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M/G/1 WITH EXCEPTIONAL SERVICE AND ARRIVAL RATE (*)

by M. KRAKOWSKI [1]

Abstract. — *We modify the model $M/G/1$ by (a) providing pioneer customers (initiators of a busy period) with exceptional service; and (b) by having an exceptional arrival rate when the server idles. The regimen is FCFS (first come first served) and exhaustive (a busy server turns idle if and only if the load is exhausted). We derive so-called omni-equations for virtual and true delay and express them is terms of a delay in $M/G/1$ and a model-specific variable. We derive the queue size from the delay using the Poisson operator, as we call it. P. D. Welch (1964) allows exceptional service but no exceptional arrivals, and derives no compositions. We think our treatment is simpler and easier to extend once we overcome the hurdle of new notation. The special arrival rate of pioneers forces us to carefully distinguish virtual from true delays.*

Keywords : M/G/1 family, exceptional service and arrivals; balanced processes; omni-equation; composition; Poisson operator.

Résumé. — *Nous modifions le modèle $M/G/1$ de la façon suivante : (a) accorder aux clients pionniers (c'est-à-dire initiateurs d'une période de service) un service exceptionnel; (b) avoir un taux d'arrivée exceptionnel quand le serveur est inoccupé. Le régime est FCFS (« first come first served », premier arrivé premier servi). Nous présentons les équations dites omni-équations pour les délais virtuels et véritables, et les exprimons en termes de délai en $M/G/1$ et d'une variable spécifique au modèle. Nous déduisons la taille de la file d'attente à partir du délai, par utilisation de ce que nous appelons l'opérateur de Poisson. P. D. Welch (1964) permet un service exceptionnel, mais pas d'arrivées exceptionnelles, et ne présente aucune composition. Nous pensons que notre traitement est plus simple et plus facile à généraliser une fois dépassé l'obstacle d'une notation nouvelle. Le taux spécial d'arrivée des pionniers nous oblige à distinguer soigneusement les délais virtuels et les délais véritables.*

Mots clés : Famille M/G/1, service exceptionnel; arrivées exceptionnelles; processus équilibrés; omni-équation; composition; opérateur de Poisson.

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BALANCED PROCESSES AND OMNI-TRANSFORMS

DEFINITION: *The omni-transform* (Krakowski, 1984, 1987) of Z is the expectation $E\psi(Z)$ of an arbitrary well-behaved function $\psi(Z)$. "Well-behaved" means that all operations needed in context are valid, in particular that $E\psi(Z)$ and $E\psi'(Z)$ are finite. Good behavior depends thus on Z, on $\psi(Z)$, on our model – and on our rigor. The inverse of $E\psi(Z)$ is (generic) Z itself, just as the inverse of $E\exp(-sZ)$, a special case of $E\psi(Z)$, can be thought of as Z rather than the distribution of Z.

Thus, the omni-equation $E\psi(A)=\alpha E\psi(B+C)+(1-\alpha)E\psi(D)$ says that A is distributed like the mixture of $(B+C)$ and D weighted α and $1-\alpha$ in turn. Omni-transforms allow us great freedom in choosing ψ and provide a handy notation for sums and mixtures of random variables (and thus for some tree structures). In particular, the relation between the distribution of virtual waiting time and the distribution of queue size become clearer. Still, any omni-equation in independent random variables can be specialized to an equation in Laplace transforms (or characteristic functions or generating functions) by the simple device of setting $\psi(B+C)=\psi(B)\psi(C)$.

DEFINITION: We call a random process Z *balanced* if $E\,dZ=0$, *i.e.* if its expected fluctuation over a "random" dt vanishes. The balance condition is weaker than stationarity. [We do not define "random dt" rigorously. We can initiate the intervals dt by a Poisson source independent of all the model's processes.]

The Omni-Method

Under wide conditions if the process Z is balanced than so is an arbitrary function $\psi(Z)$. The essence of the omni-method is to analyze and balance $\psi(Z)$ rather than just Z. Not surprisingly, the conservation method and omni-method go well together. Though one often balances $\exp(-sZ)$ and $\exp(-itZ)$ and z^N the idea of balancing $\psi(Z)$ seems recent (Krakowski, 1984).

We start with the omni-equation for regular M/G/1 in integrated form (July 1986):

M/G/1 $$E\psi(u)=(1-\rho)\psi(0)+\rho\,\mathrm{E}\,\psi(u+\mathscr{R}x) \qquad (1.1)$$

u being virtual delay; $\mathscr{R}x$ residue of service time x; and $\rho=\lambda/\mu$. The operator E makes it immaterial whether both instances of u are the same process or equivalent processes; unless said otherwise all variables in an omni-equation are generic and independent; both instances of u are mutually independent and we say they are *free copies* of a generic delay u. [We can write: $E\psi(u_1)=(1-\rho)\psi(0)+\rho E\psi(u_2+\mathscr{R}x)$ with i.i.d. u_1 and u_2 but we think (1.1) is more readable and more esthetic.]

Specializing $\psi(u)$ to u, u^2, u^k or $\exp(-su)$ we get successive moments of u and its LST. If $\psi(u)=1$ for $0\leqq u\leqq t$, and $\psi(u)=0$ otherwise, we get the convolution equation

$$\Pr(u\leqq t)=(1-\rho)\Pr(0\leqq t)+\rho\Pr(u+\mathscr{R}x\leqq t)$$

$$=(1-\rho)+\rho\Pr(u+\mathscr{R}x\leqq t) \quad (1.2)$$

The probability that a r.v. satisfies a set of inequalities is, as known, the expectation of an indicator. But we need not explicitly specify this indicator in order to write down the corresponding convolution: we just write $E\psi(u)=\Pr(u\leqq t)$. The distinction between probabilistic and analytic procedures is now blurred; omni-equations become convolution equations by letting $E\psi(u)=\Pr(u\leqq t)$.

Omni-Convention

Omni-equation, *i.e.* equations with such terms as $E\psi(Z)$ or $EZ^k\psi(Z)$, are easily told by sight. The omni-convention mentally applies the expectation operator E to each side of an omni-equation. (We retain E if ambiguity threatens.) This convention is akin to the summation convention for matrices and tensors. The Omni-Convention turns (1.1) into

$$\psi(u)=(1-\rho)\psi(0)+\rho\psi(u+\mathscr{R}x) \quad (1.1\,a)$$

Equation (1.1), or (1.1 *a*), states: The process u is a mixture of 0, weighted $1-\rho$, and of "a clone of **u** plus $\mathscr{R}x$", weighter ρ. We can refer to (1.1 *a*) or (1.1), as well as (1.2), as a convolution equation.

True or virtual customers and continuous observers find stochastically equivalent states in all models with a steady poissonian arrival rate (Wolff, 1982).

A GENERAL COMPOSITION LEMMA

LEMMA: *Let $0<\alpha<1$ and let the independent random variables $U\geqq 0$, $Z\geqq 0$ and $g\geqq 0$ satisfy the two equations (mind the omni-convention!)*

$$\psi(U)=(1-\alpha)\psi(0)+\alpha\psi(U+A) \tag{2.1}$$

$$\psi(Z)=(1-\alpha)\psi(g)+\alpha\psi(Z+A) \tag{2.2}$$

Such equations, which state a mixture property, occur often in $M/G/1$ family when service time x is transposed into residual service $\mathscr{R}x$. Then the composition

$$\psi(Z)=\psi(U+g) \tag{2.3}$$

holds which says that Z is distributed like the sum of U and g (or that generic Z is the sum of generic U and generic g). Only the r.v. A enters both (2.1) *and* (2.2) *but does not explicitly enter* (2.3).

Proof: Since (2.1) and (2.2) are convolutions it is plausible to attempt the derivation of (2.3) by specializing $\psi(U)=e^{-sU}$. Thus (2.1) and (2.2) become

$$Ee^{-sU}=1-\alpha+\alpha Ee^{-sU}Ee^{-sA} \tag{2.1 a}$$

$$Ee^{-sZ}=(1-\alpha)Ee^{-sg}+\alpha Ee^{-sZ}Ee^{-sA} \tag{2.2 a}$$

Since $\alpha<1$ we have $1-\alpha Ee^{-sU}>0$, and therefore (2.1 a) and (2.2 a) imply

$$Ee^{-sU}=(1-\alpha)/[1-\alpha Ee^{-sU}] \tag{2.4}$$

$$Ee^{-sZ}=(1-\alpha)Ee^{-sg}/[1-\alpha Ee^{-sA}] \tag{2.5}$$

It follows that

$$Ee^{-sZ}=Ee^{-(U+g)} \tag{2.6}$$

which says, as does (2.3), that Z is distributed like the sum of U and g. Thus (2.6) is equivalent to (2.3).

Note: Equation (2.3) holds for the basic multiple vacation model if $\alpha=\rho=\lambda/\mu$, $Z=u=$ virtual delay, $U=u_R=$ virtual delay in regular M/G/1, $g=\mathscr{R}v=$ residual vacation v; and $A=\mathscr{R}x=$ residual service; (2.2) occurs also in M/G/1 *with Initial Quorum* ("Heyman model", Krakowski, July 1986 and November 1986) and other models. It explains much of compositions' ubiquity in the M/G/1 family.

If $U=u_R$, virtual delay in a regular M/G/1, and $A=\mathscr{R}x$ and $\alpha=\rho$ then (2.1) becomes

$$\psi(u_R)=(1-\rho)\psi(0)+\alpha\psi(u_R+A) \tag{2.7}$$

and the composition (2.3) becomes

$$\psi(Z)=\psi(u_R+g) \tag{2.8}$$

Indeed, (2.1) can always be interpreted as an omni-equation for a regular M/G/1.

POSITIVE DELAY IN M/G/1 WITH EXCEPTIONAL SERVICE AND ARRIVAL RATE

We now balance $\psi(u)$ for a modified M/G/1 whose pioneers (initiators of a busy period) receive exceptional service x_0 instead of the regular x; and whose arrival rate is λ_0 instead of the regular λ when server idles. (We keep x and λ free of subscripts when server works to stress their regularity.) Service is exhaustive and customers' service and delay become known when they join the queue.

The virtual delay u is a state variable since it is defined at each time instant, unlike an arriver's prospective delay w_a, defined at arrival instants, or a departer's (into service) retrospective delay w_d, defined at departure instants. Though we balance state variables, we relate virtual delay to true delay, not a state variable. The reason for defining virtual delay is to have a balanced state variable with the same or related distribution as true delay.

The expected fluctuation $E\,d\psi(u)$ during a random dt has several components:

(*a*) aging when server works adds

$$E\,d\psi(u)\,|_{\text{aging}}=P_*\,E[\psi(u_*-dt)-\psi(u_*)]=-dt\,P_*\,E\,\psi'(u_*)$$

A busy server works off his load at rate $du_*=-dt$; $P_*=1-P_0=\Pr$ (server busy). The asterisk in subscript position says that the server works.

(*b*) Arriving pioneers add

$$E\,d\psi(u)\,|_{\text{pioneers}}=dt\,\lambda_0\,P_0\,E[\psi(x_0)-\psi(0)]$$

(*c*) Arriving non-pioneers add

$$E\,d\psi(u)\,|_{\text{non-pioneers}}=dt\,\lambda\,P_*\,E[\psi(u_*+x)-\psi(u_*)]$$

(*d*) In an exhaustive model a departure causes no finite change in $d\psi(u)$ as it occurs when service residue is down to zero.

The changes (*a*) through (*d*) must sum to zero for a balanced $\psi(u)$:

$$-P_* \psi' u_*)+\lambda_0 P_0 [\psi(x_0)-\psi(0)]+\lambda P_* [\psi(u_*+x)-\psi(u_*)]=0 \quad (3.1)$$

With $\psi(u)=u$, $\rho_0=\lambda_0 \bar{x}_0$, *and* $\rho=\lambda \bar{x}$, we get from (3.1)

$$P_0=\frac{1-\rho}{1+\rho_0-\rho} \quad \text{and} \quad P_*=\frac{\rho}{1+\rho_0-\rho} \quad (3.2)$$

The process we have balanced, $\psi(u)$, does not enter (3.1) though $\psi(u_*)$ does. This is common; thus Little's theorem results from balancing "aggregate sojourn of all incumbents", a variable absent in the theorem's statement (Krakowski, 1973).

The renewal equation (Krakowski, Sep. 1984, 1987) sates that

$$\psi(x)-\psi(0)=\bar{x}\psi'(\mathscr{R} x) \quad (3.3)$$

where $\mathscr{R} x$ = residue of service x (remaining service time of x).

Outline of Proof: We balance $\psi(S)$, S being an incumbent's remaining service. Note that $dt=-dS$. Aging add $E d\psi(S)|_{\text{aging}}=-E\psi'(\mathscr{R} x)\,dt$. Renewals, of rate $1/\bar{x}$, add $E d\psi(S)|_{\text{renewal}}=E[\psi(x)-\psi(0)]/\bar{x}$. The two contributions sum to zero and yield (3.3). (Krakowski, September 1984 and 1987.)

"Shifted" by u_*, the positive virtual delay, (3.3) becomes

$$\psi(u_*+x)-\psi(u_*)=\bar{x}\psi'(u_*+\mathscr{R} x) \quad (3.3\,a)$$

With the aid of (3.3) and (3.3 *a*) equation (3.1) becomes

$$P_* \psi'(u_*)=P_0 \rho_0 \psi'(\mathscr{R} x_0)+P_* \rho\psi'(u_*+\mathscr{R} x) \quad (3.1\,a)$$

Defining $\varphi=\psi'$ we typographically integrate (3.1 *a*):

$$P_* \varphi(u_*)=P_0 \rho_0 \varphi(\mathscr{R} x)+P_* \rho\varphi(u_*+\mathscr{R} x) \quad (3.1\,b)$$

Dividing throughout by P_* we get

$$\varphi(u_*)=(P_0 \rho_0/P_*)\varphi(\mathscr{R} x)+\rho\varphi(u_*+\mathscr{R} x) \quad (3.1\,c)$$

Letting $\varphi(.)=1$ we find $P_0 \rho_0/P_*=1-\rho$, thus confirming (3.2). We rewrite (3.1 *c*), while reverting to ψ in place of the equally general φ, as

$$\psi(u_*)=(1-\rho)\psi(\mathscr{R} x_0)+\rho\psi(u_*+\mathscr{R} x)$$

which is the integrated form of (3.1). We thus have a convolution relation for the virtual delay of non-pioneers; the delay of pioneers is, of course, zero.

The arguments x_0 and x in the differential equation (3.1) are replaced in the integrated equation (3.4) by $\mathscr{R}\,x_0$ and $\mathscr{R}\,x$. In omni-equations service residue tends to enter integrated equations while service itself tends to go with differential equations; the use of the renewal equation makes this clear. LST (Laplace-Stieltjes Transform) equations are also often formally (but not perforce numerically) simplified when transposed from service-form to residue-form. But the idea of integrating an omni-equation seems to lack so far a specific counterpart in LST equations.

From (2.8) with $g=\mathscr{R}\,x_0$, $Z=u_*$ and $U=u_R$ we get the composition

$$\psi(u_*)=\psi\, u_R+\mathscr{R}\,x_0) \tag{3.5}$$

which solves, in omni-form, for positive delay u_*, if the corresponding virtual delay (*i.e.* with same λ, μ and x) for regular M/G/1 is known.

The general virtual delay u is a mixture of 0 and u_* with weights P_0 and P_*:

$$\psi(u)=P_0\,\psi(0)+P_*\,\psi(u_*) \tag{3.6}$$

From (3.5) and (3.6) follows the virtual delay in the extended Welch problem

$$\psi(u)=P_0\,\psi(0)+P_*\,\psi(u_R+\mathscr{R}\,x_0) \tag{3.7}$$

with $P_0=(1-\rho)/(1+\rho_0-\rho)$ and $P_*=\rho/(1+\rho_0-\rho)$ from (3.2).

Example 1: We get successive moments of u_* from $\psi(u_*)=u_*^k$ in (3.5). For $k=1$

$$E\,u_*=E\,x_0+\frac{\rho}{1-\rho}\,E\,\mathscr{R}\,x \tag{3.8}$$

Example 2: With $\psi(u)=e^{-su}$ in (3.4) we find the Laplace Transform of w_*:

$$E\,e^{-su_*}=\frac{(1-\rho)\,E\,e^{-s\,\mathscr{R}x_0}}{1-\rho\,E\,e^{-s\,\mathscr{R}x}} \tag{3.9}$$

We specialize (3.4) by letting

$$\psi(u_*+\mathscr{R}\,x)=\mathscr{L}\,(u_*)\,.\,\mathscr{L}\,(\mathscr{R}\,x) \tag{3.10}$$

This turns (3.4) into

$$\mathscr{L}\,(u_*)=(1-\rho)\,\mathscr{L}\,(\mathscr{R}\,x_0)+\rho\,\mathscr{L}\,(u_*)\,.\,\mathscr{L}\,(\mathscr{R}\,x) \tag{3.11}$$

and

$$\mathscr{L}\,(u_*)=\frac{(1-\rho)\,\mathscr{L}\,(\mathscr{R}\,x_0)}{1-\rho\,\mathscr{L}\,(\mathscr{R}\,x)} \tag{3.12}$$

$\mathscr{L}\,(\mathscr{R}\,x)$ can stand for $\exp(-s\,\mathscr{R}\,x)$ or $\exp(s\,\mathscr{R}x)$ or $\exp(-is\,\mathscr{R}\,x)$. In other contexts we do likewise with generating functions.

Example 3: We find a convolution equation for the cdf of u_* by setting $E\,\psi(u_*)=\Pr(u_*\leqq t)$ in (3.4) [*i.e.* $\psi(u_*)=1$ if $u_*\leqq t$ and $\psi(u_*)=0$ if $u_*>t$]. Then

$$\Pr(u_*\leqq t)=(1-\rho)\Pr(\mathscr{R}\,x_0\leqq t)+\rho\Pr(u_*+\mathscr{R}\,x\leqq t) \tag{3.13}$$

We go on to find w, a customer's delay. If the arrival rate, as in our model, is not steady the virtual delay and true delay have different distributions.

DELAY OF CUSTOMERS IN M/G/1 WITH EXCEPTIONAL SERVICE AND ARRIVALS

Let π_{a0} = fraction of arrivals into an empty system, and π_{a*} = fraction of arrivals into a busy system. Clearly

$$\pi_{a0}=f_0/(f_0+f_*) \qquad \text{and} \qquad \pi_{a*}=f_*/(f_0+f_*) \tag{4.1}$$

where $f_0=\lambda_0\,P_0$ is the arrival rate while server idles, and $f_*=\lambda\,P_*$ is the arrival rate when server works. From (3.2) and (4.1) we have

$$\pi_{a0}=\lambda_0\,(1-\rho)/[\lambda\rho+\lambda_0\,(1-\rho)] \qquad \text{and} \qquad \pi_{a*}=\lambda\rho/[\lambda\rho+\lambda_0\,(1-\rho)] \tag{4.2}$$

The prospective delay w_a of an arrival is a mixture of 0 and w_{a*}, the positive prospective delay, with weights $\pi_0=\pi_{a0}$ and $\pi_*=\pi_{a*}$:

$$\psi\,(w_a)=\pi_0\,\psi\,(0)+\pi_*\,\psi\,(w_{a*}) \tag{4.3}$$

In our model virtual non-pioneers and true non-pioneers find stochastically equivalent states, in particular equivalent delays. This key property is stated as

$$\psi\,(u_*=\psi\,(w_{a*}) \tag{4.4}$$

Of course, both virtual and true pioneers find the system empty. In other variants of M/G/1, e.g. in the *initial quorum* problem (also known as Heyman's N-Policy; *cf.* Krakowski, July 1986: Novemer 1986) property (4.4) may also play a key role.

It often pays to know which among the spans virtual delay, true delay, load, and clearing time are identical or equivalent or related: Always? If server works?

If w_R and $\mathscr{R} x_0$ are known then equations (3.6) and (4.4) yield a solution for w_{a*}:

$$\psi(w_{a*})=\psi(w_R+\mathscr{R} x_0) \tag{4.5}$$

From (4.4) and (4.5) we have the solution for w_a

$$\psi(w_a)=\pi_0 \psi(0)+\pi_* \psi(w_R+\mathscr{R} x_0) \tag{4.6}$$

π_0 and π_* are as in (4.2). The equations for w_a and u, (4.6) and (3.7), differ but in their weights. If $\lambda_0=\lambda$ then $\pi_0=P_0$ and $\pi_*=P_*$ and $\psi(u)=\psi(w_a)$; virtual and true customers find stochastically equivalent views, and we have Welch's model with exceptional service but steady source. If also $x_0=x$ then our model is a regular M/G/1.

In the next section the retrospective delay w_d is more suitable for analysis than the prospective delay w_a. Let π_{d0} = probability that a customer departing the system leaves it empty; let $\pi_{d*}=1-\pi_{d0}$. We have $\pi_{a0}=\pi_{d0}$: a customer is as likely to find the server idle as to leave him idle (since the transitions $0 \to 1$ and $1 \to 0$ are equally frequent); likewise $\pi_{a*}=\pi_{d*}$: a customer is as likely to find the server busy as to leave him busy (jumps $j \to j+1$ being as frequent as $j+1 \to j$ for $j \geqq 1$). Thus

$$\pi_0=\pi_{d0}=\pi_{a0} \quad \text{and} \quad \pi_*=\pi_{d*}=\pi_{a*} \tag{4.7}$$

Since, as easily seen, in our model

$$\psi(w_d)=\psi(w_a) \quad \text{and} \quad \psi(w_{d*})=\psi(w_{a*}) \tag{4.8}$$

equations (4.5) and (4.6) become

$$\psi(w_{d*})=\psi(w_R+\mathscr{R} x_0) \tag{4.9}$$

$$\psi(w_d)=\pi_0 \psi(0)+\pi_* \psi(w_R+\mathscr{R} x_0) \tag{4.10}$$

POISSON OPERATOR AND QUEUE SIZE

Consider a random time interval Z and an independent Poisson source of rate σ.

DEFINITION: The Poisson operator $\mathscr{P}_\sigma$ acting on Z, $\mathscr{P}_\sigma Z$, is the number of events generated during Z by a poissonian source of intensity σ. We can view $\mathscr{P}_\sigma$ as an independent "Poisson clock" of rate σ which assigns a non-negative integer, its count, to time intervals. The basic properties of $\mathscr{P}_\sigma$ are

$$(a)\quad \Pr(\mathscr{P}_\sigma Z=j)=\frac{E e^{-\sigma Z}(\sigma Z)^j}{j!}$$

(Gross and Harris 1985, Sect. 5.1) (5.1)

(b) For disjoint intervals A and B, $\mathscr{P}_\sigma(A+B)=\mathscr{P}_\sigma A+\mathscr{P}_\sigma B$ and hence $\psi(\mathscr{P}_\sigma(A+B))=\psi(\mathscr{P}_\sigma A+\mathscr{P}_\sigma B)$; clearly, $\mathscr{P}_\sigma 0=0$

(c) If $\psi(Z)=\alpha\psi(A)+\beta\psi(B)$, A and B being independent time-spans, then

$$\psi(\mathscr{P}_\sigma Z)=\alpha\psi(\mathscr{P}_\sigma A)+\beta\psi(\mathscr{P}_\sigma B);\qquad \alpha\geqq 0,\quad \beta\geqq 0,\quad \alpha+\beta=1.$$

In words: If Z is a mix (lottery) of independent time-spans A and B then $\mathscr{P}_\sigma Z$ is a like-weighted mix of $\mathscr{P}_\sigma A$ and $\mathscr{P}_\sigma B$.

$\mathscr{P}_\sigma Z$ is a well behaved random variable if Z is; hence an arbitrary function $\varphi(\mathscr{P}_\sigma Z)$ is a special case of the general omni-function $\psi(Z)$; and $E\varphi(\mathscr{P}_\sigma Z)$ is thus the omni-transform of $\mathscr{P}_\sigma Z$. This observation, along with (b) and (c) above, enables us to transpose omni-equations for Z into omni-equations for $\mathscr{P}_\sigma Z$. We thus have

THEOREM: *An omni-equation in independent time-spans stays valid if each span is formally subjected to the same Poisson operator* $\mathscr{P}_\sigma$.

Example 1: Number of Poisson Events During Service and Service Residue

The equation of a renewal process generated by the r.v. x is (5.2) or (5.2a) below:

$$\psi(x)-\psi(0)=\bar{x}\,\psi'(\mathscr{R}x) \qquad (3.3)=(5.2)$$

$$\psi(x)-\psi(0)=\bar{x}\lim\,[\psi(\mathscr{R}x+dt)-\psi(\mathscr{R}x)]/dt \qquad (5.2a)$$

Applying formally the Poisson operator $\mathscr{P}_\lambda$ to each argument in (5.2a) we have

$$\psi(\mathscr{P}_\lambda x)-\psi(\mathscr{P}_\lambda 0)=\bar{x}\lim\,[\psi(\mathscr{P}_\lambda\mathscr{R}x+\mathscr{P}_\lambda dt)-\psi(\mathscr{P}_\lambda\mathscr{R}x)]/dt \qquad (5.3)$$

Let $k^d = \mathscr{P}_\lambda x$ and $h^d = \mathscr{P}_\lambda \mathscr{R} x$; (5.3) becomes

$$\psi(k) - \psi(0) = \bar{x} \lim [\psi(h + \mathscr{P}_\lambda dt) - \psi(h)]/dt \tag{5.4}$$

Now, $\psi(\mathscr{P}_\lambda dt)$ is, within $0.(dt)^2$, a mixture of 1 weighted $\lambda\, dt$ (*i.e.* probability of one event during dt), *and* 0 *weighted* $1 - \lambda\, dt$ (*i.e.* probability of no event during dt):

$$\psi(\mathscr{P}_\lambda dt) = \lambda\, dt\, \psi(1) + (1 - \lambda\, dt)\, \psi(0) \tag{5.5}$$

Shifting (5.5) by h we have

$$\psi(h + \mathscr{P}_\lambda dt) = \lambda\, dt\, \psi(h+1) + (1 - \lambda\, dt)\, \psi(h) \tag{5.5a}$$

Equations (5.4) and (5.5 a) imply, with $\rho = \lambda \bar{x}$,

$$\psi(k) - \psi(0) = \rho [\psi(h+1) - \psi(h)] \tag{5.6}$$

Thus $\psi(k) = z^k \Rightarrow E z^k - 1 = \rho [z . E z^h]$ and

$$E z^h = \frac{1 - E z^k}{\rho [1 - z]} \tag{5.7}$$

$\psi(k) = k \Rightarrow \bar{k} = \rho$, as it should; $\psi(k) = k^2 \Rightarrow E k^2 = \rho [2 \bar{h} + 1]$ and

$$\bar{h} = \frac{E k^2}{2 \bar{k}} - \frac{1}{2} \tag{5.8}$$

Example 2: Transposing Delay in Regular M/G/1 into Queue Size

What makes M/G/1 so tractable is that (Krakowski, 1974; Gross and Harris, 1985)

$$\text{M/G/1} \qquad \psi(u) = \psi(w_a) = \psi(w_d) \tag{5.9}$$

$$\text{M/G/1} \qquad \psi(n) = \psi(q_a) = \psi(q_d) \tag{5.10}$$

where w_a is the prospective delay of an arrival and w_d is the retrospective delay of a customer departing the queue for service; and where n = virtual queue size, q_a = queue size found by an arrival, and q_d = queue size seen by a customer departing the queue for service. Hence the equation for delay w_d in M/G/1 is, from (1.1 a) and (5.2 a),

$$\text{M/G/1} \qquad \psi(w_d) = (1 - \rho)\, \psi(0) + \rho \psi(w_d + \mathscr{R} x) \tag{5.11}$$

Applying formally the operator $\mathscr{P}_\lambda$ throughout (5.3) we get

M/G/1 $$\psi(\mathscr{P}_\lambda w_d) = (1-\rho)\,\psi(\mathscr{P}_\lambda 0) + \rho\psi(\mathscr{P}_\lambda w_d + \mathscr{P}_\lambda \mathscr{R} x) \tag{5.12}$$

Equation (5.12) stays true if we replace $\mathscr{P}_\lambda$ by any $\mathscr{P}_\sigma$ but it lacks interest if $\sigma \neq \lambda$.

As known, in *regular* M/G/1 a customer who departs his queue and enters service leaves behind a queue sized $\mathscr{P}_\lambda w_d = q_d$ so that in view of (5.10)

M/G/1 $$\psi(\mathscr{P}_\lambda w_d) = \psi(q_d) = \psi(q_a) = \psi(n) \tag{5.13}$$

(5.13) and (5.12) imply an equation for virtual queue size

M/G/1 $$\psi(n) = (1-\rho)\,\psi(0) + \rho\psi(n+h) \tag{5.14}$$

Of course, the queue sizes q_d and q_a can replace n in (5.14).

QUEUE SIZE IN M/G/1 WITH EXCEPTIONAL SERVICE AND ARRIVALS

Applying $\mathscr{P}_\lambda$ to both arguments in (4.9) we have

$$\psi(\mathscr{P}_\lambda w_{d*}) = \psi(\mathscr{P}_\lambda w_R + \mathscr{P}_\lambda \mathscr{R} x_0) \tag{5.15}$$

where q_{d*} is the queue size left behing by a non-pioneer entering service, so that

$$\psi(q_{d*}) = \psi(n_R + h_0) \tag{5.16}$$

with $h_0 \overset{d}{=} \mathscr{P}_\lambda \mathscr{R} x_0$.

Since $\psi(q_{d*}) = \psi(n_*)$ we have likewise

$$\psi(n_*) = \psi(n_R + h_0) \tag{5.17}$$

If n_R and h_0 are known, (5.16) solves for n_*, the queue-size when server works. To get the unconditioned queue size n we observe that for virtual customers

$$\psi(n) = P_0\,\psi(0) + P_*\,\psi(n_*) \tag{5.18}$$

and, from (5.18) and (5.17) we have, with P_0 and P_* given by (3.2),

$$\psi(n) = P_0\,\psi(0) + P_*\,\psi(n_R + h_0) \tag{5.19}$$

For customers who depart the queue (and enterservice)

$$\psi(q_d)=\pi_0\,\psi(0)+\pi_*\,\psi(q_{d*}) \tag{5.20}$$

and since $\psi(q_{d*})=\psi(n_*)$ we have, with π

$$\psi(q_d)=\pi_0\,\psi(0)+\pi_*\,\psi(n_R+h_0) \tag{5.21}$$

Thus n and q_d are each a mixture of 0 and n_R+h_0 but with different weights.

Note: We can formally apply the Poisson operator $\mathscr{P}_\lambda$ throughout (4.10) and thus immediately get (5.21); but since the arrival rate is not steady the interpretation of the transformed arguments is somewhat subtler. Take into account that $\psi(\mathscr{P}_\sigma 0)=\psi(0)$ whatever the value of σ.

Using the Poisson operator to turn delay-equations into size-equations may be new.

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