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SPECIFYING THE SYSTEMATIC RISK OF PORTFOLIOS: A CLOSED FORM SOLUTION (*)

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Abstract. — *In this note, we examine a particular quadratic program which arises in a variety of financial allocation problems and derive a closed form solution for its first order Lagrangian conditions. Our technique bypasses the standard matrix inversion, thus reducing computational effort. As a consequence, larger size portfolios can now be analyzed.*

Keywords: Systematic risk; Quadratic program; pseudo-inverse.

Résumé. — *Nous examinons dans cet article un programme quadratique particulier qui se présente dans les divers problèmes d'affectation financière et nous calculons une solution explicite de ses lagrangiennes du premier ordre. Notre technique évite l'inversion matricielle habituelle, réduisant ainsi le travail de calcul. Il en résulte que des portefeuilles de plus grande taille peuvent maintenant être analysés.*

1. INTRODUCTION

In finance, a number of important allocation problems can be modeled as a mathematical programming problem in which the objective is to minimize the variance of a performance measure while attaining a target level for that measure and exhausting all funds. For instance, this paradigm arises in portfolio theory when a manager seeks to select a portfolio with certain market characteristics which minimizes his/her systematic risk (Morse, 1982) as well as when a manager seeks to allocate country risks in a portfolio of international loans (Morse, 1983). In addition, this model also arises as a integral part of a larger model which analyses the tradeoffs between attaining

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between attaining a portfolio of minimum systematic risk for 15 given stocks versus purchasing only n of those stocks and banking 15-n fixed transaction costs (McInish, Morse and Saniga, 1984).

In each of the cases cited above, the financial problem is modeled as the following quadratic program:

$$\left.\begin{aligned} &\text{Min} \sum_{i=1}^{n} d_i x_i^2, \\ &s.\,t. \sum_{i=1}^{n} \beta_i x_i = \beta, \\ &\sum_{i=1}^{n} x_i = 1, \end{aligned}\right\} \qquad \text{(VSR)}$$

where x_i is the fraction of funds invested in stock i, β_i is the calculated beta of stock i, d_i is the variance of that beta, and β is the target beta of the portfolio. At this time, we note that β_i is a standard means of measuring the volatility of a given security. More precisely, it is a regression coefficient that captures the covariance with a market index (Sharpe, 1981). In addition, we note that the constraint $x_i \geqq 0$ is included in the above model for each stock which cannot be sold short.

The classical technique for solving the above model is to first form the Lagrangian $L(x, \lambda)$, then to find (x^*, λ^*) such that $\nabla L(x^*, \lambda^*) = 0$ and finally to check if the appropriate nonnegativity conditions are satisfied by x^*. This approach has two basic drawbacks. First, to find (x^*, λ^*) we need to invert an $(n+2) \times (n+2)$ matrix. Second, given (x^*, λ^*), we need to check the appropriate nonnegativity conditions. As seen in (McInish *et al.*, 1984), this approach can be computationally expensive especially if it needs to be repeated a number of times, as is the case in the above work, and/or if the matrix inversion technique employed does not exploit the special structure of the $\nabla L(x^*, \lambda^*)$ as is the case in most of the finance literature.

In this paper, we derive a closed form solution for just x^* which bypasses the above matrix inversion in solving for (x^*, λ^*). As a result, we eliminate a need for a major portion of the calculational effort expended in (McInish *et al.*, 1984) and as a consequence make that analyses more amenable to practical implementation. In addition, we are no longer restricted to studying portfolios consisting of at most 15 to 20 stocks.

2. THE DERIVATION

In this section, we derive the formula for x^*.

First, consider the case where each $d_i=1$. In this case, the (VSR) model reduces to the problem of finding the element of minimum norm in the feasible region. Since the feasible region is of the form $Ax=b$, the optimal solution is, by definition, just $A^\dagger b$ where $A^\dagger$ is the Moore-Penrose pseudo inverse of A. [*See* either (Ben-Israel and Greville 1974) or (Luenberger 1969).] In fact, when the rows of A are linearily independent, which is the case in practice, $A^\dagger = A^T(AA^T)^{-1}$. [*See* Proposition 1, page 165 (Luenburger 1969).] Thus, in this case, $x^*=A^T(AA^T)^{-1}b$.

Next, consider the case where some $d_i \neq 1$. Then, under the substitution $y_i=\sqrt{d_i}\,x_i$, the (VSR) model reduces to the above case from which it follows that $y^*=A^T(AA^T)^{-1}b$ where:

$$A=\begin{bmatrix} \beta_1/\sqrt{d_1},\ldots,\ \beta_n/\sqrt{d_n} \\ 1,\ldots,\ 1 \end{bmatrix} \quad \text{and} \quad b=\begin{bmatrix} \beta \\ 1 \end{bmatrix}.$$

Thus, for all i, $x_i^*=y_i^*/\sqrt{d_i}$. In other words,

$$x_i^*=(\alpha+\gamma\beta_i)/\Delta d_i$$

where

$$\Delta=\left(\sum_{j=1}^{n}(1/d_j)\right)\left(\sum_{j=1}^{n}(\beta_j^2/d_j)\right)-\left(\sum_{j=1}^{n}(\beta_j/d_j)\right)^2,$$

$$\alpha=\sum_{j=1}^{n}((\beta_j^2-\beta\beta_j)/d_j)$$

and

$$\gamma=\sum_{j=1}^{n}((\beta-\beta_j)/d_j).$$

Note, these formula hold as long as the feasible region is nonempty and the rows of A are linearily independent. In other words, under the following assumptions:

$$\text{Min}\{\beta_i\}\leq\beta\leq\text{Max}\{\beta_i\} \quad \text{(feasibility)} \tag{1}$$

and

$$\text{Min}\{\beta_i\}\neq\text{Max}\{\beta_i\} \quad \text{(independence).} \tag{2}$$

In practice, these two assumptions are normally satisfied.

Finally, we conclude with a remark about nonnegativity. It follows from the Cauchy-Schwartz inequality and the above that x_i^* is nonnegative if and only if $\alpha+\gamma\beta_i$ is nonnegative. Thus, in each portfolio, $-\alpha/\gamma$ can be viewed as a guide for selling short.

In summary, we address a control problem which arises in several practical contexts in the field of finance. Our contribution is a solution technique which can significantly lower the computational costs inherent in solving those financial problems.

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