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AN AXIOMATIC APPROACH TO DISTANCE ON PARTIAL ORDERINGS (*)

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Abstract. — *In this paper we develop a set of axioms which uniquely determine a distance measure on the set of all partial orderings. The results presented generalize the previous work of Bogart and of Kemeny and Snell. While the matrix representation used herein differs from that of the above mentioned authors, our distance measure is shown to be equivalent to theirs when applied to the appropriate ranking subspaces.*

Keywords: Partiel ranking, axioms, distance measures.

Résumé. — *Dans cet article est développé un ensemble d'axiomes qui définissent une distance entre ordres partiels. Les résultats présentés généralisent le travail antérieur de Bogart et de Kemeny et Snell. Alors que la représentation matricielle introduite ici diffère de celle de ces auteurs, on montre que notre distance est équivalente à la leur dans les applications à des sous espaces appropriés de classification.*

1. INTRODUCTION

In recent years various authors have examined the problem of combining individual preferences to form a compromise or consensus of opinion. Problems of this nature arise frequently in such areas as marketing strategy design based on consumer opinion, design of voting systems, allocation of priorities to R & D projects etc. Selected references are Black [1], Davis *et al.* [4], and Riker [7].

In each of these examples an individual's preferences are expressed in terms of a ranking of a set of available alternatives (projects, candidates in an election, etc.), and in many instances the ranking is of the ordinal rather than the cardinal type. That is, the information available to the ranker/voter is of such a nature that only an expression of preference (not degree thereof), can be given. It is the problem of combining ordinal rankings which we address in this paper.

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Much of the work on ordinal ranking theory has concentrated on the axiomatic characterization of appropriate measures of distance between rankings. Some of the early research along these lines was initiated by Kemeny and Snell [5]. Kemeny and Snell present a model for combining preference rankings into a group consensus. They specifically address the case of *weak* (and linear) orderings, meaning that every pair of objects is compared although ties are permitted. Bogart [2] and [3] later generalized the model of Kemeny and Snell to include *partial* orderings. It is important to point out, however, that by Bogart's definition, the partial orderings do not include weak orderings. More clearly, in Bogart's representation, for any two objects, one is either preferred to the other or else they are not compared – ties are not allowed. So, while Bogart's work is a form of generalization of the Kemeny-Snell model it does not provide a mechanism for dealing with the case in which rankings can be in any one of the three forms (linear, weak or partial). In a linear order all objects are compared and no ties are permitted. A weak ordering has all objects compared but ties are allowed. In a partial ordering all objects do not have to be compared. A precise definition is given in the next section. For a more complete description of the basic algebraic models for preference, the reader is referred to Chapter 19 in *The Handbook of Mathematical Psychology* [6].

The model presented in the following sections provides a preference matrix representation of rankings which accomodates all *three* possibilities described above. While this representation differs from those given in Kemeny and Snell [5] and Bogart [2], we prove that the distance between any two weak orders in the Kemeny and Snell sense is the same as the Kemeny and Snell distance, and that the distance between any two partial orders in the Bogart sense is the same as the Bogart distance. Our model, thus, is equivalent to theirs, yet it allows us to compare weakly ordered pairs (ties) and uncompared pairs (partials). In addition, our general model is derived from a set of axioms which uniquely characterize this extended distance measure.

2. CHARACTERIZATION

As in [2], [3] and [5] we begin by examining some conditions which our generalized distance function d should satisfy. First we define a partial ordering.

([3]) That is, if x is preferred to y and y is preferred to z than we cannot have z preferred to x. In notation terms then we require that:

$$\forall n \geqq z, \qquad \text{if} \quad x_i \gtrsim x_{i+1}, \quad 1 \leqq i \leqq n-1, \quad \text{yet } x_n \gtrsim x_1,$$

then

$$x_i \sim x_{i+1}, \qquad 1 \leqq i \leqq n-1.$$

DEFINITION 2.1: A *partial ordering* (ranking) A of a set of objects is a subset of pairs (x, y) of the objects (possibly a proper subset), in which *either* one of the objects is preferred to the other *or* the two objects are tied, and *no intransitivities* (3) are permitted.

We shall use the notation $x > y$ to denote "x is preferred to y" and $x \sim y$ to denote "x and y are tied".

AXIOM 1: *$d(A, B) \geqq 0$ with equality if and only if $A = B$.*

AXIOM 2: *$d(A, B) = d(B, A)$.*

AXIOM 3: *$d(A, C) \leqq d(A, B) + d(B, C)$ with equality if and only if ranking B is between A and C.*

(Ranking B is said to be *between* A and C (we represent this by $[A, B, C]$) if for each pair of objects i and j either A prefers i and C prefers j (or vice versa) or the judgement of B either agrees with A or agrees with C.)

These axioms are the usual conditions for a metric with the additional requirement that distance be additive on "lines". We next wish to assume that the measure of distance does not in any way depend upon the labeling of the objects to be ranked.

AXIOM 4: *If A' results from A by a permutation of the objects and B' results from B by the same permutation, then $d(A', B') = d(A, B)$.*

We also require that if two rankings are in agreement except for at most one pair of objects a and b, then this distance is the same as if these two objects were the only ones under consideration.

AXIOM 5: *Suppose A and B differ only for exactly one pair of objects x and y. Then:*

$$d(A, B) = d(\{(x, y)_A\}, \{(x, y)_B\}),$$

where $(x, y)_A$ is the relation between the objects x and y in A. (Either $(x > y)$, $(x \sim y)$ or φ in case x & y are not compared).

Note that Axiom 5 is similar to the "agreement on segments" axiom of Kemeny and Snell, but is substantially weaker.

AXIOM 6: *Let A_1 be an ordering in which object a is preferred to b and no objects are ranked between a and b. Let:*

$$A = A_1 \setminus \{(a > b)\} \text{ and let } A_2 = A \cup \{(a \sim b)\}.$$

Then $d(A, A_1) = d(A, A_2)$.

This axiom states that the amount of disagreement from any weak ordering of a pair of objects to an empty ordering is constant.

Our final condition is simply a normalization or scaling convention.

AXIOM 7: *The minimum positive distance is one.*

Having stated the seven reasonable conditions (these are similar to the axioms of Kemeny and Snell and of Bogart), we wish the distance function to satisfy, we now ask the question: Are these axioms consistent and, if so, do they characterize a unique distance function? For the case of two objects, the answer is affirmative as shown by the following lemma.

LEMME 2.1: *For all orderings of $n=2$ objects, the distances are determined by the axioms.*

Proof: For $n=2$ there are four possible rankings, $A=(a>b)$, $B=(b>a)$, $C=(a\sim b)$ and $D=\varphi$. Of the sixteen possible pairs, we can form out of these four rankings, Axiom 1 asserts that:

$$d(A, A)=d(B, B)=d(C, C)=d(D, D)=0,$$

which reduces this number to twelve pairs. Also since $d(X, Y)=d(Y, X)$ by Axiom 2 for any two rankings X and Y, it follows that only six of these twelve pairs are relevant. Since $[A, C, B]$ and $[A, D, B]$, Axiom 3 asserts that:

$$d(A, B)=d(A, C)+d(C, B)=d(A, D)+d(D, B).$$

Since the permutation which transforms A into B leaves C and D unchanged, Axiom 4 asserts $d(A, C)=d(B, C)$ and $d(A, D)=d(B, D)$. From Axiom 6, we have $d(C, D)=d(A, D)$, so that $d(A, B)=2d(A, C)=2d(A, D)=2d(C, D)$, thus it is sufficient to determine $d(A, C)$. From the above however it is clear that this is the minimum positive distance; hence $d(A, C)=1$ by Axiom 7, and the result is established.

We next show that if two rankings agree on ties then the distance between them is determined by the axioms.

LEMME 2.2: *If A and B are two rankings such that objects a and b are tied in A if and only if they are tied in B (i. e. $(a\sim b)\in A \Rightarrow (a\sim b)\in B$), then the axioms uniquely determine the distance $d(A, B)$.*

Proof: The proof will be by induction on "n", the number of comparisons (of the form $a>b$) in $A\cup B$ but not in $A\cap B$. If $n=0$ then $A=B$ and $d(A, B)$ is determined by Axiom 1. If $n=1$ then A and B differ only for exactly one pair of objects and thus $d(A, B)$ is determined by Axiom 5 and Lemma 1.

Assume that $n=k$ (and that $d(X, Y)$ is determined for all rankings X, Y (having the same ties) for which the cardinality $|X \cup Y \setminus (X \cap Y)| < k$).

Since $[A, A \cap B, B]$, by Axiom 3 we have $d(A, B)=d(A, A \cap B)+d(B, A \cap B)$. The number of comparisons in $A=A \cup (A \cap B)$ but not in $A \cap B=A \cap (A \cap B)$ is less than or equal to k. Similarly $|B \setminus (A \cap B)| \leqq$ k. If we have equality, e.g. $|B \setminus (A \cap B)|=k$, then $A \subseteqq B$ (and $|A \setminus (A \cap B)|=0$). Hence the distances $d(A, A \cap B)$ and $d(B, A \cap B)$ [and thus $d(A, B)$] are determined by induction unless A is a subset of B or vice versa.

Suppose $A \subseteqq B$. We will construct R such that $[A, R, B]$. Let b and c be objects such that $(b > c) \in B$ but $(b > c) \notin A$. Pick an object a to be a maximal element relative to A among those objects above c relative to B that are not above c relative to A. Next choose an object d minimally among those objects not below a relative to A but below a relative to B, and not above c relative to A. Let $R=A \cup \{(a > d)\}$. Then R is transitive by construction, between A and B, and $|A \cup R \setminus A \cap R|=1$ and $|R \cup B \setminus R \cap B|=k-1$, thus by induction $d(A, B)=d(A, R)+d(R, B)$ is determined (4).

Employing the preceeding lemmas, we are now able to show that the axioms uniquely determine the distance between any two rankings.

THEOREM 2.1: *Let A and B be any partial orderings. Then, $d(A, B)$ is determined by the axioms.*

Proof: We start with A and construct a sequence of rankings where in each step we break some tie between two objects (in A that is not in B) by placing them in relative positions as in B or not comparing them if they are not compared in B. The last ranking so constructed, A_n, has no ties unless the corresponding two objects are tied in B. Clearly each A_i, $i=1, \ldots, n$ is between A_{i-1} and B where $A_0 \equiv A$.

We apply exactly the same procedure to B and obtain a sequence of rankings:

$$B=B_0, B_1, B_2, \ldots, B_m,$$

where B_m has no ties unless the corresponding two objects are tied in A.

Consider the following sequence:

$$A \equiv A_0, A_1, A_2, \ldots, A_n, B_m, B_{m-1}, \ldots, B_1, B_0 \equiv B.$$

(4) Steps in the above proof are similar to those of Bogart [2].

Since each ranking is between its predecessor and B we have:

$$d(A, B)=\sum_{i=1}^{n} d(A_{i-1}, A_i)+d(A_n, B_m)+\sum_{j=1}^{m} d(B_{j-1}, B_j),$$

where $d(A_n, B_m)$ is determined by Lemma 2 and the other distances are determined by Axiom 5 and Lemma 1. Thus $d(A, B)$ is determined by the axioms.

3. REPRESENTATION FOR PARTIAL RANKINGS

3.1. Matrix representation

We represent a ranking A of n objects by an $n \times n$ matrix $A=\{\alpha_{ij}\}$ where:

$$\alpha_{ij}=\begin{cases} 1, & \text{if } i \text{ is preferred to } j, \\ 1/2, & \text{if } i \text{ and } j \text{ are tied,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Note:
(i) if $i > j$ then $\alpha_{ij}=1$, $\alpha_{ji}=0$;
(ii) if $i \sim j$ then $\alpha_{ij}=\alpha_{ij}=\alpha_{ji}=1/2$;
(iii) $\alpha_{ii}=1/2$ for all i.

3.2. The distance function

THEOREM 3.1: *The unique distance d which satisfies Axioms* 1-7 *is given by:*

$$d(A, B)=\sum_{ij} |\alpha_{ij}-\beta_{ij}|.$$

Proof: Axiom 1 follows immediately since the distance is an absolute value functional. Axioms 2, 4, 5 and 7 are immediately obvious as well. To see that Axiom 3 is true let (α_{ij}), (β_{ij}), (γ_{ij}) denote the matrix representations for rankings A, B, C respectively. Since

$$|\alpha_{ij}-\gamma_{ij}| \leqq |\alpha_{ij}-\beta_{ij}|+|\beta_{ij}-\gamma_{ij}|,$$

for each pair (i, j) then $d(A, C) \leqq d(A, B)+d(B, C)$. Also if $\alpha_{ij} \leqq \beta_{ij} \leqq \gamma_{ij}$ for all (i, j) then:

$$|\alpha_{ij}-\gamma_{ij}|=|\alpha_{ij}-\beta_{ij}|+|\beta_{ij}-\gamma_{ij}|$$

$$\Rightarrow \quad d(A, C)=d(A, B)+d(B, C),$$

when B lies between A and C. With regard to Axiom 6:

$$d(A,A_1)=d(\{(a>,b)\},\{\varphi\})=1=\frac{1}{2}+\frac{1}{2}=d(\{(a\sim b)\},\{\varphi\}).$$

We now show that this unique distance function is actually an extension of the distance functions derived by Bogart and Kemeny and Snell.

THEOREM 3.2: *Let A and B be complete weak orderings. Then the distance given by Theorem* 3.1 *agrees with the Kemeny-Snell distance i.e.* $d(A, B)=d_{KS}(A, B)$. See [5].

Proof: $d_{KS}(A, B)=(1/2)\sum_{ij}|a_{ij}-b_{ij}|$ where a_{ij} is the $K-S$ matrix representation. Hence:

$$d_{KS}(A, B)=\frac{1}{2}\sum_{ij}(2\,\delta_{ij}^{AB}+\tau_{ij}^{AB}),$$

where:

$$\delta_{ij}^{AB}=\begin{cases}1, & \text{if } A \text{ ranks } i\,(j) \text{ higher than } j\,(i) \text{ and } B \text{ ranks } j\,(i) \text{ higher than } i\,(j),\\ 0, & \text{otherwise.}\end{cases}$$

and

$$\tau_{ij}^{AB}=\begin{cases}1, & \text{if } A\,(B) \text{ prefers one object to the other and } B\,(A) \text{ ties them}\\ 0, & \text{otherwise.}\end{cases}$$

Thus:

$$d_{KS}(A, B)=\sum_{ij}\left(\delta_{ij}^{AB}+\frac{\tau_{ij}^{AB}}{2}\right)=\sum_{ij}|\alpha_{ij}-\beta_{ij}|=d\,(A, B).$$

THEOREM 3.3: *Let A and B be partial linear orderings. Then the distance given by Theorem* 3.1 *agrees with the Bogart distance, i.e.* $d\,(A, B)=d_B(A, B)$.

Proof: In [2] it is shown that $d_B(A, B)=I(A)-I(B)$, where:

$$I(A)_{ij}=\begin{cases}1, & \text{if } (i>j)\in A,\\ 0, & \text{otherwise.}\end{cases}$$

If A and B are partial linear orderings the only elements of the matrix not zero and not one, lie on the diagonal. Thus:

$$d\,(A, B)=\sum_{ij}|\alpha_{ij}-\beta_{ij}|=\sum_{i\neq j}|\alpha_{ij}-\beta_{ij}|+\sum_{i=j}|\alpha_{ij}-\beta_{ij}|$$

$$=\sum_{i\neq j}|\alpha_{ij}-\beta_{ij}|+0=|I(A)-I(B)|=d_B(A, B).$$

3. CONCLUSION

The importance of the ordinal ranking problem and the associated measurement of agreement or disagreement between rankings have been well established in the literature. The works of Kemeny and Snell [5] and of Bogart [2] represent an important and interesting component of the literature in this area. In contrast to previous axiomatic models, the model presented in this paper allows a comparison of linear, weak and partial orderings. In addition, the distance function, uniquely determined by a set of axioms, is in agreement with, and thus is an extension of the distance functions of the above mentioned authors.

The concept of a *consensus* has not been discussed herein, since the *median ranking* is defined in precisely the same manner as in Kemeny and Snell [5] and Bogart [2]. The problem of determining the median of a set of partial orderings is, however, problematic (even more so than for weak orderings where no efficient solution procedures exist), and will not be discussed herein.

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