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A NOTE ON CHECKING SCHEDULES WITH FINITE HORIZON (*)

by Bruno VISCOLANI (1)

Abstract. – *This communication analyzes the problem of determining an optimum checking schedule over a finite horizon, for an equipment subject to failure, in order to minimize the expected cost due to inspections and failure. We refer to a known formulation of the problem, discussing the correctness of its proposed solution, and exhibiting some misunderstood aspects of it.*

Keywords : Reliability; inspection schedules; calculus of variations.

Résumé. – *Cette communication analyse le problème qui consiste à déterminer une politique d'inspection optimale pour un équipement sujet à panne sur un horizon fini. Le critère d'optimalité est de minimiser le coût espéré des inspections et de la panne éventuelle. Nous considérons une formulation connue du problème, discutons la correction de la solution proposée et montrons des aspects mal compris de ce modèle.*

Mots clés : Fiabilité; politiques d'inspection; calcul des variations.

1. INTRODUCTION

In [7] J. B. Keller proposes an approach to determining an "optimum checking schedule for a system subject to random failures". His formulation of the problem is a simple case in the calculus of variations. Thus it has been proposed as an example of economic application in two more recent textbooks ([6], p. 50; [2], p. 460), because of the interesting and easy interpretation of its necessary conditions. Other authors [8; 3; 4; 5], which are interested in suboptimal inspection policies for application purposes, refer to the same paper [7]. Kamien and Schwartz ([6], p. 50) and Guerraggio and Salsa ([2], p. 460) change the original infinite horizon problem into a finite horizon one, but fail to realize some consequences on the existence of an optimal solution.

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2. OPTIMUM CHECKING SCHEDULE PROBLEM

Let Z be the occurrence time of the first failure of a machine which starts working at time 0. Z is a positive random variable with distribution F and continuously differentiable p.d.f. f. The status (working or failed) of the machine is inspected at some epochs, at a fixed cost c_0 per each inspection, without interfering with the machine running. $L(X)$ is the loss from a failure, where X is the time elapsed after the failure until an inspection detects it and L is a positive and strictly increasing function. C is the total cost incurred for checking and failure in the interval $[0, Z+X]$, where $Z+X$ is the epoch of detection of the first failure. A checking schedule S is defined as an increasing sequence of positive time points:

$$S=\{y_k : k\geqq 1\}, \qquad 0<y_k<y_{k+1}, \quad k\geqq 1. \tag{1}$$

If the schedule $S=\{y_k\}$ is adopted, then y_1 is the epoch of the first inspection, y_2 is the epoch of the second one, . . .

Then the problem may be stated as follows:

P : determine the checking schedule $S=\{y_k\}$

which minimizes the expectation

$$E(C)=E(c_0 M+L(X))=E_Z(c_0 E(M\mid Z)+E(L(X)\mid Z)), \tag{2}$$

where M is the number of inspections necessary to detect the failure.

Keller [7] assumes that the checking schedule depends on a smooth function $n(t)$, the number of checks per unit time.

Then, after setting

$$x(t)=\int_0^t n(w)\,dw, \qquad t\geqq 0, \tag{3}$$

he restates the problem P as the following infinite horizon problem in the calculus of variations:

$$\begin{aligned} P_K : \quad & \min J(x(t))=\int_0^\infty [c_0 x(t)+L(1/2x'(t))]\,f(t)\,dt \\ & \text{s.t. } x(0)=0, \\ & \qquad x'(t)>0, \qquad t\geqq 0, \\ & \qquad \text{no condition on } x(t) \text{ when } t\to\infty. \end{aligned}$$

3. THE FINITE HORIZON CASE

Now, also finite horizon versions of the problem P_K are worth studying, as proposed in ([6], p. 50; [2], p. 460). However, some care must be taken when reformulating P_K with a finite horizon. In fact the problem may not have an optimal solution in general.

Following the authors of [6; 2], let us assume that $L(x)=c_1 x$, $c_1>0$. The relevant interval is [0, T], where $T>0$ is fixed and $F(T)\leqq 1$. The most natural problem is then ([6], p. 50; [2], p. 460)

$$P': \quad \min J(x(t))=\int_0^T [c_0 x(t)+c_1/2x'(t)] f(t)\,dt,$$

$$\text{s. t. } x(0)=0, \qquad x(T) \text{ free.}$$

The Euler equation reads, after integration, as follows:

$$c_1 f(t)/(x'(t))^2=2c_0[a-F(t)], \qquad 0\leqq t\leqq T, \tag{4}$$

where a is an integration constant [7; 6]. Thus, it is necessary first that

$$a\geqq F(T)=\max\{F(t),\ t\leqq T\}, \tag{5}$$

because the first member of the equation (4) is non-negative. We notice that, if $f(\hat{t})=0$ for some $\hat{t}<T$, then also the second member of (4) must vanish at $\hat{t}$, so that $a=F(\hat{t})$ and hence, by (5), $F(\hat{t})=F(T)$, i.e. $f(t)=0$, $\hat{t}\leqq t\leqq T$. On the other hand, if $f(\hat{t})=0$ and $f(s)>0$, for some $\hat{t}$ and s, $0<\hat{t}<s<T$, then there does not exist any solution to (4) and the problem has not any optimal solution.

A second necessary condition is the "transversality condition" [6; 2], $-c_1 f(T)/2(x'(T))^2=0$, *i.e.*:

$$f(T)/(x'(T))^2=0. \tag{6}$$

Two cases are possible for it:

(i) $f(T)=0$, then any solution of (4), with $a=F(T)$, satisfies (6) too;

(ii) $f(T)\neq 0$, then no optimal solution exists.

In fact (ii) is the unique interesting case, in view of the typical distributions considered in reliability theory ([1]; [9], pp. 353-358) and in view of the independent choice of the parameter T. The different behavior of the finite horizon problem, from the infinite horizon one, is due to the fact that the cost of the inspections is not enough for bounding their frequency towards the end of the relevant interval.

4. UPPER BOUNDED FINAL STATE

Let us consider here the new finite horizon problem

$$P'':\quad \min J(x(t)) = \int_0^T [c_0 x(t) + c_1/2\, x'(t)] f(t)\, dt,$$
$$\text{s.t. } x(0)=0 \qquad x(T) \leqq x_T,$$

where $x_T > 0$ is fixed and all the assumptions of Section 4 on problem P' still hold.

The Euler equation is again (4), whereas the proper transversality condition, corresponding to the terminal condition $x(T) \leqq x_T$, is now ([10], pp. 31-36):

$$f(T)/(x'(T))^2 \geqq 0, \qquad (=0 \text{ if } x(T) < x_T). \tag{7}$$

If $f(T) > 0$, as we should assume in general, then an optimal solution must satisfy the terminal condition as an equality:

$$x(T) = x_T. \tag{8}$$

On the other hand, $f(T) > 0$ implies that $f(t) > 0$, $0 \leqq t \leqq T$, otherwise the Euler equation would not admit any solution.

Then (4) has the solutions, in terms of x',

$$x'_a(t) = [c_1 f(t)/2 c_0 (a - F(t))]^{1/2}, \qquad 0 \leqq t \leqq T, \tag{9}$$

for all $a > F(T)$.

Let x_a be the state function whose derivative is x'_a and which verifies $x_a(0) = 0$. If there exists such an $a > F(T)$ that (8) is satisfied, $x_a(T) = x_T$, then $x_a(t)$ is the global minimum and it is unique. In fact the sufficient condition ([10], p. 43) that $[c_0 x(t) + c_1/2\, x'(t)] f(t)$, be convex as a function of (x, x'), is satisfied. Notice that

$$x'_a(t) < x'_{F(T)}(t), \qquad 0 \leqq t < T, \tag{10}$$

for all $a > F(T)$, so that we have the inequality

$$x_a(T) = \int_0^T x'_a(t)\, dt < x_{F(T)}(T). \tag{11}$$

In the opposite, exceptional case that $f(T) = 0$ and $f(t) > 0$, $0 \leqq t < T$, (7) is satisfied by any solution of (4) with $a = F(T)$, if it exists; again that solution would be a global minimum of P''.

Had we imposed the constraint

$$x'(t) \leqq k, \qquad k>0 \text{ fixed}, \tag{12}$$

instead of $x(T) \leqq x_T$, we would have found a situation similar to the one just discussed. Nevertheless, the problem, with the new constraint (12), is better formulated and discussed in terms of the Optimal Control Theory and the Maximum Principle.

Example 1: Let $F(t)=1-e^{-\lambda t}$, $t \geqq 0$, where $\lambda>0$ is fixed: then $f(t)=\lambda e^{-\lambda t}$, $t \geqq 0$, and T may be any positive number. The Euler equation has the solutions

$$n_a(t)=[c_1 \lambda e^{-\lambda t}/2 c_0 (a-1+e^{-\lambda t})]^{1/2}, \qquad 0 \leqq t \leqq T,$$

for all $a>1-e^{-\lambda T}=F(T)$, because $f(T) \neq 0$. The transversality condition implies $\int_0^T n_a(t)\,dt=x_T$. We can see that, if $T=10$, $\lambda=0.03$, $c_0=10$, $c_1=15$ and $x_T=5$, then the optimal solution exists and has $a \cong 0.262$. If $x_T \geqq 5.5$ approximately, and the remaining parameters are the same as above, then no optimal solution exists.

Example 2: Let $F(t)=\lambda t$, $0 \leqq t \leqq \lambda^{-1}$, where $\lambda>0$ is fixed: then $f(t)=\lambda$, $0 \leqq t \leqq \lambda^{-1}$ and T must satisfy the condition $T<\lambda^{-1}$. The Euler equation has the solutions

$$n_a(t)=[c_1/2 c_0 (a/\lambda-t)]^{1/2}, \qquad 0 \leqq t \leqq T,$$

for all $a>\lambda T$, because $f(T) \neq 0$. An optimal solution exists iff $x_T<(2 c_1 T/c_0)^{1/2}$. The transversality condition is

$$a/\lambda=c_1 (T+c_0 x_T^2/2 c_1)^2/2 c_0 x_T^2.$$

When an optimal "density" $n(t)$ exists, it depends on the parameters x_T, c_0, c_1 and T, but not on λ. The optimal value of the objective functional is:

$$J^*=\lambda c_1 \{ T^2+c_0 x_T^2 T/c_1-(c_0 x_T^2/2 c_1)^2/3 \}/2 x_T.$$

We can see that, if $T=10$, $c_0=10$, $c_1=15$, then no optimal solution exists iff $x_T \geqq 5.48$ approximately.

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