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A PRIMAL-DUAL ALGORITHM FOR A CONSTRAINED FERMAT-WEBER PROBLEM INVOLVING MIXED NORMS (*)

by H. IDRISSI [1], O. LEFEBVRE [1] and C. MICHELOT [1]

Abstract. — *The aim of this paper is to present a primal-dual algorithm for finding a solution to the constrained Fermat-Weber problem involving mixed norms. Affine, polyhedral and other convex constraints are permissible. The procedure, with very simple updating rules generates sequences globally converging to an optimal solution and a dual solution. It can be viewed as a decomposition method giving the possibility to make parallel computations. Numerical results are reported.*

Keywords : Mathematical programming, Location theory, Fermat-Weber problem, Optimality conditions, Primal-dual algorithm, Partial inverse method.

Résumé. — *Nous présentons un algorithme primal-dual pour trouver la solution sous contraintes d'un problème de Fermat-Weber avec normes mixtes. Des contraintes affines et polyédriques peuvent être prises en compte ainsi que d'autres types de contraintes convexes. L'algorithme qui engendre des suites globalement convergentes vers une solution primale et une solution duale utilise des règles de mise à jour très simples. Il peut être considéré comme une méthode de décomposition permettant d'effectuer des calculs parallèles. On donne des résultats numériques.*

Mots clés : Problème d'optimisation, Problème de Fermat-Weber, Conditions d'optimalité, Algorithme primal-dual, Méthode de l'inverse partiel.

1. INTRODUCTION

The classical single facility location problem or so-called Fermat-Weber problem, has been studied extensively from both theoretical and computational points of view.

It is to find a location x of a new facility which minimizes the sum of the transportation costs, assumed to be proportional to the distance, between x and the known locations of a finite family of existing facilities.

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A lot of various resolution methods have been proposed to deal with the unconstrained problem with distances measured by l^p-norms, $1 \leq p < +\infty$, polyhedral norms and mixed norms.

However, while more realistic, the convex constrained problem of locating the new facility x in some given region only, has received much less attention. Moreover most methods are not completely satisfying and are often difficult to implement particularly due to the nondifferentiability of the problem.

Five kinds of approaches have been investigated:

– Linear programming methods for problems involving only l^1-norm and linear constraints [13, 19].

– Methods of fitted functions, with the aim to eliminate the nondifferentiability, which solve an approximated problem by standard nonlinear programming routines [7, 10].

– Dual methods solving a dual problem either by a nonlinear programming package or by a dual decomposition procedure [4, 8, 9].

– Subgradient methods [1].

– Other methods which minimize the objective function on the boundary of the set of points which are visible from the unconstrained optimal location [2, 16, 18].

These approaches do not really take into account (except l^1-norm and l^2-norm cases) the special structure of the location problem – e. g. the fact that the objective function is a sum of convex functions involving norms – and most of the time do not easily permit mixing of different kinds of norms and constraints (linear and nonlinear). Moreover some of them cannot be used in dimension $N > 2$.

The purpose of this paper is to present a primal-dual algorithm for solving a mixed norm problem with possibly different kinds of convex constraints (affine, polyhedral and other convex constraints) with simple updating rules. The procedure, providing dual variables, allows to make a sensitivity analysis from an economical point of view. It extends results of [11] to the constrained case and can be viewed as a primal-dual decomposition method implementable on a parallel computer.

To begin with, we recall the mathematical formulation of the problem which is the following:

$$\mathscr{P} \qquad \begin{aligned} &\text{Minimize} \sum_{i=1}^{n} \omega_i \gamma_i (x-a_i) \\ &x \in C = \bigcap_{j=0}^{m} C_j \end{aligned}$$

where for each $i=1, 2, \ldots, n$, $a_i \in \mathbb{R}^N$ is the known location of an existing facility, ω_i is a positive weight and γ_i is a norm.

The set of constraints is defined as an intersection of a family of closed convex sets C_j, $j=1, 2, \ldots, m$ with a linear subspace C_0 (possibly equal to the whole space $\mathbb{R}^N$).

If $m=1$ and $C_0=\mathbb{R}^N$ we obtain a problem with a single constraint (e. g. a polyhedron described by its extreme points). If $m>1$, and if each C_j is a half-space we obtain a problem with a polyhedral constraint defined by a family of linear equations and inequalities as studied in [4], [8] and [9]. When $C_0=\mathbb{R}^N$, $m=n$ and $C_j=\{x/\gamma_j(x-a_j)\leqq r_j\}$ we obtain the case studied in [2], [16] and [18]. If $n=1$, problem $\mathscr{P}$ is reduced to the classical problem of finding the projection x of a_1 (with respect to the norm γ_1) onto the convex set C, the most usual case involving the Euclidean norm.

However, it is to be noted that our formulation does not handle nonconvex constraints as considered in [3].

The paper is organized as follows:

– In section 2, we give a dual formulation of the problem and optimality conditions.

– In section 3, the primal-dual algorithm is described via the Partial Inverse Method recently introduced by Spingarn [17].

– In section 4, details about the implementation are discussed and numerical results are reported.

2. DUAL PROBLEM AND OPTIMALITY CONDITIONS

First of all, let us recall that the conjugate f^* of a closed convex function f defined on $\mathbb{R}^N$ and valued in $\overline{\mathbb{R}}$ is defined by

$$f^*(y)=\sup_x(\langle x, y\rangle - f(x)),$$

$\langle .,. \rangle$ denoting the usual inner product. Moreover the equality $f(x)+f^*(y)=\langle x, y \rangle$ holds if and only if y belongs to the subdifferential $\partial f(x)$ of f at x.

In the sequel we always assume that

$$C_0 \cap \left(\bigcap_{j=1}^{m} C_j^0 \right) \neq \varnothing$$

where C_j^0 denotes the relative interior of C_j or merely C_j if C_j is polyhedral. This constraint qualification assumption will be useful later.

Now problem $\mathscr{P}$ is equivalent to the following unconstrained optimization problem

$$\underset{x \in \mathbb{R}^N}{\text{Minimize}}\, \varphi(x) = \sum_{i=1}^{n} \omega_i \gamma_i (x - a_i) + \sum_{j=0}^{m} \chi_{C_j}(x)$$

where $\chi_{C_j}(.)$ is the indicator function of C_j defined by $\chi_{C_j}(x)=0$ if $x \in C_j$ and $\chi_{C_j}(x)=+\infty$ otherwise.

A dual problem to $\mathscr{P}$ can be obtained by different techniques but an extremely fruitful and general concept is to embed the given problem $\mathscr{P}$ in a family of convex problems $\mathscr{P}(u)$ depending on parameters whose effects on the problem are of interest

$$\mathscr{P}(u): \quad \operatorname*{Inf}_{x} \Phi(x; u)$$

where the parameter vector u ranges over a linear space U and Φ is a proper, closed convex function satisfying $\Phi(x; 0) = \varphi(x)$.

Then problem $\mathscr{P}$ can be rewritten as

$$\mathscr{P}: \quad \operatorname*{Inf}_{x} \Phi(x; 0)$$

and the dual $\mathscr{D}$ to $\mathscr{P}$ is defined by

$$\mathscr{D}: \quad \operatorname*{Sup}_{v} -\Phi^*(0; v)$$

Φ^* meaning the conjugate of Φ with respect to x and u. The following result (*see* [5] or [15]) expresses the duality and gives links between $\mathscr{P}$ and $\mathscr{D}$.

THEOREM: *Assume that $\mathcal{P}$ and $\mathcal{D}$ both have a finite solution and that there is no duality gap. Then x is a solution to $\mathcal{P}$ and v is a solution to $\mathcal{D}$ if and only if* $\Phi(x; 0) + \Phi^*(0; v) = 0$.

In our location problem it is natural to perturb the known location of existing facilities (hence the objective function) and to perturb the constraints too.

For that, let $u = (\alpha, \beta)$ be a parameter vector which lies in $U = (\mathbb{R}^N)^n \times (\mathbb{R}^N)^{m+1}$ equipped with the classical Euclidean structure and consider the function Φ:

$$\Phi(x; (\alpha, \beta)) = \sum_{i=1}^{n} \omega_i \gamma_i (x - a_i + \alpha_i) + \sum_{j=0}^{m} \chi_{C_j}(x + \beta_j)$$

with

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \qquad \alpha_i \in \mathbb{R}^N, \quad i = 1, 2, \ldots, n$$

$$\beta = (\beta_0, \beta_1, \ldots, \beta_m), \qquad \beta_j \in \mathbb{R}^N, \quad j = 0, 1, \ldots, m$$

By a straightforward calculation we obtain:

$$\Phi^*(0; (q, r)) = \begin{cases} \sum_{i=1}^{n} [\langle a_i, q_i \rangle + \omega_i \gamma_i^* (q_i/\omega_i)] + \sum_{j=1}^{m} \chi_{C_j}^*(r_j) \\ \qquad \text{if } \sum_{i=1}^{n} q_i + \sum_{j=0}^{m} r_j = 0 \quad \text{and} \quad r_0 \in C_0^{\perp} \\ +\infty \text{ otherwise.} \end{cases}$$

As we have $\gamma_i^* = \chi_{B_i^0}$, B_i^0 denoting the unit ball associated with the dual norm γ_i^0 of the norm γ_i, we obtain the dual problem

$$\mathcal{D} \qquad \begin{aligned} &\text{Maximize} - \sum_{i=1}^{n} \langle a_i, q_i \rangle - \sum_{j=1}^{m} \chi_{C_j}^*(r_j) \\ &\sum_{i=1}^{n} q_i + \sum_{j=0}^{m} r_j = 0 \\ &r_0 \in C_0^{\perp} \\ &\gamma_i^0(q_i) \leqq \omega_i, \qquad i = 1, 2, \ldots, n. \end{aligned}$$

It is very easy to see that Φ is proper, closed convex jointly in x, α and β and verifies $\Phi(x; 0) = \varphi(x)$.

As $\varphi(x)$ tends to infinity when x tends to infinity in $\mathbb{R}^N$, problem $\mathcal{P}$ has at least a finite solution x_0. Then we have $0 \in \partial\varphi(x_0)$. Moreover, thanks to the constraint qualification assumption we get (*see* [14])

$$\partial\varphi(x_0)=\sum_{i=1}^{n}\omega_i\,\partial\gamma_i(x_0-a_i)+\sum_{j=0}^{m}\partial\chi_{C_j}(x_0).$$

Therefore there exist $q\in(\mathbb{R}^N)^n$, $q_i\in\omega_i\,\partial\gamma_i(x_0-a_i)$ for $i=1,2,\ldots,n$, $r\in(\mathbb{R}^N)^{m+1}$, $r_0\in C_0^{\perp}$, $r_j\in\partial\chi_{C_j}(x_0)$ for $j=1,2,\ldots,m$, such that $\sum_{i=1}^{n}q_i+\sum_{j=0}^{m}r_j=0$. This implies that (q,r) is dual feasible. Using the properties of conjugate functions we obtain $\varphi(x_0)=-\sum_{i=1}^{n}\langle a_i,q_i\rangle-\sum_{j=1}^{m}\chi_{C_j}^{*}(r_j)$ which proves that (q,r) is a dual solution and that there is no duality gap.

Consequently x is a primal solution and (q,r) is a dual solution if and only if

$$\Phi(x;0)+\Phi^*(0;(q,r))=0$$

or in other words

$$\sum_{i=1}^{n}\omega_i\,\gamma_i(x-a_i)+\sum_{j=0}^{m}\chi_{C_j}(x)+\sum_{i=1}^{n}[\langle a_i,q_i\rangle+\omega_i\,\gamma_i^*(q_i/\omega_i)]+\sum_{j=0}^{m}\chi_{C_j}^{*}(r_j)=0$$

providing that $x\in\bigcap_{j=0}^{m}C_j$, $(q_i/\omega_i)\in B_i^0$ for each i, $r_0\in C_0^{\perp}$, and $\sum_{i=1}^{n}q_i+\sum_{j=0}^{m}r_j=0$ which assures that all the terms in the preceding equation are finite.

Using the last equality and rearranging the terms in the summations we obtain

$$\sum_{i=1}^{n}\omega_i\,[\gamma_i(x-a_i)+\gamma_i^*(q_i/\omega_i)-\langle(q_i/\omega_i),x-a_i\rangle]$$

$$+\sum_{j=0}^{m}[\chi_{C_j}(x)+\chi_{C_j}^{*}(r_j)-\langle x,r_j\rangle]=0.$$

Observing that $\partial\chi_{C_j}(\cdot)=N_{C_j}(\cdot)$, where $N_{C_j}(x)$ denotes the normal cone to C_j at x, the optimality conditions become

$$x \in \bigcap_{j=0}^{m} C_j$$

$$q_i \in \omega_i \, \partial\gamma_i(x-a_i), \qquad i=1,2,\ldots,n$$

$$r_j \in N_{C_j}(x), \qquad j=1,2,\ldots,m$$

$$\sum_{i=1}^{n} q_i + \sum_{j=1}^{m} r_j \in C_0^{\perp}.$$

3. PRIMAL-DUAL ALGORITHM

In this section our aim is to give an algorithm based on duality. To this end, let us introduce the space $H=(\mathbb{R}^N)^{n+m}$ whose elements are denoted by $\hat{x}=(x_l)$, $x_l \in \mathbb{R}^N$, equipped with the scalar product $\langle \hat{x}, \hat{y} \rangle = \sum_{l=1}^{n+m} \langle x_l, y_l \rangle$. The optimality conditions previously given can be written as:

$$\text{find } \hat{x} \in A, \qquad \hat{p} \in B, \qquad \hat{p} \in T(\hat{x})$$

where

$$A=\{\hat{x} \in H, \hat{x}=(x, x, \ldots, x), x \in C_0\}$$

$$B=\left\{\hat{p} \in H, \sum_{l=1}^{n+m} p_l \in C_0^{\perp}\right\}$$

$$T=\prod_{l=1}^{n+m} T_l$$

$$T_i(\hat{x})=\omega_i \, \partial\gamma_i(x_i-a_i), \qquad i=1,2,\ldots,n$$

$$T_{n+j}(\hat{x})=N_{C_j}(x_{n+j}), \qquad j=1,2,\ldots,m.$$

We can easily see that T, defined on H, is a maximal monotone multifunction and that A and B so defined are two complementary subspaces of H.

Now from Spingarn [17], these optimality conditions are equivalent to

$$\text{find } \hat{x} \in A, \qquad \hat{p} \in B, \qquad 0 \in T_A(\hat{x}+\hat{p})$$

where T_A is the partial inverse of T with respect to A.

T_A is a (maximal monotone) multifunction defined by its graph Γ in the following way:

$$\Gamma(T_A)=\{(\hat{x}_A+\hat{p}_B, \hat{p}_A+\hat{x}_B), \hat{p}\in T(\hat{x})\}$$

$\hat{z}_A$ and $\hat{z}_B$ denoting the orthogonal projection of $\hat{z}$ onto A and B respectively.

This means that if $\hat{z}$ is a zero of T_A [i. e. satisfies $0\in T_A(\hat{z})$] then the pair $(\hat{z}_A, \hat{z}_B)$ satisfies the optimality conditions and $\hat{z}_A$ (resp. $\hat{z}_B$) is a solution to $\mathscr{P}$ (resp. to $\mathscr{D}$). Such a zero of T_A can be approached by making use of the proximal point algorithm which generates from any starting point $\hat{z}_0$ the sequence

$$\hat{z}^{k+1}=(I+T_A)^{-1}(\hat{z}^k) \tag{3.1}$$

The projections $\hat{x}^k=\hat{z}_A^k$ and $\hat{p}^k=\hat{z}_B^k$ of $\hat{z}^k$ onto A and B give two sequences $\{\hat{x}^k\}$ and $\{\hat{p}^k\}$ which converge to a primal and a dual solution respectively. In fact, the main problem as underlined in [16] is to know if this method is implementable. To see that, we need to express iteration (3.1) in terms of the multifunction T rather than T_A.

The proximal iteration (3.1) can be rewritten as

$$\hat{x}^{k+1}+\hat{p}^{k+1}=(I+T_A)^{-1}\quad(\hat{x}^k+\hat{p}^k)$$

which is equivalent to

$$\hat{x}^k-\hat{x}^{k+1}+\hat{p}^k-\hat{p}^{k+1}\in T_A(\hat{x}^{k+1}+\hat{p}^{k+1}).$$

Put

$$\hat{p}'^k=\hat{x}^k-\hat{x}^{k+1}+\hat{p}^{k+1}$$
$$\hat{x}'^k=\hat{p}^k-\hat{p}^{k+1}+\hat{x}^{k+1}.$$

It follows that

$$\hat{x}^k+\hat{p}^k=\hat{x}'^k+\hat{p}'^k$$
$$\hat{x}^{k+1}=(\hat{x}'^k)_A \qquad \text{and} \qquad \hat{p}^{k+1}=(\hat{p}'^k)_B.$$

By the definition of T_A, we get

$$\hat{p}'^k\in T(\hat{x}'^k)$$

meaning that $\hat{x}'^k$ is the image of $\hat{z}^k=\hat{x}'^k+\hat{p}'^k$ under the proximal mapping associated with T or equivalently that $\hat{p}'^k$ is the image of $\hat{z}^k$ under the proximal mapping associated with T^{-1}.

Now, examining the definition of T, the relation $\hat{p}'^k \in T(\hat{x}'^k)$ is equivalent to:

$$p_i'^k \in \omega_i \partial\gamma_i(x_i'^k - a_i), \qquad i=1, 2, \ldots, n$$
$$p_{n+j}'^k \in N_{C_j}(x_{n+j}'^k), \qquad j=1, 2, \ldots, m$$

and we have

$$x^k - a_i + p_i^k - p_i'^k \in N_{B_i^0}(p_i'^k/\omega_i), \qquad i=1, 2, \ldots, n$$
$$x^k + p_{n+j}^k - x_{n+j}'^k \in N_{C_j}(x_{n+j}'^k), \qquad j=1, 2, \ldots, m.$$

Then on one hand, for $i=1, 2, \ldots, n$, $p_i'^k$ is uniquely determined by

$$p_i'^k = \omega_i \operatorname{Proj}_{B_i^0}[(x^k - a_i + p_i^k)/\omega_i]$$

and we get the corresponding $x_i'^k$ by $x_i'^k = x^k + p_i^k - p_i'^k$.

On the other hand, for $j=1, 2, \ldots, m$, $x_{n+j}'^k$ is uniquely determined by

$$x_{n+j}'^k = \operatorname{Proj}_{C_j}(x^k + p_{n+j}^k)$$

and we get the corresponding $p_{n+j}'^k$ by $p_{n+j}'^k = x^k + p_{n+j}^k - x_{n+j}'^k$.

To summarize, the algorithm for solving the constrained problem is the following:

Starting point: choose

. $x^0 \in C_0$,

. $\hat{p}^0 \in H$ such that $\sum_{l=1}^{n+m} p_l^0 \in C_0^\perp$.

Step k: compute for $i=1, 2, \ldots, n$

$$. \; p_i'^k = \omega_i \operatorname{Proj}_{B_i^0}[(x^k - a_i + p_i^k)/\omega_i]$$
$$. \; x_i'^k = x^k + p_i^k - p_i'^k$$

compute for $j=1, 2, \ldots, m$

$$. \; x_{n+j}'^k = \operatorname{Proj}_{C_j}(x^k + p_{n+j}^k)$$
$$. \; p_{n+j}'^k = x^k + p_{n+j}^k - x_{n+j}'^k$$

and update

$$x^{k+1}=\mathrm{Proj}_{C_0}\left[\left(\sum_{t=1}^{n+m} x_t'^k\right)\Big/(n+m)\right]$$

$$p_l^{k+1}=p_l'^k-\mathrm{Proj}_{C_0}\left[\left(\sum_{t=1}^{n+m} p_t'^k\right)\Big/(n+m)\right], \qquad l=1,2,\ldots,n+m.$$

4. IMPLEMENTATION AND NUMERICAL RESULTS

First, let us remark that x^{k+1} and $\hat{p}^{k+1}$ can be determined without computing $x_i'^k$, $i=1,2,\ldots,n$ and $p_{n+j}'^k$, $j=1,2,\ldots,m$. Indeed, since $\hat{p}^k+\hat{x}^k=\hat{p}'^k+\hat{x}'^k$, we have $(\hat{x}'^k)_A=\hat{x}^k-(\hat{p}'^k)_A$ and then

$$x^{k+1}=x^k-\delta^k$$

$$p_i^{k+1}=p_i'^k-\delta^k, \qquad i=1,2,\ldots,n$$

$$p_{n+j}^{k+1}=p_{n+j}^k+x^k-x_{n+j}'^k-\delta^k, \qquad j=1,2,\ldots,m$$

with

$$\delta^k=\mathrm{Proj}_{C_0}\left[\left(\sum_{i=1}^{n} p_i'^k+mx^k+\sum_{j=1}^{m}(p_{n+j}^k-x_{n+j}'^k)\right)\Big/(n+m)\right].$$

This remark is significant when large size problems for which the dual variables are numerous, must be solved using a micro-computer with limited memory.

Another crucial point is the problem of computing the projections onto the balls B_i^0 and onto the convex sets C_j. While any convex constraint can theoretically be handled, in practice C_j is often either a polyhedron, possibly a halfspace, or a ball associated with the Euclidean norm (*see* for instance [4, 16]). In dimension two, specific routines can be used for finding the nearest point of a polyhedron. Concerning the projections onto B_i^0 we refer the reader to [11]. In dimension $N>2$, a very efficient procedure which has been given by Mifflin [12] can be used.

Finally, as in the unconstrained case [11], our method provides a rule to stop the iterations via lower and upper bounds of the optimal value of the objective function φ. As an illustration we shall study three types of constraints.

In the following and according to the notation of section 2, the vector of dual variables $\hat{p}^k$ will be decomposed into two vectors $q^k \in (\mathbb{R}^N)^n$ and $r^k \in (\mathbb{R}^N)^m$ of dual variables associated with facilities and constraints respectively.

4.1. Affine constraints

Consider the case $m=0$ and $C_0=\{x \in \mathbb{R}^N, Mx=b\}$ where M is an $s \times N$ matrix with full row rank s. Owing to a change of variables we can suppose without loss of generality that $b=0$. Then the dual problem $\mathcal{D}$ becomes

$$\mathcal{D} \quad \begin{array}{l} \text{Maximize} - \sum_{i=1}^{n} \langle a_i, q_i \rangle \\ \sum_{i=1}^{n} q_i \in C_0^{\perp} \\ \gamma_i^0(q_i) \leqq \omega_i, \qquad i=1, 2, \ldots, n. \end{array}$$

Applied to this particular case, the algorithm generates a primal feasible sequence $\{x^k\}$ and a dual infeasible sequence $\{q^k\}$.

Consequently a (converging) upper bound M^k of the optimal value is straightforwardly given by

$$M^k = \sum_{i=1}^{n} \omega_i \gamma_i (x^k - a_i).$$

Put

$$\alpha^k = \operatorname{Max}\{\gamma_i^0(q_i^k)/\omega_i, i=1, 2, \ldots, n\}.$$

Without restriction, we can suppose that $\alpha_k > 0$. Then the "modified" dual variables q_i^k/α^k are dual feasible and a (converging) lower bound of the optimal value is given by

$$m^k = -\left(\sum_{i=1}^{n} \langle a_i, q_i^k \rangle\right) \Big/ \alpha^k.$$

This allows us to get a rational rule to stop the algorithm with little computational effort.

4.2. Polyhedral constraints defined by extreme points

Consider the case for which $m=1$, $C_0=\mathbb{R}^N$ and C_1 is a (bounded) polyhedron whose set of extreme points denoted by $\operatorname{Ext}(C_1)$ is supposed to be known.

It may be seen that the conjugate of the indicator function of C_1 is given by

$$\chi_{C_1}^*(r)=\operatorname{Max}\{\langle r,e\rangle, e\in\operatorname{Ext}(C_1)\}.$$

Hence the dual $\mathscr{D}$ becomes

$$\mathscr{D}\quad\begin{cases}\text{Maximize } -\sum_{i=1}^{n}\langle a_i, q_i\rangle-\operatorname{Max}\{\langle r,e\rangle, e\in\operatorname{Ext}(C_1)\}\\ \sum_{i=1}^{n} q_i+r=0\\ \gamma_i^0(q_i)\leqq\omega_i, \qquad i=1,2,\ldots,n.\end{cases}$$

In this case, while the algorithm is neither primal feasible nor dual feasible, a stopping rule can be defined by making use of a lower bound m^k and an upper bound M^k of the optimal value in the following way.

Let $\alpha^k=\operatorname{Max}\{\gamma_i^0(q_i^k)/\omega_i,\ i=1,2,\ldots,n\}$

$$y^k=\operatorname{Proj}_{C_1}(x^k).$$

We define m^k and M^k by

$$M^k=\sum_{i=1}^{n}\omega_i\gamma_i(y^k-a_i),$$

$$m^k=-\left(\sum_{i=1}^{n}\langle a_i, q_i^k\rangle+\operatorname{Max}\{\langle r,e\rangle, e\in\operatorname{Ext}(C_1)\}\right)\Big/\alpha^k.$$

4.3. Polyhedral constraints defined by inequalities

We consider now the case $m>1$, $C_0=\mathbb{R}^N$ and $C_j=\{x, \langle x,u_j\rangle\leqq b_j\}$, $j=1,2,\ldots,m$, where u_j is a vector of $\mathbb{R}^N$, $u_j\neq 0$ and $b_j\in\mathbb{R}$.

As the conjugate $\chi_{C_j}^*$ of the indicator function of C_j is given by

$$\chi_{C_j}^*(r_j)=\begin{cases}\lambda_j b_j & \text{if } r_j=\lambda_j u_j, \quad \lambda_j\geqq 0\\ +\infty & \text{otherwise}\end{cases}$$

the dual problem $\mathscr{D}$ becomes

$$\mathscr{D} \qquad \begin{aligned} &\text{Maximize } -\sum_{i=1}^{n} \langle a_i, q_i \rangle - \sum_{j=1}^{m} \lambda_j b_j \\ &\sum_{i=1}^{n} q_i + \sum_{j=1}^{m} \lambda_j u_j = 0 \\ &\gamma_i^0(q_i) \leqq \omega_i, \qquad i=1, 2, \ldots, n \\ &\lambda_j \geqq 0, \qquad j=1, 2, \ldots, m. \end{aligned}$$

Applied with such constraints the algorithm is again primal infeasible and dual infeasible. A lower bound of the optimal value is straightforwardly given by

$$m^k = -\left(\sum_{i=1}^{n} \langle a_i, q_i^k \rangle + \sum_{j=1}^{m} \lambda_j b_j\right) \Big/ \alpha^k$$

with $\alpha^k = \text{Max}\{\gamma_i^0(q_i^k)/\omega_i,\ i=1, 2, \ldots, n\}$.

An upper bound could also be obtained by projection of the sequence $\{x^k\}$ onto $C = \bigcap_{j=1}^{m} C_j$ in the same way as in the previous case. However, from a practical point of view, the choice to represent a polyhedron by inequalities rather than by its extreme points is often made to avoid computing the projections onto C (e. g. in dimension $N>2$). In such a case, it would be better to generate an upper bound differently. For instance if one has at hand a point x_0 satisfying $\langle x_0, u_j \rangle < b_j$ for all j, we can explicitly obtain the unique point y^k of the boundary of C which belongs to the segment of line joining x_0 and x^k, and evaluate the objective function at y^k.

4.4. Numerical results

The algorithm was programmed in FORTRAN and has been implemented using a MATRA 550-CX computer at the Dijon University Computing Center. It was tested on several problems in the plane. Some results are summarized in Tables I to III. The times indicated in seconds do not include input/output times. Even when lower and upper bounds, as studied in the previous section, were available the stopping rule used was

$$|||\hat{z}^{k+1} - \hat{z}^k||| < 10^{-4}$$

with $|||\hat{z}||| = \max_{l=1,\ldots,n+m} \|z_l\|$, $\|.\|$ denoting the Euclidean norm in $\mathbb{R}^2$.

The tests concerning a family of problems (numbered from 1 to 16) involving the Euclidean norm proposed by Schaefer and Hurter [16] and Watson-Gandy [18] are reported in Table I. The results are also compared with those obtained by Hansen, Peeters and Thisse [2] for five of these problems. Even taking into account the difference of computer our results are always efficient while generating in addition the dual variables.

In Table II we give tests concerning the following problem involving three polyhedral norms and polyhedral constraints. We consider the set $\{a_1, a_2, \ldots, a_6\}$ with $a_1=(0,1)$, $a_2=(0,2)$, $a_3=(2,3)$, $a_4=(2,0)$, $a_5=(3,2)$, $a_6=(3,1)$ and the weights $\omega_1=1+\sqrt{2}$, $\omega_2=\omega_3=\ldots=\omega_6=1$. The distance between a point x and facility a_i is measured by an octagonal norm (the uunit ball of which is a regular octagon inscribed in a circle with radius one) for $i=1,2$, by the norm $\gamma(x)=(1/2)\ |x^1|+|x^2|$ for $i=3,4$, and by the l^1-norm for $i=5,6$.

Constraint C imposed on the solution is the intersection of five half-planes defined (using notations of section 4.3) by $u_1=(1,-2)$, $b_1=-3$, $u_2=(+1,-1)$, $b_2=-1$, $u_3=(1,1)$, $b_3=5$, $u_4=(0,1)$, $b_4=7/2$, $u_5=(-1,0)$, $b_5=0$. This set is a polyhedron whose set of extreme points is $\mathrm{Ext}(C)=\{e_1, e_2, \ldots, e_5\}$ with $e_1=(0,3/2)$, $e_2=(1,2)$, $e_3=(2,3)$, $e_4=(3/2, 7/2)$, $e_5=(0,7/2)$.

The set $S_{\mathcal{P}}$ of optimal locations is a segment of line joining e_2 and the point with coordinates (1/3, 5/3). If C is considered as a single constraint represented by $\mathrm{Ext}(C)$ it can be easily verified that the optimal dual solution is made up of the dual variables $q_1=(1, \sqrt{2}+1)$, $q_2=(1, 1-\sqrt{2})$, $q_3=(-1/2, -1)$, $q_4=(-1/2, 1)$, $q_5=(-1,-1)$, $q_6=(-1,+1)$ associated with a_1, a_2, a_3, a_4, a_5 and a_6 respectively, and of the dual variable $r=(1,-2)$ associated with C. It is interesting to note that the optimal location obtained always belongs to the relative interior of $S_{\mathcal{P}}$.

The difference of behavior of the algorithm when constraints are represented in different ways is also and more especially illustrated by the following example. We consider the set $\{a_1, a_2, a_3\}$ with $a_1=(-2,2)$, $a_2=(-3,-1)$, $a_3=(0,0)$, the weights $\omega_1=\omega_2=\omega_3=1$ and as constraints the segment of line joining the points $e_1=(2,2)$ and $e_2=(-3,-3)$. The set $S_{\mathcal{P}}$ of optimal locations is the segment of line $[x_1^*, x_2^*]$ with $x_1^*=(0,0)$ and $x_2^*=(-1,-1)$.

The algorithm has been tested with the three following possible representations of the constraints.

TABLE I

Problem No.	n	Optimal Value	Optimal Location	CPU time (in sec.)			
				Prox	W−G	H−P−T	S−H
1	3	5.4019	(0.8660, 0.4993)	0.06	0.093	0.15	8.6
2	3	6.3949	(0.8348, 0.4489)	0.04	0.157	0.31	8.9
3	4	12.8984	(0.7640, 1.2139)	0.10	0.123	0.11	10.8
4	4	6.0864	(0.5503, 0.5000)	0.08	0.270	–	–
5	5	8.6004	(0.8574, 0.5139)	0.10	0.146	–	–
6	5	8.5528	(0.9300, 0.6337)	0.10	0.175	–	–
7	8	18.3344	(1.1714, −0.0001)	0.18	0.249	0.14	17.9
8	10	130.7435	(6.9613, 4.9030)	0.24	0.273	–	–
9	10	174.7036	(5.9691, 3.9414)	0.28	0.233	–	–
10	15	254.7334	(6.9611, 4.9029)	0.42	0.542	–	–
11	15	200.5979	(6.9610, 4.9029)	0.42	0.478	–	–
12	15	260.7525	(6.5406, 4.9529)	0.50	0.402	–	–
13	15	239.5073	(6.9073, 4.0937)	0.50	0.575	–	–
14	20	324.5954	(6.9612, 4.9029)	0.70	0.565	–	–
15	20	32.7745	(2.9309, 5.5709)	0.54	0.654	–	–
16	25	98.7151	(5.4542, 6.1593)	1.82	0.741	–	–

Prox: results of the primal-dual algorithm with a MATRA 550-CX.
W−G: results of Watson-Gandy's algorithm with a CDC 6500.
H−P−T: results of Hansen, Peeters and Thisse's algorithm with a IBM 370/158.
S−H: results of Schaefer and Hurter's algorithm with a CDC 6400.
–: not available results.

TABLE II

	Constraints with inequalities	Constraints with extreme points
Number iterations. . .	206	131
CPU time (sec.). . . .	1.8	1.18
Optimal value.	13.4142	13.4142
Optimal location. . . .	$x=(0.6707,\ 1.8353)$	$x=(0.5301,\ 1.7650)$
Dual variables.	$q_1=(\ 1.0000,\ \ 2.4142)$ $q_2=(\ 1.0000,\ -0.4142)$ $q_3=(-0.5000,\ -0.9999)$ $q_4=(-0.5000,\ \ 1.0000)$ $q_5=(-1.0000,\ -0.9999)$ $q_6=(-1.0000,\ \ 1.0000)$ $r_1=(\ 1.0000,\ -2.0000)$ $r_2=(-2.10^{-6},\ 5.10^{-6})$ $r_3=(-2.10^{-6},\ 5.10^{-6})$ $r_4=(-2.10^{-6},\ 5.10^{-6})$ $r_5=(-2.10^{-6},\ 5.10^{-6})$	$q_1=(\ 1.0000,\ \ 2.4142)$ $q_2=(\ 1.0000,\ -0.4142)$ $q_3=(-0.5000,\ -1.0000)$ $q_4=(-0.5000,\ \ 1.0000)$ $q_5=(-1.0000,\ -1.0000)$ $q_6=(-1.0000,\ \ 1.0000)$ $r=(\ 1.0000,\ -2.0000)$

– Representation 1:

$$m=1, C_0=\mathbb{R}^2;$$
$$C_1=\{x, x=\lambda e_1+(1-\lambda)e_2, \lambda\in[0, 1]\}.$$

– Representation 2:

$$m=4, C_0=\mathbb{R}^2;$$
$$C_1=\{x, x^1-x^2\leqq 0\};$$
$$C_2=\{x, -x^1+x^2\leqq 0\};$$
$$C_3=\{x, x^1\leqq 2\};$$
$$C_4=\{x, -x^1-x^2\leqq 6\}.$$

– Representation 3:

$$m=2, C_0=\{x, x^1-x^2=0\}$$
$$C_1=\{x, x^1\leqq 2\}$$
$$C_2=\{x, -x^1-x^2\leqq 6\}.$$

Results are presented in Table III. It is worth noting that with the representation 3, the location problem is solved very efficiently. This is due to the fact that in this case the convergence of the algorithm has been proved to be finite [6]. Indeed the interior, with respect to the subspace C_0, of $S_{\mathscr{P}}$ is non empty and the dual solution is unique, given by $q_1=(1, -1)$, $q_2=(1, 1)$,

TABLE III

Constraints	Representation 1	Representation 2	Representation 3
Number of iterations. . . .	54	51	15
CPU time (sec.).	0.14	0.16	0.06
Optimal location.	$x=(-0.50411, -0.50418)$	$(-0.58803, -0.58812)$	$(-0.53694, -0.53694)$
Optimal value. . .	8.00	8.00	8.00
Dual variables. .	$q_1=(0.9998, -0.9998)$ $q_2=(0.9998, 1.0000)$ $q_3=(-1.0000, -0.9998)$ $r=(-0.9994, 0.9994)$	$q_1=(1.0000, -1.0000)$ $q_2=(1.0000, 0.9998)$ $q_3\doteq(-0.9998, -1.0000)$ $r_1=(4.9408, -4.9408)$ $r_2=(-5.9416, 5.9416)$ $r_3=(1.10^{-4}, -1.10^{-4})$ $r_4=(1.10^{-4}, -1.10^{-4})$	$q_1=(1., -1.)$ $q_2=(1., 1.)$ $q_3=(-1., -1.)$ $r_1=(0., 0.)$ $r_2=(0., 0.)$

$q_3=(-1, -1)$, $r_1=r_2=(0, 0)$. Consequently the main Theorem of [6] works and the algorithm has finite termination. Note that again the optimal location found lies in the relative interior of $S_{\mathscr{P}}$.

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