ_m so that some figure of merit (say total profit) is optimized.

The customers of priority class $m=1$, because they have a pre-emptive priority over all other customers, see a system with arrival rate $\lambda_1 (p_1)$, service rate μ_1 and balking number B_1 (and hence system utilization $\rho_1=\lambda_1 (p_1)/\mu_1$). The results for this system are well known [Gross and Harris (1974)]:

$$P_0^1=(1-\rho_1)/(1-\rho_1^{B_1+1}), \tag{3.1}$$

$$L_1=\bar{n}_1=\rho_1/(1-\rho_1)-(B_1+1)\,\rho_1^{B_1+1}/(1-\rho_1^{B_1+1}), \tag{3.2}$$

$$\lambda'_1=\lambda_1\,(1-\rho_1^{B_1})/(1-\rho_1^{B_1+1}), \tag{3.3}$$

$$W_1=L_1/\lambda'_1=\rho_1\,[(1-\rho_1^{B_1})-B_1\,\rho_1^{B_1}\,(1-\rho_1)]/\lambda\,(1-\rho_1^{B_1})\,(1-\rho_1), \tag{3.4}$$

where P_0^1 is the probability that a customer of class 1 will have to wait before being served, L_1 is the mean number of class 1 customers in the queue, λ'_1 is the number of class 1 customers served per period ($\lambda'_1<\lambda_1$) and W_1 is the expected system time of class 1 customers. The expected waiting time for class 1 customers is $W_1^q=W_1-1/\mu_1$.

Class 2 customers see a system which includes customers of classes 1 and 2. The arrival rate of class 1 customers is λ'_1 and none of these class 1 customers leave the system. The arrival rate of class 2 customers is $\lambda_2 (p_2)$ and class 2 customers will depart if the line length exceeds B_2. Now if $B_1<B_2$ a customer of class 1 will never be in a priority line of length B_2 and our assumption that only customers of class 2 leave the system is correct at all times. If

$B_1 \geqq B_2$ we introduce a small error. Now, if $\mu_2 = \mu_1 = \mu$ we have a system with arrival rate ${}_2\lambda = \lambda'_1 + \lambda_2$, a service rate μ, ${}_2\rho = {}_2\lambda/\mu$, and balking number B_2.

Using the same procedures as for the previous problem we have the same results as Equations (3.1) to (3.4) with ${}_2\rho$ replacing ρ_1, ${}_2\lambda$ replacing λ_1 and B_2 replacing B_1. Note that this is for customers of both classes. We use a left subscript m (for class m) for systems including all customer classes up to and including class m.

We have the conditions:

$$\lambda'_m = {}_m\lambda' - {}_{(m-1)}\lambda', \tag{3.5}$$

$$\mathrm{L}_m = {}_mL - {}_{(m-1)}L, \tag{3.6}$$

$$\mathrm{W}_m = ({}_mW_m\,\lambda' - {}_{(m-1)}W_{(m-1)}\,\lambda')/\lambda'_m, \tag{3.7}$$

where the right subscript m indicates the values for class m *alone*. Using these conditions and computing priority class by priority class we have:

$${}_m\lambda = {}_{(m-1)}\lambda' + \lambda_m, \tag{3.8}$$

$${}_mP_0 = (1 - {}_m\rho)/(1 - {}_m\rho^{B_m+1}), \tag{3.9}$$

$${}_mL = {}_m\rho/(1 - {}_m\rho) + (B_m + 1)\,{}_m\rho^{B_m+1}/(1 - {}_m\rho^{B_m+1}), \tag{3.10}$$

$${}_m\lambda' = {}_m\rho\,(1 - {}_m\rho^{B_m})/(1 - {}_m\rho^{B_m+1}), \tag{3.11}$$

$${}_mW = {}_mL/{}_m\lambda'. \tag{3.12}$$

Similar results are obtained for the related reneging system with reneging rates k_m. In this pre-emptive non-resume system, as in the balking system just described, a high priority customer sees a system which serves only class 1 arrivals. Using the results given by Kœnigsberg (1980) we have:

$$P_0^1 = [{}_1F_1\,(1,\ b_i;\ y_1]^{-1}, \tag{3.13}$$

$$\mathrm{L}_1 = P_0^1\,\rho_1\,[{}_1F_1\,(2,\ b_1 + 1;\ y_1)] \tag{3.14}$$

$$\lambda'_1 = \lambda_1\,(1 - P_0^1)/\rho_1, \tag{3.15}$$

where:

$$\rho_1 = \lambda_1/\mu_1, \qquad b = \mu_1/k_1, \qquad y = \lambda_1/k_1,$$

${}_1F_1\,(a,\ b;\ y)$ is the Confluent Hypergeometric Function (Abromowitz and Stegun, 1965) given by:

$${}_1F_1\,(a,\ b;\ y) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b)}{\Gamma(a)\,\Gamma(b+n)}\,(y^n/n!), \tag{3.16}$$

and $\Gamma(z)$ is the Gamma Function given by:

$$\Gamma(z+1)=\int_0^\infty u^z e^{-u} du = z\Gamma(z); \qquad (3.17)$$

when z is an integer

$$\Gamma(z+1)=z!=z\Gamma(z). \qquad (3.18)$$

Customers of class 2 see a system which includes customers of classes 1 and 2, just as in the balking case. The arrival rate of class 1 customers is λ_1' and no class 1 customers leave this "equivalent" system. The arrival rate of class 2 customers is λ_2 and class 2 customers have a reneging rate k_2. Thus we have a system with arrival rate ${}_2\lambda=\lambda_1'+\lambda_2$, a service rate μ, ${}_2\rho={}_2\lambda/\mu$ and reneging rate k_2.

Then:

$${}_2P_0=[{}_1F_1(1, b_2; y_2]^{-1}, \qquad (3.19)$$

$${}_2L={}_2P_0\,{}_2\rho[\rho[{}_1F_1(2, b_2+1; y_2)], \qquad (3.20)$$

$${}_2\lambda'={}_2\lambda(1-{}_2P_0)/{}_2\rho. \qquad (3.21)$$

Equations (3.5) to (3.7) apply in computing P_0^2, L_2, λ_2' and W_2.

In the case of reneging there is no restriction on the relative magnitudes of k_1 and k_2. Thus we could, for example, consider a system with only a small number of low priority customers who leave quickly when they discover that the waiting line is quite large (such customers represent opportunity traffic, such as "standby" passengers for an airline).

(b) Non-priority systems

Consider two classes of customers distinguished by demand rates $\lambda_i(\rho_i)$ and balking numbers B_i (with $B_2 \geqq B_1$) and a single priority class. Service is first-come first-served with an exponential distribution of service times. For this system we have:

$$P_0=[(1-\rho^{B_1+1})/(1-\rho)+\rho^{B_1}\rho_2(1-\rho^{B_2-B_1})/(1-\rho_2)]^{-1}, \qquad (3.22)$$

$$\lambda_1'=\lambda_1(1-\rho^{B_1})/(1-\rho), \qquad (3.23)$$

$$\lambda_2'=\lambda_2(1-P_0\,\rho^{B_1}\rho_2^{B_2-B_1}), \qquad (3.24)$$

$$L(<B_1)=P_0[(\rho-B_1\rho^{B_1}+(B_1-1)\rho^{B_1+1})/(1-\rho)^2, \qquad (3.25)$$

$$L(\geqq B_1)=P_0\,\rho^{B_1}B_1-(B_1-1)\rho_2$$
$$-(B_2+1)\rho_2^{B_2-B_1+1}B_2+\rho_2^{B_2-B_1+2}/(\lambda-\rho_2), \qquad (3.26)$$

$$L_2 = L(\leqq B_1) + L(>B_1), \tag{3.27}$$

$$W_2 = (L_2/\lambda_2') - 1/\mu, \tag{3.28}$$

where $\rho = (\lambda_1 + \lambda_2)/\mu$, $L(<B)$ is the mean number of customers in the system when there are fewer than B_1 customers in the system (of both classes) and $L(\geqq B_1)$ the same when there are B_1 or more customers in the system.

Table shows the results for the two systems (priority and non-priority) for the same values of λ_i, B_i, and μ. When customers of class 1 have priority they have shorter waiting times and a higher retention than under the non-priority system. Customers of class 2 show the reverse behaviour. We can also note that the total number of customers ($\lambda' = \lambda_1' + \lambda_2'$) is different. By imposing a priority system the firm retains more class 1 customers at the expense of losing a slightly larger portion of class 2 customers. The net balance is positive; i. e. the total number of customers retained is larger.

TABLE

$\lambda_1 = 1/3$, $\lambda_2 = 2/3$; $B_1 = 2$, $B_2 = 6$; $\mu = 3/2$

	Priority System	Non priority System
λ_1'	.320,4	.226,5
λ_2'	.637,8	.662,0
$\lambda' = \lambda_1' + \lambda_2'$	.958,2	.888,5
L_1	.252,4	.447,9
L_2	1.276,7	1.140,9
W_1^q	.121,1	1.310,8
W_2^q	1.264,9	.828,7
$\lambda_1' W_1^q + \lambda_2' W_i^q$	.845,5	.845,5

IV. ECONOMIC ANALYSIS

The firm's objective is to maximize its profit. The expected profit can be written as

$$E(\pi) = p_1 \lambda_1' + p_2 \lambda_2' - C(\lambda_1', \lambda_2', \mu). \tag{4.1}$$

We can note, that in the balking case with priorities:

$$\lambda_1' = \lambda_1'(p_1, \mu, B_1) = \lambda_1(1 - \beta_1), \tag{4.2}$$

$$\lambda_2' = \lambda_2'(p_2, \mu, B_2 : p_1, B_1) = \lambda_2(1 - \beta_2), \tag{4.3}$$

where β_i is the fraction of customers of class i who leave the system without being served.

(a) With priorities

In the short run the firm can only control price and the optimizing conditions obtained from Equation (4.1) are:

$$\frac{\partial E(\pi)}{\partial p_1}=\lambda_1'+\left(p_1-\frac{\partial C}{\partial \lambda_1'}\right)\frac{\partial \lambda_1'}{\partial p_1}+\left(p_2-\frac{\partial C}{\partial \lambda_2'}\right)\frac{\partial \lambda_2'}{\partial p_1}=0, \tag{4.4}$$

$$\frac{\partial E(\pi)}{\partial p_2}=\lambda_2'+\left(p_2-\frac{\partial C}{\partial \lambda_2'}\right)\frac{\partial \lambda_2'}{\partial p_2}=0. \tag{4.5}$$

From Equation (3.7) we obtain:

$$MC_2=\frac{\partial C}{\partial \lambda_2'}=p_2\left(1+\frac{1}{\varepsilon_2'}\right), \tag{4.6}$$

where $\varepsilon_2'=(\partial\lambda_2'/\partial p_2)/(\lambda_2'/p_2)$ is the price elasticity of the resulting demand (as indicated by actual sales) (2). We substitute in Equation (4.4) and note that under our assumptions:

$$\partial C/\partial\lambda_2'=\partial C/\partial\lambda_1',$$

thus we have:

$$p_2\left(1+\frac{1}{\varepsilon_2'}\right)=p_1\left(1+\frac{1}{\varepsilon_1'}-\frac{p_2\,\lambda_2'\,\varepsilon_{21}'}{p_1\,\lambda_1'\,\varepsilon_1'\,\varepsilon_2'}\right)=\partial C/\partial\lambda_1', \tag{4.7}$$

where $\varepsilon_{21}'=(\partial\lambda_2'/\partial p_1)/(\lambda_2'/p_1)$ is the cross-elasticity of demand. Because $\varepsilon_{21}'>0$ and $\varepsilon_1<0$, $\varepsilon_2<0$ (in fact $\varepsilon_1<-1$, $\varepsilon_2<-1$) the term including cross-elasticity is positive.

The exact form of the price solution will differ somewhat between the balking and reneging cases (in the form of λ_i' and ε_i'), but a short-run solution exists. The extension to more than two priority classes is straight-forward but may present computational difficulties, particularly for the reneging case.

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(2) We note that because $\lambda_i'=\lambda_i(1-\beta_i)$:

$$\varepsilon_i'=(\partial\lambda_i/\partial p_i)/(\lambda_i/p_i)-p_i(\partial\beta_i/\partial p_i)/(1-\beta_i),$$
$$\varepsilon_i'=\varepsilon_i-p_i(\partial\beta_i/\partial p_i)/(1-\beta_i),$$

where ε_i is the price elasticity of the initial demand (orders placed). Since $\partial\beta_i/\partial p_i<0$, final demand is less elastic than initial demand.

In the long run, the firm can adjust its capacity μ. Thus we have the additional condition:

$$\frac{\partial E(\pi)}{\partial \mu}=\left(p_1-\frac{\partial C}{\partial \lambda'_1}\right)\frac{\partial \lambda'_1}{\partial \mu}+\left(p_2-\frac{\partial C}{\partial \lambda'_2}\right)\frac{\partial \lambda'_2}{\partial \mu}-\frac{\partial C}{\partial \mu}=0. \qquad (4.8)$$

Again, $\partial C/\partial \lambda'_1=\partial C/\partial \lambda'_2$.

The short-run equations and solutions show that the priority problem is equivalent to the problem of price discrimination by a monopolist. The high priority customer pays a higher price to obtain a higher quality of service; i. e., a shorter expected waiting time. We would normally expect the demand function of the higher priority customer to be less elastic than that of the lower priority customer; $\varepsilon_1>\varepsilon_2>\varepsilon_3$... In the price discrimination case, the prices p_i and quantities λ'_i are set so that the marginal revenue is the same in all markets and the marginal revenue is equal to the marginal cost. That is indeed the case here.

Because the term containing cross-elasticity in Equation (4.7) is positive, we will find in all cases that $p_1>p_2$ if $\varepsilon'_1>\varepsilon'_2$. The normal price discrimination results hold—i. e., the price is lower in the more elastic market. In this case the monopolist justifies the price differential by providing a higher standard of service to those willing to pay a higher price.

(b) No priorities

In this case Equations (4.4) holds but Equation (4.5) becomes:

$$\frac{\partial E(\pi)}{\partial p_2}=\lambda'_2+\left(p_1-\frac{\partial C}{\partial \lambda'_1}\right)\frac{\partial \lambda'_1}{\partial p_2}+\left(p_2-\frac{\partial C}{\partial \lambda'_2}\right)\frac{\partial \lambda'_2}{\partial p_2}=0. \qquad (4.5\,A)$$

Writting $a_i=(\partial C/\partial \lambda'_i)=MC_i$ and $\partial \lambda'_i/\partial p_j=b_{ij}$ we have, when $MC_1=MC_2$:

$$MC_1=p_1+\frac{\lambda'_1 b_{22}-\lambda'_2 b_{12}}{b_{11}b_{22}-b_{21}b_{12}}=p_2+\frac{\lambda'_2 b_{11}-\lambda'_1 b_{21}}{b_{11}b_{22}-b_{21}b_{12}}=MC_2, \qquad (4.9)$$

we note that $b_{ii}<0$ and $b_{ij}>0$ so the numerator in Equation (4.9) is always negative. The denominator is positive when $b_{11}b_{22}>b_{21}b_{12}$. (e. g. only when the product of own elasticities exceeds the product of cross-elasticities).

The equilibrium equations for profit maximization hold for the priority, non-priority and bribing systems. The results (i. e. the selection of the prices and production rates) will depend on the form of the demand function.

All of the models present an alternate way of depicting the interaction between buyers and sellers. They introduce factors other than price. In particular, the utility function of the buyer is composed of preferences with respect to both price and quality of service.

REFERENCES

1. M. Abromowitz and I. A. Stegun, *Handbook of Mathematical Functions*, New York, Dover, 1965.
2. A. De Vany, *Uncertainty, Waiting Time, and Capacity Utilization: A Stochastic Theory of Product Quality*, J. Polit. Econ., Vol. 84, June, 1976, pp. 523-541.
3. A. De Vany and T. R. Saving, *Product Quality, Uncertainty and Regulation: The Trucking Industry*, Amer. Econ. Rev., Vol. 67, September, 1977, pp. 583-594.
4. *See*, for example, D. Gross and C. M. Harris, *Fundamentals of Queueing Theory*, New York, John Wiley and Sons, 1974.
5. L. Kleinrock, *Optimum Bribing for Queue Position*, Operat. Res., Vol. 15, January/February 1967, pp. 304-318.
6. L. Kleinrock, *Queueing Systems Vol. II: Computer Applications*, New York, Wiley-Interscience, 1976.
7. E. Kœnigsberg, *Uncertainty, Capacity and Market Share in Oligopoly: A Stochastic Theory of Product Quality*, J. Business, The University of Chicago, 1980, Vol. 53, No. 2, pp. 151-164).