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USING LINEAR PROGRAMMING IN PETRI NET ANALYSIS (*)

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Abstract. — *The algebraic representation of polyhedral sets is an alternative tool for the analysis of structural and local properties of Petri nets. Some aspects of the issues of reachability, boundedness and liveness of a net are analyzed and characterized by the means of linear system of inequalities. In most cases, linear programming instead of integer programming can be used to check these properties therefore enabling the validation of very large nets.*

Keywords : Petri nets; linear programming.

Résumé. — *On utilise des techniques simples d'Algèbre linéaire pour l'analyse de certaines propriétés des réseaux de Pétri. On montre que ces propriétés (invariants, réseau borné, réseau vivant, places implicites) peuvent être analysées en utilisant la Programmation Linéaire (de complexité polynomiale) et non la Programmation en nombres entiers. Pour les propriétés citées, le calcul (en général prohibitif) d'une base d'invariants n'est jamais nécessaire. La taille du réseau n'est donc pas un obstacle pour l'analyse de ces propriétés.*

Mots clés : Réseaux de Pétri; programmation linéaire.

I. INTRODUCTION

Petri nets have become a widely used tool for modelling and analysing large, complex and discrete event systems. They appear in computer science as well as in operations research for modelling a quite large number of problems such as information processing, communication network design, scheduling and control of manufacturing processes. Here we focus on some aspects of the analysis of Petri nets. An important issue is for example to know whether it is able to realize and complete all the tasks for which it has been designed. Among the related properties which are commonly analyzed are the boundedness and the liveness of the net in order to certify that neither traps nor deadlocks may occur. These two properties are linked to the issue

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of reachability, which in turn is one of the key concept in mathematical system theory. In that sense it is quite natural to view the evolution of the marking on the places of the graph as a linear time invariant discrete system. If M_k is the vector of marks on the space of places of the net at time k, s_k is a control vector of firings on the space of transitions and C is the incidence matrix of the graph, then the dynamic state equation is:

$$M_{k+1}=M_k+C\,s_k$$

In fact, linear system theory is of limited help when dealing with Petri nets. Some partial characterization of controllability and reachability are given in Murata [6] but these results are not sufficient and cannot be exploited practically. On the other hand, linear algebra and in particular some duality results for systems of linear inequalities gave rise to a few interesting propositions about boundedness, liveness and reduction techniques (*cf.* [2, 5]). This approach constitutes a valid alternative to the classical analysis where we must build the reachability tree, *i. e.* the tree of all feasible firing sequences from a given initial state. However the computational cost remains high because the conditions have to be tested on the set of integers [8]. Peterson [7] has mentionned the possibility to relax the integrality condition and to use linear programming for the detection of semi-flows in the net. We now extend this statement, proving its validity for the conditions of boundedness, liveness and for reduction techniques. Duality is then used to yield some insights on the geometry of the reachable set.

II. LINEAR ALGEBRA APPROACH

2.1. Notation and key lemma

We use the same notation as in Brams [2].

A Petri net is a four-tuple $R=\langle P, T, \text{Pré}, \text{Post}\rangle$ where:

P is a finite set of places with $|P|=m$

T is a finite set of transitions with $|T|=n$,

Pré: $P\times T\rightarrow N$ is the forward incidence mapping

Post: $P\times T\rightarrow N$ is the backward incidence mapping.

A marked net is a couple $\langle R; M\rangle$ where R is a Petri Net and $M: P\rightarrow N$ is a marking mapping. We denote $M(p)$ the marking of place $p\in P$, which is also the number of tokens available on place p.

The incidence matrix C is defined as follows:

$$\forall (p, t) \in P \times T, \quad C(p, t) = \text{Post}(p, t) - \text{Pré}(p, t)$$

If M' is an accessible marking from M, then we have the fundamental relationship:

$$M' = M + Cs \tag{1}$$

where $s(t)$ is the number of times transition t has been fired. Petri net analysis using linear algebra is based on the above equation.

In the following, we consider some weighting or valuation function $f: P \to N$. In fact, it can be viewed as a linear functional defined on the primal space of marking vectors, N^m. Then f is a dual vector associated to the primal equation (1).

Before proceeding further, we need the two following lemma:

LEMMA 1: *Let C be any matrix with coefficients in Z. Then we have:*

$$\Omega_0 = \{f: f \in N^m,\ C^T f = 0\} \neq \{0\}$$
$$\Leftrightarrow \quad \Omega'_0 = \{f: f \in R^m, f \geqq 0,\ C^T f = 0\} \neq \{0\}$$

Proof:

$\Rightarrow$ is trivial.

$\Leftarrow$ Ω'_0 is a closed polyhedral convex cone. Hence,

$$\forall f \in \Omega'_0, \quad f = \sum_{i=1}^{l} \lambda_i f^i,\ \lambda_i \geqq 0 \qquad \text{for} \quad i = 1, \ldots, l$$

where $\{f^i\}_{i=1}^{l}$ is a set of generators of the cone.

A set of generators can be found by solving all homogeneous linear systems of $m-1$ linearly independent equations with m unknowns built from the rows of $\begin{bmatrix} C^T \\ I \end{bmatrix}$ (*cf.* [3, 9]). Consequently, as C and I have integer coefficients, it is always possible to compute solutions in Q^m (elementary pivoting operations with integer coefficients). Multiplying the rational values by a common denominator, we can then always obtain f^i in N^m. Hence $f^i \in \Omega_0$ and $\Omega_0 \neq \{0\}$.

LEMMA 2: *Let C be any matrix with coefficients in Z. Then we have:*

$$\Omega_1 = \{f: f \in N^m, f > 0,\ C^T f \leqq 0\} \neq \varnothing$$
$$\Leftrightarrow \quad \Omega'_1 = \{f: f \in R^m, f \geqq e,\ C^T f \leqq 0\} \neq \varnothing$$

where e is an m-vector of ones.

Proof:

$\Rightarrow$ is trivial.

$\Leftarrow$ $\Omega_1' \neq \varnothing$. Then Ω_1' is an unbounded convex polyhedron ($f \in \Omega_1'$, then $rf \in \Omega_1'$, $\forall r \geqq 1$). Hence, any vector f in Ω_1' can be written:

$$f = \sum_{i=1}^{l} \lambda_i f^i + \sum_{j=1}^{l'} r_j f'^j$$

with

$$\lambda_i \geqq 0, \; i=1, \ldots, l, \qquad \sum_{i=1}^{l} \lambda_i = 1$$

$$r_j \geqq 0, \qquad j=1, \ldots, l'$$

f^i, $i=1, \ldots, l$, are the extreme points of Ω_1'.

f'^j, $j=1, \ldots, l'$, are the extreme rays of Ω_1'.

Again by standard arguments on the computation of vertices and extreme rays of a polyhedron in R^n given by a set of inequalities with rational coefficients, we obtain that f^i and $f'^j \in Q^m$, $\forall i, j$.

We can then take any f^i and multiply it by the common denominator $\lambda_0 > 0$ to get $\lambda_0 f^i \in \Omega_1$ (in fact, $\lambda_0 > 1$, then $\lambda_0 f^i \in \Omega_1'$). Hence, $\Omega_1 \neq \varnothing$.

Remarks: (i) these results still hold if the matrix C has coefficients in Q.

(ii) to check if Ω_0 (respectively Ω_1) is empty one only need to check if Ω_0' (resp. Ω_1') is empty. This can be done by using Linear Programming techniques which can handle very large size problems.

2.2. Boundedness

Boundedness is an important desirable property for Petri nets. It means that the number of tokens in every place is bounded whatever happens. We know (*see* [2] for example) that:

$$R \text{ is bounded} \quad \Leftrightarrow \quad \exists f \in N^m, f > 0, \; C^T f \leqq 0$$

If such an f exists, then we have:

$$M(p) \leqq \frac{f^T M_0}{f(p)}, \qquad \forall p \in P$$

In view of lemma 2, we now have:

PROPOSITION 1:

$$R \text{ is bounded} \iff \exists f \in R^m,\ f \geqq e,\ C^T f \leqq 0.$$

COROLLARY: *A bound for the number of tokens in place p* is given by.

$$\text{Min } M_0^T f \quad \left| \begin{array}{l} C^T f \leqq 0 \\ f \geqq e \\ f(p) = 1 \end{array} \right.$$

which is a linear program.

2.3. p-semi-flows and liveness

p-semi-flows are also important in the analysis of Petri nets [5]. The set of semi-flows on a net R is precisely the set Ω_0 defined in Lemma 1. The purpose of this section is to illustrate the fact that any homogeneous linear relation that must be satisfied on the set of semi-flows can be tested equivalently on the set Ω_0'. Indeed, this is again because we are able to find a set of integer-valued generators for the cone Ω_0'.

For instance, the invariance property of semi-flows is valid on Ω_0': for any accessible marking M, we have:

$$\forall f \in \Omega_0', \quad f^T M = f^T M_0$$

and if $f(p) \neq 0$, $M(p) \leqq M_0^T f / f(p)$, $\forall f \in \Omega_0'$.

A bound on $M(p)$ can therefore be computed by solving the linear program:

$$\text{Min } M_0^T f \quad \left| \begin{array}{l} C^T f = 0 \\ f(p) = 1 \\ f \geqq 0 \end{array} \right.$$

A necessary condition for liveness can be simplified by rising the same argument:

PROPOSITION 2: *If* $\langle R;\ M \rangle$ *is live, then:*

$$\forall f \in \Omega_0', \quad f^T M \geqq f^T \text{pré}(.,t), \quad \forall t = 1, \ldots, n$$

To check this condition, we only need solve the n linear programs:

$$\left|\begin{array}{ll}\operatorname{Min} f^T (M - \text{pré}(., t)), & t=1, \ldots, n \\ C^T f = 0 & \\ f \geqq 0 & \end{array}\right.$$

Hence, if $\langle R; M \rangle$ is live, the optimal value of the above linear program is equal to zero for any t.

2.4. Reduction techniques

Reduction techniques are used to simplify large scale Petri nets into "equivalent" smaller nets. Simplification of implicit places is such a technique (*see* [1]).

p is an implicit place iff there exists $\bar{f}: p \to Z$ such that:

- $\bar{f}(p) > 0$ and $\bar{f}(q) \leqq 0, \quad \forall q \neq p.$
- $\forall M_0 \in \bar{M}_0, \bar{f}^T M_0 \geqq 0$
- $\forall t \in T, \bar{f}^T \text{Pré}(., t) \leqq \operatorname{Min}\{\bar{f}^T M_0, M_0 \in \bar{M}_0\}$
- $\exists c \in N^m, f^T C = c$

where $\bar{M}_0$ is a set of initial markings.

Again, it suffices to check that the system below has a solution:

$$\begin{array}{rl} f \in R^m \quad \text{and} & C^T f \geqq 0 \\ f(p) = 1 & \\ f(q) \leqq 0, & \forall q \neq p \\ f^T M_0 \geqq 0, & \forall M_0 \in \bar{M}_0 \\ f^T \text{Pré}(., t) \leqq f^T M_0, & \forall M_0 \in \bar{M}_0, \quad \forall t \in T \end{array}$$

This can be done by standard Linear Programming.

III. A BOUND ON THE REACHABLE SET

In this section, we give an analytical expression of a set which always contains the reachable set after K iterations have been fired (for any K). When the reachable set is bounded, this set is also bounded and we retrieve the bound given in the previous section. When the reachable set is not bounded, it allows one to give bounds on the places after K transitions have been fired.

Let M be an accessible marking from M_0 after K transitions. Then, there exists $s \in N^m$ such that, $M - M_0 = Cs$, $e^T s = K$

Now, let $\Omega(K)$ be the convex set defined by:

$$\Omega(K) = \{M \in R^m;\ M \geqq 0, \text{ there exist } s \in R^n \text{ such that } Cs = M - M_0,\ e^T s = K,\ s \geqq 0\}$$

Introducing dual variables $f \in R^m$ and $v \in R$, we characterize $\Omega(K)$ by its bounding hyperplanes:

$$\Omega(K) = \{M \in R^m;\ M \geqq 0,\ f^{iT}(M - M_0) + K v^i \geqq 0,\ i = 1, \ldots, r\}$$

where (f^i, v^i), $i = 1, \ldots, r$ are the extreme rays of the polyhedral cone:

$$\mathscr{E} = \left\{ \begin{pmatrix} f \\ v \end{pmatrix} \in R^{m+1};\ C^T f + e\, v \geqq 0 \right\}$$

$\Omega(K)$ always contains the reachable set after K transitions since the integrality constraint has been dropped.

To retrieve the boundedness results, we observe that $M(p)$ is bounded by the optimal value of the following LP:

$$\begin{array}{l|c} \text{Max} & M(p) \\ & -M + Cs = -M_0 \\ & e^T s = K \\ & s \geqq 0 \\ & M \geqq 0 \end{array}$$

The dual problem is:

$$\begin{array}{l|c} \text{Min} & M_0^T f - Kv \\ & C^T f + e\, v \leqq 0 \\ & f(p) \geqq 1 \\ & f \geqq 0 \end{array}$$

Let (f^i, v^i) be the vertices of the dual feasible set. Three cases must be considered:

(*a*) There exists an (f, v) dual feasible such that $v > 0$. Then, for large K, the optimal value of the dual problem becomes negative, which implies, because $M(p) \geqq 0$, that this value is $-\infty$ and the primal is infeasible. In other words the reachable set is empty.

(*b*) for all the vertices, $v^i < 0$.

Then, for K large enough, the optimal value is $M_0^T \bar{f} - K\bar{v}$ where $(\bar{f}, \bar{v})$ is a vertex such that $\bar{v} = \underset{i}{\text{Min}}\, v^i$. Therefore, $M(p)$ is not bounded when $K \to \infty$.

(*c*) for all the vertices, $v^i \leqq 0$ and for at least one, say v^j, $v^j = 0$. Then, for K large enough, the optimal value is $\min \{M_0^T f^j, j \text{ s. t. } v^j = 0\}$. Observe that as $f \geqq 0$, we have at the optimal solution $f(p) = 1$ and we retrieve the bound obtained in the precedent section.

IV. CONCLUSION

As we have just seen, all the properties we have investigated (boundedness, semi-flows, liveness and simplification of implicit places) can be checked by using standard Linear Programming instead of Integer Programming as used in some packages like Ovide [8] and without computing a basis of invariants. This means that for that kind of analysis, very large Petri Nets can be handled whereas the use of Integer Programming leads to serious limitations on the size of the net.

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