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RANDOM INSPECTION SCHEDULES WITH NON-DECREASING INTENSITY (*)

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Abstract. – *The problem of minimizing expected cost until detection of failure is addressed here, using random checking (inspection) schedules. Non-homogeneous Poisson checking processes with continuous non-decreasing intensity are investigated. An approximation of the original problem, in the form of an optimal control problem, is discussed. An optimal solution is proved to exist and then it is characterized, using Pontryagin's Maximum Principle.*

Keywords : Reliability; random checking schedules; non-homogeneous Poisson process; optimal control.

Résumé. – *Nous considérons le problème de minimiser le coût espéré jusqu'à la détection d'une panne, en employant des politiques d'inspection aléatoires. Nous étudions des processus d'inspection de Poisson non-homogènes avec intensité continue non-décroissante. Nous discutons une approximation du problème original, qui a la forme d'un problème de contrôle optimal. Nous prouvons l'existence d'une solution optimale et la caractérisons par le Principe du Maximum de Pontryagin.*

Mots clés : Fiabilité; politiques d'inspection aléatoires; processus de Poisson; contrôle optimal.

1. INTRODUCTION

We consider a variant of the classical problem of "minimizing expected cost until detection of failure", which is relevant for situations in which a human being, who may want his failure to remain undetected as long as possible, is the subject of the possible failure. The original problem, concerning an industrial system subject to random failures, has been treated by Barlow, Hunter and Proschan ([1]; [2], pp. 108-116), who obtained optimal deterministic checking schedules. In their formulation, an event (failure) can

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occur at a random time and its occurrence has relevant consequences for the (industrial) system under study. Hence it is important to detect the failure as soon as possible and this can be done by checking (inspecting) the state of the system from time to time. Other authors have addressed the same problem [10, 7, 8, 9], with the purpose of finding approximations of optimal inspection schedules. Now, if a human being is the subject of the possible failure, and if his interests conflict with those of the system, then from the system viewpoint it is desirable that the failing subject be unable to foresee the inspection times. Therefore deterministic checking schedules are no longer useful. As an example, let us consider the problem of inspecting the behavior of a taxpayer, who may try to avoid paying taxes on the revenues from a certain economic activity. We may assume that this illegal behavior begins possibly at a random epoch and continues, unless it is detected by an inspection. In some cases (e. g. "scontrino fiscale" in Italy) we may assume also that the illegality which has occurred at a certain time may be detected at that time only: afterward no trace of it can be observed, although a real loss of taxes has occurred. If the taxpayer knew the time of an inspection before it takes place, then he would resume paying the tax just before the inspection. In this way, one would pay the taxes only on a minor part of his activity. Therefore a random inspection schedule is needed here and the loss of taxes associated with an illegal behavior will increase (linearly perhaps) with the delay in detecting it after it begins. In view of this sort of application, we consider stochastic checking schedules and search for approximations of optimal ones. The following assumptions define the system and its behavior.

The system starts working at time 0 and the first system failure occurs at a time T, where T is a positive random variable with probability distribution function F and density f:

$$\Pr\{T \leqq t\} = F(t) = \int_0^t f(s)\,ds, \qquad t \geqq 0. \tag{1}$$

The first failure is relevant if and only if it occurs by a fixed finite time $t_1 > 0$. The density f is a continuous function on $[0, t_1]$ and

$$0 < F(t_1) < 1. \tag{2}$$

Thus the event of a failure occurring in $[0, t_1]$ is possible, but not certain. Inspecting the state of the system (which can be either "failed" or "working") has a constant cost c_0, takes a negligible time and does not influence the system performance. Moreover, let $l(x)$ be the cost due to the delay x from a failure to its detection, where $l(x)$ is assumed to be a strictly increasing, concave and continuously differentiable function. If $T \in [0, t_1]$ and if X is the failure detection delay, then $T+X$ is the time of discovery of the first failure and we stop the checking process immediately after it. If $T > t_1$, then the failure need not be detected and the checking process ends on the first check after the instant t_1. Thus the "detection delay loss' is

$$L(X, T) = l(X)\, 1_{\{T \leqq t_1\}}, \tag{3}$$

where 1_E is the indicator function of the event E.

A checking (or inspection) schedule

$$S = \{ y_k : k \geqq 1 \}, \quad 0 < y_k < y_{k+1}, \qquad k \geqq 1, \tag{4}$$

is an increasing sequence of points in time. In these terms, the final time of the checking process is y_M, where

$$M = M(S, T) = \min \{ k : y_k > T \wedge t_1, y_k \in S \}, \tag{5}$$

where $A \wedge B = \min(A, B)$. If $T \leqq t_1$, then y_M is the time of detection of the first failure, $y_M = T + X$, where X is the detection delay. Otherwise, if $T > t_1$, the failure is unimportant and y_M can very well be less than T, or also more than T, as it is when $T \leqq t_1$. The expected total cost resulting from the inspections and the possible (first) failure is then

$$E[C] = E[c_0 M + l(X)\, 1_{\{T \leqq t_1\}}] \tag{6}$$

and we want to determine an inspection schedule $S = \{ y_k \}$ minimizing it.

In the following, we denote by $\dot{a}(t)$ the derivative of any function $a(t)$, depending on the time t, $\dot{a}(t) = da(t)/dt$. Moreover, we denote by $l'(x)$ the derivative of the function $l(x)$, $l'(x) = dl(x)/dx$.

2. NON-DECREASING INTENSITY POISSON CHECKING PROCESSES

Let $N(t)$ denote the Poisson process with intensity $n(t) \geqq 0$, and denote by $S_p(n)$ the checking schedule

$$S_P(n) = \{ Y_1, Y_2, \ldots, Y_k, \ldots \}, \tag{7}$$

where Y_k is the occurrence time of the k-th event concerning $N(t)$. Then $N(t)$ is the number of checks during the interval $[0, t]$. We will refer to $N(t)$ as to the Poisson checking process (PCP). The expected number of checks in the interval $(t, t+s]$ is (*see* [4], p. 48)

$$E[N(t+s)-N(t)]=\int_t^{t+s} n(w)\,dw, \qquad t, \quad s \geqq 0. \tag{8}$$

Here, we restrict our attention to a special family of monotonic intensity functions. Assume that the Poisson checking process intensities are continuous, non-decreasing, piecewise continuously differentiable. Assume further that the derivative of the intensities is bounded in $[0, t_1]$ and null in $[t_1, +\infty)$.

The conditional expected number of inspections is, by (5),

$$E[M \mid T]=1+N(T \wedge t_1), \tag{9}$$

whereas the conditional expected detection delay loss is, by (3),

$$E[L(X, T) \mid T]=E[l(X) \mid T]\, 1_{\{T \leqq t_1\}}. \tag{10}$$

There is no simple way of expressing (10), but the following Lemma gives us a useful upper bound.

LEMMA 2.1: If $N(t)$ is a time-dependent Poisson checking process with non-decreasing intensity $n(t)$, if T is the first failure time, independent of all checking times, then $E[X \mid T] \leqq 1/n(T)$. Moreover, if $l(x)$ is an increasing and concave function, then

$$E[l(X) \mid T] \leqq l(1/n(T)). \tag{11}$$

Proof: The occurrence of the first check after t at a time y depends only on the intensity $n(w)$, $w \in (t, y]$, whereas it is independent of the occurence of other checks in the interval $(0, t]$. Thus the delay of the first check after an arbitrary time instant $t>0$, independent of $\{Y_k\}$, has the same distribution as an inter-checking interval which starts from the time t. Hence its mean is (*see* [4], p. 49)

$$E[X \mid T=t]=\int_0^{+\infty} \exp\left[-\int_t^{t+x} n(w)\,dw\right] dx \leqq \int_0^{+\infty} \exp(-n(t)\,x)\,dx=1/n(t).$$

Now, since $l(x)$ is concave, Jensen's inequality ([5], p. 153) implies that $E[l(X) \mid T] \leqq l(E[X \mid T])$ and hence the thesis follows, because l is increasing. ■

Notice that the smaller $n(t_1)-n(t)$, the tighter the above inequality. This is interesting in particular in those applications, where a physical bound to the increasing rate of the inspection intensity is present. In the special case in which the intensity is constant, *i.e.* when the PCP is homogeneous, the equality sign holds in (11).

From Lemma 2.1, we obtain that the expected total cost satisfies the inequality

$$E[C] \leqq c_0 \{1+[1-F(t_1)]E[N(t_1)]\} + \int_0^{t_1}\left[c_0\int_0^t n(w)\,dw + l(1/n(t))\right]f(t)\,dt. \quad (12)$$

After setting

$$x_1(t)=n(t) \qquad \text{and} \qquad x_2(t)=\int_0^t n(w)\,dw, \quad (13)$$

and

$$f_0(x_1, x_2, t) = -[c_0 x_2 + l(1/x_1)]f(t) - c_0[1-F(t_1)]x_1, \quad (14)$$

we have that (12) can be rewritten as

$$E[C] \leqq c_0 - \int_0^{t_1} f_0(x_1(t), x_2(t), t)\,dt. \quad (15)$$

In the following, instead of minimizing $E[C]$, we discuss the problem of minimizing its upper bound, the right hand side of inequality (15). Furthermore, we assume that there exist:

(i) a fixed upper bound $\bar{u}>0$ to the rate of growth of the checking intensity, $\dot{x}_1(t)$;

(ii) a fixed value $a>0$ for the initial checking intensity. Therefore, the approximate inspection schedule problem may be stated as the following optimal control problem:

NDI: $\max J(u)$

$$J(u)=\int_0^{t_1} f_0(x_1(t), x_2(t), t)\,dt,$$

subject to:

$$\begin{gathered} \dot{x}_1(t)=u(t), \quad x_1(0)=a, \qquad a>0, \\ \dot{x}_2(t)=x_1(t), \quad x_2(0)=0, \\ u(t)\in[0,\bar{u}], \quad \bar{u}>0. \end{gathered}$$

Here x_1 and x_2 are the state functions defined in (13), whereas the control function u is the rate of growth of the checking intensity. From the NDI constraints it follows that

(*a*) the state function $x_1(t)$, the checking intensity, is positive, monotonically non-decreasing, has a bounded derivative and is bounded itself,

$$a \leqq x_1(t) \leqq a+\bar{u}t, \qquad 0 \leqq t \leqq t_1; \tag{16}$$

(*b*) the state function $x_2(t)$, the expected number of checks up to the instant t, is also bounded,

$$at \leqq x_2(t) \leqq (a+\bar{u}t/2)\,t, \qquad 0 \leqq t \leqq t_1. \tag{17}$$

In the following, we will refer to the optimal solution of NDI as the "optimal solution", although it is only an approximate optimal solution of our original problem.

3. OPTIMAL SOLUTION

The Hamiltonian function of the problem NDI is

$$H(\mathbf{x}, u, \mathbf{p}, t) = p_0 f_0(\mathbf{x}, t) + p_1 u + p_2 x_1, \tag{18}$$

where $\mathbf{p}(t) = (p_1(t), p_2(t))$ is the adjoint function and $p_0 \in \{0, 1\}$ is a constant.

THEOREM 3.1: If $(\mathbf{x}^*(t), u^*(t))$ is an optimal pair of the problem NDI, then we have that

$$\mathbf{p}(t_1) = 0 \qquad \text{and} \qquad p_0 = 1; \tag{19}$$

$$u^*(t) = \begin{cases} \bar{u}, & \text{if } p_1(t) > 0, \\ 0, & \text{if } p_1(t) < 0; \end{cases} \tag{20}$$

if u^* is continuous at t, then

$$\dot{p}_1 = -\partial H/\partial x_1 = -l'(1/x_1) f/x_1^2 + c_0(1-F(t_1)) - p_2, \tag{21}$$

$$\dot{p}_2 = -\partial H/\partial x_2 = c_0 f. \tag{22}$$

Proof: The conditions in the statement of the Theorem are Pontryagin's necessary conditions ([11], p. 85), after some straightforward simplifications. In particular, $\mathbf{p}(t_1) = 0$, because $\mathbf{x}(t_1)$ is free, and then $p_0 = 1$, because we must have $(p_0, p_1(t_1), p_2(t_1)) \neq (0, 0, 0)$. ■

From (19) and (22) we get

$$p_2(t)=c_0[F(t)-F(t_1)]. \tag{23}$$

Hence the adjoint equation (21) becomes

$$\dot{p}_1=c_0(1-F(t))-l'(1/x_1)f/x_1^2. \tag{24}$$

We notice that

$$\dot{p}_1(t)\geqq 0 \quad \text{iff} \quad c_0 x_1^2(t)/l'(1/x_1(t))\geqq r(t), \tag{25}$$

where $r(t)=f(t)/(1-F(t))$, known as the "failure rate function" ([12], p. 10), is defined for all $t\in[0, t_1]$, because of (2).

The equations (19) to (22) are only necessary conditions for an optimal solution and Pontryagin's maximum principle does not guarantee that a couple $(\mathbf{x}^*(t), u^*(t))$, satisfying them, is optimal, nor that an optimal solution exists at all. The theorems of Arrow ([11], p. 107) and of Filippov-Cesari ([11], p. 132) give positive answers to the last two questions.

THEOREM 3.2. (sufficiency): If $(\mathbf{x}^*(t), u^*(t))$ satisfies the necessary conditions stated by Theorem 3.1 and if $g(y)=l''(y)/y+2l'(y)\geqq 0$, $y\geqq 0$, then $(\mathbf{x}^*(t), u^*(t))$ is an optimal solution of the checking schedule problem NDI.

Proof: We verify the assumptions of Arrow's sufficiency theorem ([11], p. 107):

If $(\mathbf{x}^*(t), u^*(t))$ is a solution to the Maximum Principle necessary conditions, with the adjoint function $\mathbf{p}(t)$ and the constant p_0, then

(i) $p_0=1$;

(ii) $\hat{H}(\mathbf{x}, \mathbf{p}, t)=\max\{H(\mathbf{x}, u, \mathbf{p}, t): 0\leqq u\leqq \bar{u}\}$

$$=f_0(\mathbf{x}, t)+p_2 x_1+\max\{0, p_1\bar{u}\}.$$

The second derivatives of $\hat{H}$ w. r. t. $\mathbf{x}$ are

$$\partial^2\hat{H}/\partial x_1\,\partial x_2=\partial^2\hat{H}/\partial x_2^2=0,$$

$$\partial^2\hat{H}/\partial x_1^2=-f(t)g(x_1)/x_1^3.$$

Thus if $g(y)\geqq 0$ $(y\geqq 0)$, then $\hat{H}$ is concave in $\mathbf{x}$ for all t. ■

Remark: If $l(y)=c_1 y\,(c_1>0)$, then $g(y)>0$, for all $y\geqq 0$.

THEOREM 3.3. (existence): The checking schedule problem NDI admits an optimal measurable solution $u^*(t)$.

Proof: We verify the assumptions of the Filippov-Cesari theorem ([11], p. 132):

(i) $(\mathbf{x}(t), u(t)) = (a, at, 0)$ is an admissible pair.

(ii) For each $(\mathbf{x}, t) \in (0, +\infty) \times [0, +\infty) \times [0, t_1]$, let $w \in N(\mathbf{x}, t) \subset \mathbb{R}^3$ iff

$$w_1 = f_0(\mathbf{x}, t) + \gamma, \qquad \gamma \leqq 0,$$
$$w_2 = u, \qquad u \in [0, \bar{u}],$$
$$w_3 = x_1.$$

Then $N(\mathbf{x}, t) = (-\infty, f_0(\mathbf{x}, t)) \times [0, \bar{u}] \times \{x_1\}$ and it is clearly convex for all $(\mathbf{x}, t)$.

(iii) The control region $[0, \bar{u}]$ is closed and bounded.

(iv) There exists a number b such that $\mathbf{x}(t) \leqq b$ for all $t \in [0, t_1]$ and all admissible pairs $(\mathbf{x}(t), u(t))$.

In fact, from (16) and (17) we obtain

$$\mathbf{x}(t)^2 \leqq (a + \bar{u}t_1)^2 + (at_1 + \bar{u}t_1^2/2)^2. \quad \blacksquare$$

4. COMMENTS

(*a*) A very special solution is the constant intensity one, $(\mathbf{x}(t), u(t)) = ((a, at), 0)$, because the resulting inspection schedule is a homogeneous Poisson process. In this case, the expected cost of the constant intensity schedule is $E[C] = c_0 - J(0)$. From the Maximum Principle conditions, given by Theorem 3.1, it follows that the constant intensity schedule is optimal if and only if

$$c_0\, a^2 / l'(1/a) \geqq r(t), \quad \text{for all } t \in [0, t_1]. \tag{26}$$

(*b*) If the choice of the initial inspection intensity $x_1(0)$ is not restricted to a given value, as in the problem just discussed, then we are led to formulate the "free initial intensity" version of problem NDI: let us denote it by FNDI. In FNDI, the initial condition

$$x_1(0) > 0 \tag{27}$$

substitutes the old $x_1(0) = a$, $a > 0$ fixed.

If $(\mathbf{x}^*(t), u^*(t))$ is an optimal pair of FNDI, then it has to be admissible, in particular $x_1^*(0) > 0$, it has to satisfy the necessary conditions stated by

Theorem 3.1 and the further condition ([11], p. 185)

$$p_1(0)=0. \tag{28}$$

(*c*) A constant intensity solution $(\mathbf{x}(t), u(t))=((a, at), 0)$ is optimal in the free initial inspection intensity problem if and only if

(i) the condition (26) holds;

(ii) there exists a solution $p_1(t)$ of the adjoint equation

$$\dot{p}_1=c_0(1-F(t))-l'(1/a)f/a^2, \tag{29}$$

which satisfies the boundary conditions $p_1(0)=p_1(t_1)=0$.

(*d*) A remark is needed for the case in which the density function f vanishes on some interval. Let

$$f(t)=0, \quad \text{for all} \ \ t\in[\alpha, \beta]\subset[0, t_1], \tag{30}$$

so that $F(t)=F(\alpha)$, $t\in[\alpha, \beta]$. In this case and for $t\in(\alpha, \beta)$, the adjoint equation (24) reads

$$\dot{p}_1(t)=c_0(1-F(\alpha)). \tag{31}$$

By (2), $F(\alpha)\leqq F(t_1)<1$, so that $\dot{p}_1(t)$ is a positive constant, and $p_1(t)$ is strictly increasing, on $[\alpha, \beta]$.

The case where $\beta=t_1$ is particularly interesting, because we obtain by (19) that

$$p_1(t)<p_1(t_1)=0, \qquad t\in[\gamma, t_1), \tag{32}$$

for some $\gamma\in[0, \alpha)$. Therefore, (20) implies that

$$u^*(t)=0, \qquad t\in(\gamma, t_1)\supset[\alpha, t_1), \tag{33}$$

i.e., the checking intensity $x_1(t)$ must be constant on $[\alpha, +\infty)$. Thus we may reformulate the problem equivalently by considering it on the smaller interval $[0, \alpha]$.

5. EXAMPLES

In order to actually determine the optimal solution of the problem NDI, we are usually forced to resort to numerical techniques. However, a few interesting qualitative results may be obtained analytically in some cases.

This happens, in particular, when the failure rate distribution belongs to the "decreasing failure rate" (DFR) family ([2], pp. 22-41), [3]. In the following examples, we first discuss the general features of the optimal solution when the failure rate distribution is DFR; then we analyze a special case of strict DFR distributions, the Weibull distributions with $\alpha<1$; next we consider the case of the exponential distributions, which belong to the intersection between DFR and "increasing failure rate" (IFR) distributions ([2], pp. 22-41), [3]. Finally, we discuss a special case of strict IFR distribution, the Weibull distribution with $\alpha=3$, which, among the Weibull distributions with $\alpha>1$, is subject to a particularly easy analysis. In all of the examples we assume that the loss function is linear,

$$l(y)=c_1 y, \qquad \text{where} \quad c_1>0. \tag{34}$$

As a consequence, we have that the condition (25) reads as

$$\dot{p}_1(t)\geqq 0 \quad \text{iff } c_0 x_1^2(t)/c_1 \geqq r(t), \tag{35}$$

where $c_0 x_1^2(t)/c_1$ is a monotonic non-decreasing function.

Example 1: Decreasing Failure Rate distributions

If F is a DFR distribution, *i.e.* if the function $r(t)$ is a monotonic non-increasing function, then one of four cases may occur:

1. $c_0 x_1^2(t_1)/c_1<r(t_1)$;
then $\dot{p}_1(t)<0$, $0<t<t_1$.

2. $c_0 a^2/c_1>r(0)$;
then $\dot{p}_1(t)>0$, $0<t<t_1$.

3. $c_0 a^2/c_1=r(0)$;
then $\dot{p}_1(t)=0$, $0\leqq t\leqq t_2$, and $\dot{p}_1(t)>0$, $t_2<t\leqq t_1$, where $t_2=\sup\{t: c_0 x_1^2(t)/c_1=r(t)=r(0)\}$.

4. $c_0 a^2/c_1<r(0)$ and $r(t_1)\leqq c_0 x_1^2(t_1)/c_1$;
then $\dot{p}_1(t)<0$, $t<t_2$, and $\dot{p}_1(t)\geqq 0$, $t\geqq t_2$, for some $t_2\in(0, t_1]$, such that

$$c_0 x_1^2(t_2)/c_1=r(t_2) \quad \text{and} \quad c_0 x_1^2(t)/c_1<r(t), \qquad t<t_2.$$

Case 1: from $p_1(t_1)=0$ it follows that $p_1(t)>0$, all t. Then $u^*(t)=\bar{u}$ and $x_1^*(t)=a+\bar{u}t$, $t\in[0, t_1]$.

The characteristic condition is

$$c_0(a+\bar{u}t_1)^2/c_1<r(t_1).$$

The initial value of the first adjoint function is

$$p_1(0)>0.$$

Case 2: from $p_1(t_1)=0$ it follows that $p_1(t)<0$, for all $t<t_1$. Then $u^*(t)=0$ and $x_1^*(t)=a$, $t\in[0, t_1]$.

The initial value of $p_1(t)$ is $p_1(0)<0$.

Case 3: if $r(t_1)<r(0)$, then

$$c_0 x_1^2(t_1)/c_1>r(t_1) \tag{36}$$

and from $p_1(t_1)=0$ it follows that $p_1(t)<0$, for all $t<t_1$. Then $u^*(t)=0$ and $x_1^*(t)=a$, $t\in[0, t_1]$.

The same result holds if $r(t_1)=r(0)$, *i.e.* if $r(t)$ is constant. In fact, if $u^*(t)>0$ for some t, then $x_1^*(t_1)>a$ and the same condition (36) would hold, giving $u^*(t)=0$ for all t.

The initial value of $p_1(t)$ is $p_1(0)\leqq 0$ and $p_1(0)=0$ iff $r(t)$ is constant.

Case 4: from $p_1(t_1)=0$ it follows that there exists a

$$t_3\in[0, t_2], \quad \text{such that } p_1(t)<0, \qquad t\in(t_3, t_1).$$

Then $u^*(t)=0$ and $x_1^*(t)=x_1^*(t_1)$, $t\in(t_3, t_1]$.

If $t_3=0$, then $u^*(t)=0$ and $x_1^*(t)=a$, $t\in[0, t_1]$.

In the opposite case, if $t_3>0$, then $p_1(t_3)=0$, because p_1 is a continuous function, and $\dot{p}_1(t)<0$ for $t\in[0, t_2)$, which is a neighborhood of t_3. Hence we obtain that $p_1(t)>0$, $t\in[0, t_3)$, so that

$$\begin{aligned} u^*(t)&=\bar{u} \quad \text{and} \quad x_1^*(t)=a+\bar{u}t, && \text{for} \quad t\in[0, t_3],\\ u^*(t)&=0 \quad \text{and} \quad x_1^*(t)=a+\bar{u}t_3, && \text{for} \quad t\in[t_3, t_1]. \end{aligned}$$

The characteristic condition is

$$c_0 a^2/c_1<r(0) \qquad \text{and} \qquad c_0(a+\bar{u}t_1)^2/c_1>r(t_1).$$

The initial value of $p_1(t)$ is $p_1(0)\geqq 0$ and $p_1(0)=0$ iff $t_3=0$.

We recall that the condition $p_1(0)=0$ is the additional necessary condition for the problem FNDI, the free initial inspection intensity one (Section 4). There are only two possible ways in which the optimal solution of NDI is also the optimal solution of FNDI. These are *Case* 3 with a constant $r(t)$, *i.e.* with exponential failure distribution, and *Case* 4, with $t_3=0$. In both cases the optimal solution with free initial inspection intensity is necessarily homogeneous and then it must be the optimal homogeneous PCP, with intensity [6]

$$x_1^*(t)=(c_1 F(t_1))^{1/2}/\{c_0[(1-F(t_1))\,t_1+F(t_1)\,E[T\mid T\leqq t_1]]\}^{1/2}.$$

Example 2: DFR Weibull failure distributions

Let $f(t)=\lambda\alpha t^{\alpha-1}\exp(-\lambda t^{\alpha})$, $\lambda>0$ and $0<\alpha<1$. The failure rate function is $r(t)=\lambda\alpha t^{\alpha-1}$: it is a decreasing function and $\lim_{t\to 0^+} r(t)=+\infty$. From the analysis of the Example 1, we obtain that

(i) $a\geqq(c_1\lambda\alpha t_1^{\alpha-1}/c_0)^{1/2}-\bar{u}t_1$, then

$$u^*(t)=\bar{u} \quad \text{and} \quad x_1^*(t)=a+\bar{u}t, \qquad t\in[0,t_3],$$

$$u^*(t)=0 \quad \text{and} \quad x_1^*(t)=a+\bar{u}t_3, \qquad t\in(t_3,t_1].$$

As for the time parameter t_3 we have that

either $t_3\geqq 0$ and $\varphi(t_3)=0$,

or $t_3=0$ and $\varphi(0)>0$,

where

$$\varphi(t)=\int_t^{t_1}\{c_0[1-F(w)]-c_1 f(w)/(a+\bar{u}t)^2\}\,dw.$$

(ii) $a<(c_1\lambda\alpha t_1^{\alpha-1}/c_0)^{1/2}-\bar{u}t_1$, then

$$u^*(t)=\bar{u} \quad \text{and} \quad x_1^*(t)=a+\bar{u}t, \qquad \text{for} \quad t\in[0,t_1].$$

The (sub-) optimal PCP is homogeneous iff (i) occurs with $t_3=0$.

Example 3: Exponential failure distributions

This is the case of Weibull distributions with $\alpha=1$.

$$f(t)=\lambda e^{-\lambda t}, \qquad \lambda>0;$$

the failure rate function is $r(t)=\lambda$, constant. From the analysis of Example 1 we obtain that

(i) $a\geqq(c_1\lambda/c_0)^{1/2}$, then

$$u^*(t)=0 \quad \text{and} \quad x_1^*(t)=a, \qquad t\in[0,t_1].$$

(ii) $a<(c_1 \lambda/c_0)^{1/2}<a+\bar{u}t_1$, then

$$u^*(t)=\bar{u} \quad \text{and} \quad x_1^*(t)=a+\bar{u}t, \qquad t\in[0, t_3],$$
$$u^*(t)=0 \quad \text{and} \quad x_1^*(t)=a+\bar{u}t_3, \qquad t\in(t_3, t_1],$$

where $t_3=((c_1 \lambda/c_0)^{1/2}-a)/\bar{u}$, so that $a+\bar{u}t_3=(c_1 \lambda/c_0)^{1/2}$.

(iii) $a\leqq(c_1 \lambda/c_0)^{1/2}-\bar{u}t_1$, then

$$u^*(t)=\bar{u} \quad \text{and} \quad x_1^*(t)=a+\bar{u}t, \qquad t\in[0, t_1].$$

The optimal solution of the free initial inspection intensity problem is

$$u^*(t)=0 \quad \text{and} \quad x_1^*(t)=(c_1 \lambda/c_0)^{1/2}, \qquad t\in[0, t_1].$$

Example 4: IFR Weibull failure distributions ($\alpha=3$)

Let $f(t)=3\lambda t^2 \exp(-\lambda t^3)$, $\lambda>0$. The failure rate function is increasing: $r(t)=3\lambda t^2$. The condition (25) is equivalent to

$$\dot{p}_1(t)\geqq 0 \quad \text{iff} \quad x_1(t)\geqq \beta t, \qquad \beta=(3 c_1 \lambda/c_0)^{1/2}.$$

We obtain that

(i) if $a\geqq \beta t_1$, then

$$u^*(t)=0 \quad \text{and} \quad x_1^*(t)=a, \qquad t\in[0, t_1];$$
$$p_1(0)<0.$$

(ii) if $a<\beta t_1$ and $\bar{u}\geqq \beta\, (a>0)$, then

$$u^*(t)=0 \quad \text{and} \quad x_1^*(t)=a, \qquad t\in[0, a/\beta],$$
$$u^*(t)=\beta \quad \text{and} \quad x_1^*(t)=a+\beta(t-a/\beta), \qquad t\in(a/\beta, t_1],$$
$$p_1(0)<0.$$

(iii) if $a<\beta t_1$ and $\bar{u}<\beta\, (a>0)$, then

$$x_1(t)<\beta t, \qquad t\in(t_2, t_1],$$

for some $t_2<t_1$; hence it follows that

$$\dot{p}_1(t)<0, \qquad t\in(t_2, t_1),$$
$$p_1(t)>0, \qquad t\in(t_3, t_1),$$

for some $t_3 < t_2$. Now, either $t_3 = 0$, and then

$$p_1(0) \geqq 0,$$
$$u^*(t) = \bar{u} \quad \text{and} \quad x_1^*(t) = a + \bar{u}t, \qquad t \in [0, t_1];$$

or $t_3 > 0$, and then

$$p_1(t) < 0, \qquad t \in [0, t_3),$$
$$u^*(t) = 0 \quad \text{and} \quad x_1^*(t) = a, \qquad t \in [0, t_3],$$
$$u^*(t) = \bar{u} \quad \text{and} \quad x_1^*(t) = a + \bar{u}(t - t_3), \qquad t \in (t_3, t_1].$$

The condition $p_1(0) = 0$, which is necessary for the problem FNDI (free initial inspection intensity), can only be satisfied if $\bar{u} < \beta$. Then we must have that

$$u^*(t) = \bar{u} \quad \text{and} \quad x_1^*(t) = a + \bar{u}t, \qquad t \in [0, t_1],$$
$$p_1(0) = p_1(t_1) = 0.$$

There exists at most one $a < \beta t_1$, such that these conditions hold.

6. CONCLUSIONS

The problem of minimizing expected cost until detection of failure, using random inspection (checking) schedules, has been the subject of the present paper. We have investigated the behavior of non-homogeneous Poisson checking processes with continuous non-decreasing intensity and obtain a rather natural approximation of the original problem, in the form of an optimal control problem. From its discussion we find that it admits an optimal solution and that the necessary conditions, given by the Pontryagin's Maximum Principle, are also sufficient to characterize any optimal solution. Then these conditions are used in a number of examples, where the distribution of the first failure time has a monotone failure rate function. The most interesting results concern the class of "decreasing failure rate" distributions. On the other hand, a more complex discussion is required by "increasing failure rate" distributions.

In order to obtain effective solutions in practical problems, numerical procedures should be investigated and tested by simulation.

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