

M. BRAGLIA

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## M/M/1 QUEUING MODEL WITH ORDINARY MAINTENANCE AND BREAKDOWNS (\*)

by M. BRAGLIA <sup>(1)</sup>

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**Abstract.** — *An M/M/1 system with ordinary maintenance and possibility of breakdowns is considered from a practical point of view. An appropriate equation for the probability of n units in the system at any instant of time is written in terms of the Green function of the conventional M/M/1 model and solved by an iterative technique which imposes a periodic regime state. Some Monte Carlo simulations are used to assess the accuracy of the iterative process, i. e. verify that it really converges to the correct solution. Special attention is turned to the analysis of the error that is introduced if the different breakdowns that may occur during the service interval are replaced by a single cumulative waste of time. Numerical results are reported for the special cases in which this cumulative breakdown is placed at the beginning or at the end of the service interval, or is uniformly distributed. To some extent, part of the analysis is deliberately approximated, even if good accuracy is always preserved. The extension to the M/G/1 model is also considered.*

**Keywords:** Queuing system, Breakdowns, Green's function.

**Résumé.** — *Dans cet article on propose une approche pratique pour étudier un système markovien de files d'attente M/M/1 avec entretien ordinaire et la possibilité de pannes. On donne une équation convenable pour la probabilité d'avoir n unités dans le système à n'importe quel temps laquelle est écrite en termes de la fonction de Green du modèle M/M/1 conventionnel et résolue avec une technique itérative qui impose un régime d'état stationnaire. Quelques simulations Monte Carlo sont employées pour prouver la précision du procès itérative, c'est-à-dire, pour vérifier la convergence à la solution correcte. Une attention spéciale est adressée à l'analyse de l'erreur qui est introduit si on considère les pannes qui peuvent passer dans l'intervalle de service comme une seule perte de temps cumulative. Les résultats numériques sont exposés dans les suivantes cas particuliers : quand on place cette perte cumulative à la fin de l'intervalle de service, au début de cet intervalle ou si les pannes sont uniformément distribuées. Jusqu'à un certain point, le calcul est intentionnellement approximatif même si une bonne précision est toujours préservée. L'extention au système M/G/1 est aussi considérée.*

**Mots clés :** Files d'attente, pannes, fonction de Green.

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(<sup>1</sup>) Università' di Brescia, Facoltà' di Ingegneria, Dipartimento di Meccanica, Via Valotti, 9, 25060 Brescia (Italy).

## 1. INTRODUCTION

The problem of extending queue models, from the simplest  $M/M/1$  to the general  $G/G/1$ , to take account of possible breakdowns of the service facility with consequent loss of time was already considered in the 1950's. Generally, the queuing process is considered as a preemptive resume priority process. That is, breakdowns are treated as fictitious units of absolute priority with respect to the real jobs [1, 4, 9, 12], and characterized by inter-arrival and service times exponentially distributed. When a breakdown occurs, *i. e.* a fictive job arrives, the real unit that is under service is temporarily abandoned. It will be reconsidered just after the "breakdown" is served [8]. With such a technique various kinds of breakdowns have been studied. In fact, a queue of fictive jobs may be permitted [7, 12] or not [11, 12]. (In the latter case a breakdown cannot occur when the system is out of order.) Analogously, a breakdown may be permitted even if there is no unit in the system, or only if the station is giving service [6, 11]. And so on. In this paper, we consider the problem of breakdowns in an  $M/M/1$  model using a quite different and, for certain aspects, new and convenient approach. Our treatment of breakdowns is rigorous only in part, but simple and sufficiently accurate for a variety of problems. To extend its validity to practical situations, an ordinary (*i. e.* programmed) maintenance is also introduced in our system model.

The approach is based on the *fundamental solution* (*i. e.* Green function) of the (time-dependent) equation for the probability of  $n$  units at time  $t$  in a conventional  $M/M/1$  system model and there is no recourse to fictitious jobs to stimulate breakdowns. Our technique is based on the solution of an appropriate equation for the probability  $p(n, t)$  of  $n$  jobs in the system at the generic instant  $t$ . In particular, it permits to calculate  $p(n, t)$  at instants of special interest, *e. g.* the beginning and the end of the ordinary maintenance, when the populations in the system are at the lowest and highest levels, respectively. The required solution is obtained by an iterative process which imposes a periodic regime of the system with a period of  $T_S + T_M$ ,  $T_S$  and  $T_M$  being the intervals of service and ordinary maintenance, respectively. However, the system is considered during its temporal evolution and, in principle,  $p(n, t)$  can be obtained at any instant.

The paper is structured as follows. In section 2 the basic model with ordinary maintenance but absence of breakdowns is presented and the results of iterative solutions and corresponding Monte Carlo simulations are

discussed. The possibility of a single breakdown with exponential repair time is introduced in section 3, while in section 4 the possibility of multiple breakdowns is considered. In the latter case the problem is solved under the assumption of a constant (average) repair time  $\tau$ . As a first step, the breakdowns are replaced by a single cumulative breakdown and the relevant repair time is placed at the end of the service time  $T_S$ . As regards the assumption of constant repair times, apart from section 3 it is preserved in all sections of this paper. In fact, on the basis of our results it appears that a possible extension of the theory to include times distributed according to a given law would complicate the calculations with scarcely significant improvements. Moreover, little of particular interest would be added to theory and results. On the contrary, the considerations which follow from the assumption of a single cumulative interruption placed at the end of the service time  $T_S$  appear very interesting. It is evident that the procedure permits a considerable simplification of the problem, *i. e.* of considering the repair time simply as an extension of the ordinary maintenance time  $T_M$ . In fact, in this way we return to the model of section 2. For this reason we have devoted section 5 to the analysis of the limits of such a simplification. The problem is faced by placing first the total repair time also at the beginning of the service time  $T_S$ , *i. e.* when the system is in the worst condition to go out of order, with a queue that is most populated. The distribution  $p(n, t)$  is then obtained in a number of cases. The comparison of the solutions relevant to the two mentioned extreme cases is found of special interest. In fact, it shows that the discrepancies are generally small. But, as with breakdowns at the end or at the beginning of  $T_S$  the distributions  $p(n, t)$  are almost the same, with better reason one must expect that this is true when the breakdowns are distributed exactly as they occur. To confirm this conclusion in section 5 it is also considered a third case with the breakdowns uniformly distributed in  $(0, T_S)$ . As we will see, there is almost no difference between this last case and that with breakdowns at the beginning of the service time. Needless to say, these results suggest a number of interesting simplifications of the theory of systems with breakdowns. In fact, they reveal that complicated rigorous approaches are not needed since they offer slight improvements over simpler properly approximated models. Section 5 is concluded with some Monte Carlo simulations to assess the accuracy of the theoretical distributions. To complete the analysis, in section 6 the attention is turned to a discussion of the possible validity of similar results for more general  $M/G/1$  systems. Finally, some conclusive remarks are made in section 7.

## 2. M/M/1 MODEL WITH ORDINARY MAINTENANCE

Consider the single-server  $M/M/1$  model with inter-arrival and service times exponentially distributed. Let  $\lambda$  and  $\mu$  be the corresponding frequencies. First of all, we want to introduce a programmed maintenance which interrupts the service for a time  $T_M$  after each service period  $T_S$ . Thus, the service rate  $\mu(t)$  will have the behaviour given in figure 1.

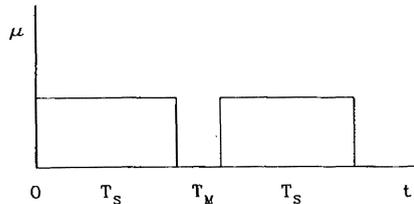


Figure 1. – Temporal behaviour of the service rate.

Our problem is that of calculating the probability  $p(n, t)$  to have  $n$  jobs in the system at the generic instant  $t$  and, particularly, at the instant  $T_S + T_M$  of highest population.

To this end, consider the fundamental solution of the conventional  $M/M/1$  model which corresponds to the initial value  $n=i$ , at time  $t=0$  [5]. We will give this solution the following form

$$\begin{aligned}
 p(n, \vartheta) &= e^{-(1+\rho)\vartheta} \left[ \rho^{(n-i)/2} I_{n-i}(2\sqrt{\rho}\vartheta) + \rho^{(n-i-1)/2} I_{n+i+1}(2\sqrt{\rho}\vartheta) \right. \\
 &\quad \left. + (1-\rho)\rho^n \sum_{l=n+i+2}^{\infty} (1/l)^{1/2} I_l(2\sqrt{\rho}\vartheta) \right] \\
 p(n, 0) &= \delta_{in}
 \end{aligned} \quad (1)$$

where, for the sake of brevity, we have introduced the quantities

$$\vartheta = \mu t, \quad \rho = \lambda/\mu, \quad \delta_{in} = \text{delta of Kronecker}$$

$$I_n = \sum_{k=0}^{\infty} \frac{(y/2)^{n+2k}}{k!(n+k)!} \quad (n > -1) \quad \left\{ \begin{array}{l} \text{Modified Bessel function} \\ \text{of the first kind} \end{array} \right\}$$

Of course,

$$p(n, \vartheta) = p(n, \vartheta | i, 0) = \text{transition probability from } i \text{ to } n \\ \text{in the reduced time } \vartheta.$$

Therefore, in the absence of maintenance, for any initial distribution  $p(l, 0)$  we can write that

$$\left. \begin{aligned} p(n, \vartheta) &= \bar{p}(n, \vartheta) \equiv \sum_{l=0}^{\infty} p(n, \vartheta | l, 0) p(l, 0) \\ \forall p(l, 0) &\geq 0 \quad \text{with} \quad \sum_{l=0}^{\infty} p(l, 0) = 1 \end{aligned} \right\} \quad (2)$$

Concerning the distribution at the end of the ordinary maintenance, that is at  $\vartheta_S + \vartheta_M = \mu T_S + \mu T_M$ , we will have

$$p(n, \vartheta_S + \vartheta_M) = \sum_{k=0}^{\infty} \bar{p}(k, \vartheta_S) \frac{(\rho \vartheta_M)^{n-k}}{(n-k)!} e^{-\rho \vartheta_M} \quad (3)$$

where  $\lambda T_M = \rho \vartheta_M$ . Thus, substitution of eq. (2) into (1) yields

$$p(n, \vartheta_S + \vartheta_M) = \sum_{k=0}^n \sum_{l=0}^{\infty} p(k, \vartheta_S | l, 0) p(l, 0) \frac{(\rho \vartheta_M)^{n-k}}{(n-k)!} e^{-\rho \vartheta_M} \quad (4)$$

The regime state of our model implies the periodicity condition

$$p(n, \vartheta_S + \vartheta_M) = p(n, 0) \quad (5)$$

In fact, *this distribution is the unknown of the problem*. It can be obtained from eq. (4) by imposing the periodicity condition. A possible and convenient technique is that of solving eq. (4) iteratively, once the transition matrix  $p(n, \vartheta_S | l, 0)$  is obtained from eq. (1). Some difficulties must be overcome particularly when calculating the Bessel functions. But good “numeral recipes” are available in the literature [10]. The procedure is simple and goes on as follows:

### First step

A generic initial distribution  $p(l, 0)$  is assumed. For instance, we may suppose that no unit is in the system at time  $t=0$ , in which case

$$p(l, 0) = {}_0p(l, 0) = \delta_{l,0}$$

Alternatively, the initial distribution may be assumed to be the steady-state distribution of the  $M/M/1$  model, that is [5]

$$p(l, 0) = {}_0p(l, 0) = \rho^l (1 - \rho)$$

As a first step, the distribution  ${}_0\bar{p}(n, \vartheta_S)$  is calculated from eq. (2). Then, from eq. (3), the distribution  ${}_0p(n, \vartheta_S + \vartheta_M)$ , at the final instant  $\vartheta_S + \vartheta_M$ , can be obtained.

### Successive steps

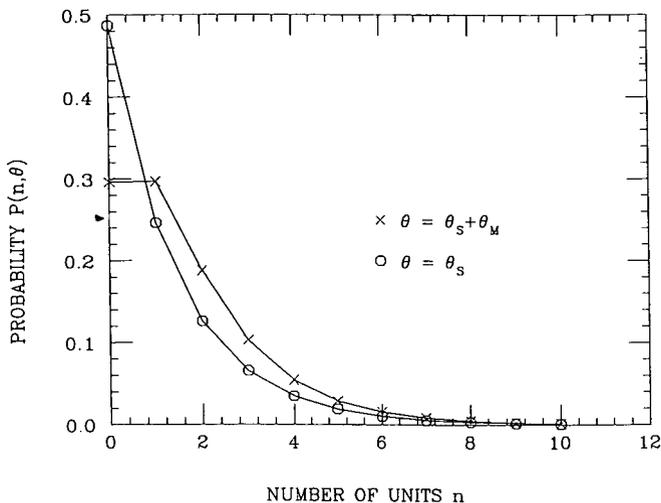
The periodicity condition (5) is imposed. It requires that

$${}_1p(l, 0) = {}_0p(l, \vartheta_S + \vartheta_M)$$

Then, we return to the first step. The iteration ends when

$${}_{(k-1)}p(l, 0) = {}_kp(l, \vartheta_S + \vartheta_M)$$

within the desired precision limits. As an example, in figure 2 we report the (steady-state) distributions  $p(n, \vartheta_S)$  and  $p(n, \vartheta_S + \vartheta_M) = p(n, 0)$  which



**Figure 2. – Probability distributions  $p(n, \vartheta_S)$  and  $p(n, \vartheta_S + \vartheta_M)$  of  $n$  costumers in the system at  $\vartheta_S$  and  $\vartheta_S + \vartheta_M$  for  $\rho = \lambda/\mu = 0.5$ ,  $\vartheta_S = 10$  and  $\vartheta_M = 1$ .**

are obtained for  $\rho = 0.5, \vartheta_S = 10, \vartheta_M = 1$ . Of course, from eqs. (2) to (4) the probability  $p(n, \vartheta)$  at any instant  $\vartheta$  from 0 to  $\vartheta_S + \vartheta_M$  can also be obtained.

In the special case when the service time  $\vartheta_S$  is higher than the relaxation time of the initial distribution  $p(l, 0)$ , further analytical steps can be made. In fact, instead eq. (2), we will have that

$$p(n, \vartheta_S) = \bar{p}(n, \vartheta_S) = (1 - \rho) \rho^n, \quad \forall p(l, 0)$$

Thus, from eq. (3), it will follow that

$$\begin{aligned} p(n, \vartheta_S + \vartheta_M) &= \sum_{k=0}^n (1 - \rho) \rho^k \frac{(\rho \vartheta_M)^{n-k}}{(n-k)!} e^{-\rho \vartheta_M} \\ &= (1 - \rho) \rho^n e^{-\rho \vartheta_M} \sum_{k=0}^n \frac{\vartheta_M^{n-k}}{(n-k)!} = p(n, 0) \end{aligned}$$

In this case, eq. (4) yields

$$p(n, \vartheta_S + \vartheta_M) = (1 - \rho) \rho^n e^{-\rho \vartheta_M} \sum_{l=0}^{\infty} \sum_{k=0}^l p(n, \vartheta_S | l, 0) \frac{\vartheta_M^{n-k}}{(l-k)!}$$

Notice, as a first immediate check, that  $p(n, \vartheta_S + \vartheta_M) = p(n, 0)$  is normalized to one, as expected. In fact,

$$\begin{aligned} p(n, \vartheta_S + \vartheta_M) &= (1 - \rho) \rho^n e^{-\rho \vartheta_M} \sum_{k=0}^n \frac{\vartheta_M^{n-k}}{(n-k)!} = (1 - \rho) \rho^n e^{-\rho \vartheta_M} \\ &\cdot \sum_{k=0}^n \frac{\vartheta_M^k}{k!} = (1 - \rho) e^{-\rho \vartheta_M} \rho^n \left( 1 + \vartheta_M + \frac{\vartheta_M^2}{2!} + \dots + \frac{\vartheta_M^n}{n!} \right) \end{aligned}$$

Then,

$$\begin{aligned} \sum_{n=0}^{\infty} p(n, \vartheta_S + \vartheta_M) &= (1 - \rho) e^{-\rho \vartheta_M} \left( 1 + \rho(1 + \vartheta_M) + \rho^2 \left( 1 + \vartheta_M + \frac{\vartheta_M^2}{2!} \right) + \dots \right) \\ &= (1 - \rho) e^{-\rho \vartheta_M} \left( \frac{1}{1 - \rho} + \left( \frac{\vartheta_M}{1 - \rho} - \vartheta_M \right) \right) \\ &+ \left( \frac{\vartheta_M^2}{2(1 - \rho)} - \frac{\vartheta_M^2}{1 - \rho} - \rho \frac{\vartheta_M^2}{2} \right) + \dots = (1 - \rho) e^{-\rho \vartheta_M} \frac{1}{1 - \rho} e^{\rho \vartheta_M} = 1 \end{aligned}$$

In the same way, it is easily proved that  $\bar{p}(n, \vartheta_S)$  is normalized to one.

To conclude it will be noted that the distributions  $p(n, 0) = p(n, \vartheta_S + \vartheta_M)$  as calculated from eqs. (4) and (5) for different values of  $\rho, \vartheta_S$  and  $\vartheta_M$ , have

been compared with the corresponding distributions obtained by Monte Carlo simulations of  $10^5$  samples, *i. e.* successive time intervals  $(0, \vartheta_S + \vartheta_M)$ . The agreement has always been quite good. In fact, no difference would generally be observed between Monte Carlo and theoretical distributions when using the scale of figure 2. This conclusion is important as not only is confirmed that our iterative process converges but even that it really converges to the correct distribution. Anyway, we will return to this subject below (see *fig. 5*).

### 3. M/M/1 MODEL WITH ORDINARY AND SINGLE EXTRAORDINARY MAINTENANCE

Up to this point, we have assumed that our system does not suffer breakdowns in the interval  $(0, \vartheta_S)$ . But we want to generalize the theory and introduce the possibility of service interruptions. In this case, even when a single breakdown is permitted, the analytical treatment becomes much heavier. So, for instance, suppose that for the only possible breakdown

$$q_R(t) dt = \gamma e^{-\gamma t} dt = \text{probability it happens in } (t, t + dt).$$

Moreover, let the duration of this extraordinary maintenance be exponentially distributed with time-constant  $\eta^{-1}$  (For the sake of brevity, when convenient we avoid to dwell on reduced times. Alternatively, say that here we assume  $\mu=1$ ). If  $p_1(n, t)$  is the probability of  $n$  jobs in the system at time  $t$ , then

$$p_1(n, T_S) = e^{-\gamma T_S} \bar{p}(n, T_S) + \int_0^{T_S} \gamma e^{-\gamma \xi} d\xi \int_0^{T_S - \xi} \eta e^{-\eta t} \left[ \sum_{k=0}^n \bar{p}(k, T_S - t) \frac{(\lambda t)^{n-k}}{(n-k)!} e^{-\lambda t} \right] dt \quad (6)$$

where  $\bar{p}(n, T_S)$  is the distribution at time  $T_S$  in the absence of breakdown [*see eq. (2)*]. Note that, if  $\xi$  is the instant of breakdown, then:

1. the excess of repair time when  $t > (T_S - \xi)$  is treated as ordinary maintenance, and

2. the repair times  $T_R$  are never permitted to exceed  $T_S + T_M - \xi$ .

Finally, note that in eq. (6) the repair time is localized at the end of the service time, that is just before  $T_M$ . In this way, we are tacitly assuming that only the value of  $T_R$  is important, while it does not matter when the breakdown happens. In other words, the service interval  $(0, T_S)$  may be

replaced by  $(0, T_S - T_R)$ . Alternatively, we could localize  $T_R$  at the beginning and assume  $(T_R, T_S)$  as service interval. We will discuss the limitations of such assumptions in section 5.

In the context of an approximate theory of  $M/M/1$  model with ordinary maintenance and breakdowns, we can further simplify the approach by introducing a mean repair time  $\tau$ , defined as follows (assume  $\gamma \neq \eta$ )

$$\begin{aligned} \tau = \langle t \rangle &= \int_0^{T_S} \gamma e^{-\gamma\xi} d\xi \int_0^{T_S-\xi} \eta t e^{-\eta t} dt \\ &= \frac{\gamma e^{-\eta T_S}}{\gamma - \eta} \left[ \frac{2\eta - \gamma}{\eta(\gamma - \eta)} (e^{-T_S(\gamma-\eta)} - 1) - T_S \right] \end{aligned}$$

and then giving eq. (6) the form

$$\begin{aligned} p_1(n, T_S) &= e^{-\gamma T_S} \bar{p}(n, T_S) + (1 - e^{-\gamma T_S}) \sum_{k=0}^n \bar{p}(k, T_S - \tau) \frac{(\lambda\tau)^{n-k}}{(n-k)!} e^{-\lambda\tau} \end{aligned}$$

But the assumption of constant repair time permits an extended analytical treatment of the problem, while preserving a relative formal simplicity of the theory. This will be done in the successive sections.

#### 4. M/M/1 MODEL WITH ORDINARY MAINTENANCE AND BREAKDOWNS WITH CONSTANT REPAIR TIMES

In principle, there is no difficulty to extend the preceding theory to include the possibility of more than one breakdown with exponential inter-occurrence and duration times. However, both from the theoretical and practical points of view, the situation can be greatly simplified if the assumption is made of constant repair time. Let us extend, then, our model to include the possibility of  $k$  breakdowns in the interval  $(0, T_S)$ , each one causing service interruption for a time  $\tau$ . Even in this case, the system may be approximated by an  $M/M/1$  model with the ordinary maintenance increased from  $\vartheta_M$  to  $\vartheta_M + k\tau$ . In this section we will accept this point of view, but we will return to the question in section 5.

At this point, for our purposes it becomes necessary to calculate the mean number of breakdowns per service interval or, even better, the probability  $Q_K(T_S)$  of  $k$  breakdowns in  $(0, T_S)$ . This will be done under the assumption that the probability of a breakdown occurring in  $(t, t+dt)$  is  $\gamma e^{-\gamma t} dt$ . Note

that the question is not of immediate solution. In fact, if  $\tau \rightarrow 0$ , we may use the Poisson law

$$Q_k(T_S) = \frac{(\gamma T_S)^k}{k!} e^{-\gamma T_S}$$

But for  $\tau \neq 0$ , it is necessary to take account that a second breakdown can only occur in the time interval  $(t+\tau, T_S)$ , if the first one occurred at time  $t$ . In fact, there is no service between  $t$  and  $t+\tau$ . This complicates the theory, particularly if more than two breakdowns are possible in  $(0, T_S)$ . A further difficulty originates from the possibility that the repair time ends between  $T_S$  and  $T_S+T_M$ . This happens when a breakdown occurs in  $(T_S-\tau, T_S)$ . In this case, only a fraction of  $\tau$  must be subtracted from  $T_S$  while the remainder must be neglected, *i. e.* considered as ordinary maintenance. Finally, it will be convenient to assume that  $\tau \leq T_M$ , in order to avoid that the repair time extends beyond  $T_S+T_M$ .

#### 4.1. Probability $Q_k$ of $k$ breakdowns in the time interval $(0, T_S)$

The probability that in  $(0, t)$  no breakdown occurs is

$$Q_0(t) = e^{-\gamma t}$$

independently of  $\tau$ .

As regards the probability of one breakdown in  $(0, t)$ , we have

$$\begin{aligned} Q_1(t) &= \int_{\xi} Q_0(\xi) \gamma d\xi Q_0(t - (\xi + \tau)) \\ &= \int_0^{t-\tau} e^{-\gamma \xi} \gamma d\xi e^{-\gamma(t - (\xi + \tau))} + \int_{t-\tau}^t e^{-\gamma \xi} \gamma d\xi \end{aligned}$$

As one can see, this equation considers the probability that a breakdown occurs in  $(\xi, \xi+d\xi)$  and no breakdown occurs both in  $(0, \xi)$  and  $(\xi+\tau, t)$ , for any  $\xi \in (0, t)$ . Moreover, it is taken into account that

$$Q_0(t - (\xi + \tau)) = \begin{cases} e^{-\gamma(t - (\xi + \tau))}, & \xi < t - \tau \\ 1, & t - \tau \leq \xi \leq t \end{cases}$$

In fact, it is certain that further breakdowns cannot occur if  $\xi \in (t - \tau, t)$ . Thus, after some calculations, it is found that

$$Q_1(t) = \gamma(t - \tau) e^{-\gamma(t - \tau)} + e^{-\gamma(t - \tau)} (1 - e^{-\gamma \tau})$$

It is interesting to observe that if we treat the second term of the right hand side as a correction, to the first order in  $\gamma\tau$  we have

$$Q_1^{(1)}(t) = \gamma t e^{-\gamma(t-\tau)}$$

which corresponds to assuming that

$$Q_1(t) = \int_0^t e^{-\gamma\xi} \gamma d\xi e^{-\gamma(t-(\xi+\tau))}$$

or, alternatively, that

$$Q_0(t - (\xi + \tau)) = e^{-\gamma(t-(\xi+\tau))}, \quad \forall \xi \in (0, t)$$

If the above expansion is extended to the second order, we have

$$Q_1^{(2)}(t) = Q_1^{(1)}(t) - \frac{1}{2}(\gamma\tau)^2 e^{-\gamma(t-\tau)}$$

which represents already a very good approximation to  $Q_1(t)$  if  $\gamma\tau$  is small.

By similar arguments, for  $t \geq \tau$ ,  $Q_2(t)$  can be given the form where terms

$$\begin{aligned} Q_2(t) &= \int_{\xi} \int_{\chi} Q_0(\xi) \gamma d\xi Q_0(\chi - (\xi + \tau)) \gamma d\chi Q_0(t - (\chi + \tau)) \\ &= \int_0^{t-2\tau} d\xi \left\{ \int_{\xi+\tau}^{t-\tau} e^{-\gamma\xi} \gamma e^{-\gamma(\chi-(\xi+\tau))} \gamma d\chi e^{-\gamma(t-(\chi+\tau))} \right. \\ &\quad \left. + \int_{t-\tau}^t \gamma e^{-\gamma\xi} e^{-\gamma(\chi-(\xi+\tau))} \gamma d\chi \right\} + \int_{t-2\tau}^{t-\tau} \gamma d\xi \int_{\xi+\tau}^t e^{-\gamma(\eta-\tau)} \gamma d\eta \\ &= \frac{\gamma^2(t-2\tau)^2}{2!} e^{-\gamma(t-2\tau)} + \{\gamma(t-2\tau) + 1\} e^{-\gamma(t-2\tau)} \\ &\quad - \{\gamma(t-\tau) + 1\} e^{-\gamma(t-\tau)} \end{aligned}$$

In this case, as first and second approximations we may assume

$$Q_2^{(1)}(t) = \frac{\gamma^2(t-\tau)^2}{2} e^{-\gamma(t-2\tau)}$$

$$Q_2^{(2)}(t) = Q_2^{(1)}(t) - \frac{1}{2}(\gamma\tau)^2 \gamma t e^{-\gamma(t-2\tau)}$$

of the order  $(\gamma\tau)^3/3!$  have also been neglected.

In the same way, for  $t \geq 3\tau$ , it is found that

$$Q_3(t) = \frac{\gamma^3(t-3\tau)^3}{3!} e^{-\gamma(t-3\tau)} + \left\{ \frac{\gamma^2(t-3\tau)^2}{2!} + \frac{\gamma(t-3\tau)}{1!} + 1 \right\} e^{-\gamma(t-3\tau)} - \left\{ \frac{\gamma^2(t-2\tau)^2}{2!} + \frac{\gamma(t-2\tau)}{1!} + 1 \right\} e^{-\gamma(t-2\tau)}$$

$$Q_3^{(1)}(t) = \frac{\gamma^3(t-2\tau)^3}{3!} e^{-\gamma(t-3\tau)}$$

$$Q_3^{(2)}(t) = Q_3^{(1)}(t) - \frac{1}{2} (\gamma\tau)^2 \frac{\gamma^2 t^2}{2} e^{-\gamma(t-3\tau)}$$

At this point, we could proceed to the calculation of the successive  $Q_k(t)$ 's. In this respect, the following recursive equation could be useful

$$Q_k(t) = \int_0^{t-(k-1)\tau} Q_0(\xi) Q_{k-1}(t - (\xi + (k-1)\tau)\gamma) d\xi$$

even if it is not difficult to infer the expression of  $Q_k(t)$  from those found for  $k \leq 3$  given above. However, in practice, it is quite reasonable to assume that breakdowns are not so frequent, in virtue of the ordinary maintenance, which is introduced just to this end. Under these conditions, a convenient approximation is that of assuming the simple form

$$Q_k(t) = Q_k^{(1)}(t) = \frac{(\gamma(t - (k-1)\tau))^k}{k!} e^{-\gamma(t-k\tau)}$$

Otherwise, we can also use the second order representation and write that

$$Q_k(t) = Q_k^{(2)}(t) = Q_k^{(1)}(t) + C_k(t)$$

$$C_k(t) = -\frac{1}{2} (\gamma\tau)^2 \frac{(\gamma t)^{k-1}}{(k-1)!} e^{-\gamma(t-k\tau)}$$

This assumption is also justified by the fact that the expressions of  $Q_k(t)$  are not complete. They require  $t > k\tau$ . For simplicity, we have not considered the possibility that  $(k-1)\tau \leq t \leq k\tau$ . In this interval, it can be proved that

$$Q_k(t) = 1 - \sum_{j=0}^{k-1} \frac{\gamma^j (t - (k-1)\tau)^j}{j!} e^{-\gamma(t-(k-1)\tau)}$$

But we do not expect these events to have significant probability and much importance under our conditions, otherwise one must use the rigorous  $Q_k(t)$ .

For our calculations, we have adopted the rigorous representations of the  $Q_k(t)$ 's and the approximations of various orders given above without observing any appreciable difference. From the numerical point of view, the normalization condition  $\sum_k Q_k=1$  is satisfied, in any case, with sufficient approximation (say, from about  $10^{-4}$  to  $10^{-5}$  for reasonable values of  $\gamma$ ,  $\tau$ ,  $T_S$ ). To give an example, in figure 3, we report some distributions  $Q_k(T_S)$  obtained for different values of  $\gamma$  and  $\tau$ .

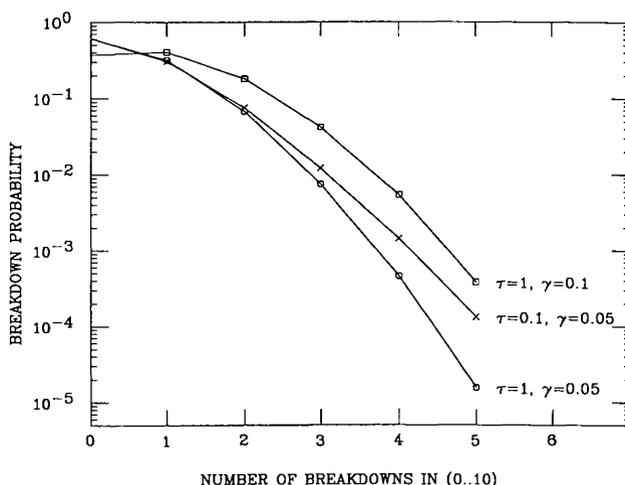


Figure 3. - Probability  $Q_k(T_S)$  of  $k$  breakdowns in the service interval  $(0, T_S)$  for different values of the repair time ( $\tau$ ) and the breakdown frequency  $\gamma$ . (Arbitrary units.)

#### 4.2. Mean time of service reduction for $k$ breakdowns

Having calculated the  $Q_k(T_S)$ 's, we are now interested to answer a further question, that is: how much is shortened, on the average, the service time? This problem can be reduced to that of calculating the mean service-time reduction relevant to the latest breakdown. In fact, if many breakdowns are possible, only the last one has a repair time  $\tau$  which may extend beyond  $T_S$ , in the interval of ordinary maintenance.

If we assume that all instants of service are equally probable for breakdown (in accord with the assumptions of section 4.1), in case of single breakdown

we write that,

$$\frac{1 - e^{-\gamma(T_S - \tau)}}{1 - e^{-\gamma T_S}} \approx \frac{T_S - \tau}{T_S} \approx \text{probability of breakdown in } (0, T_S - \tau)$$

$$1 - \frac{T_S - \tau}{T_S} = \frac{\tau}{T_S} \approx \text{probability of breakdown in } (T_S - \tau, T_S)$$

In the first case, the service reduction time is  $\tau$  as the breakdown is completely repaired in  $(0, T_S)$ . On the contrary, in the second case, part of the repair time falls into the ordinary maintenance interval  $(T_S, T_S + T_M)$  with an average reduction of service that, for small  $\tau$ 's, is given by

$$\int_{T_S - \tau}^{T_S} (T_S - \xi) \frac{1}{\tau} d\xi = \frac{\tau}{2}$$

$d\xi/\gamma$  being the probability of breakdown in  $(\xi, \xi + d\xi)$ ,  $\forall \xi \in (T_S - \tau, T_S)$ . Thus, the mean time  $\tau_1$  of service that is lost because of a single breakdown can be given the approximate form

$$\tau_1 = \frac{T^* - \tau}{T^*} \tau + \frac{\tau}{2} \frac{\tau}{T^*} = \tau - \frac{\tau}{2} \frac{\tau}{T^*} \approx \tau \exp\left(-\frac{\tau}{2T_S}\right)$$

where  $T^* = T_S$ . Notice that  $\tau_1 \approx T_S/2$  for  $T_S = \tau$  and  $\tau_1 = \tau$  in the limit  $T_S \rightarrow \infty$ .

When assuming that two breakdowns are possible, only the second one may eventually end in the ordinary maintenance. In this case, the equation for  $\tau_1$  is assumed to remain the same, but with  $T^* = (T_S - \tau)/2$ . Thus,

$$\tau_2 = \tau + \tau_1(T^*) = 2\tau - \frac{\tau}{2} \frac{\tau}{T^*} = 2\tau - \frac{\tau}{2} \frac{2\tau}{T_S - \tau} \approx \tau + \tau \exp\left(-\frac{\tau}{T_S - \tau}\right)$$

In the general case we will have that  $T^* = (T_S - (k-1)\tau)/k$  and

$$\tau_k = (k-1)\tau + \tau_1(T^*) = k\tau - \frac{\tau}{2} \frac{\tau}{T^*}$$

$$\approx (k-1)\tau + \tau \exp\left(-\frac{k\tau}{2(T_S - (k-1)\tau)}\right), \quad k \geq 1$$

$$\tau_0 = 0, \quad k = 0$$

This loss of service will occur with probability  $Q_k(T_S)$ . Note that, in principle, the (extended) exponential form of  $\tau_k$  we have written here can be used for any  $T_S > (k-1)\tau$ , even if we are tacitly assuming that  $\tau_k > k\tau$  and not

too small. On the contrary, for  $(k - 1)\tau \leq T_S \leq k\tau$  with  $k = 1$  and  $2$ , the behaviour of  $\tau_k$  is much better represented by the equation

$$\tau_k = (k - 1)\tau + \frac{T_S - (k - 1)\tau}{(k + 1)\tau}$$

In fact, if  $T_S = \vartheta$  is the (iterative) solution of the equation

$$\exp\left\{-\frac{k\tau}{2(T_S - (k - 1)\tau)}\right\} = \frac{T_S - (k - 1)\tau}{(k + 1)\tau}, \quad k = 1, 2$$

a very good representation of  $\tau_k$  is found to be the following

$$\tau_k = (k - 1)\tau + \begin{cases} \frac{T_S - (k - 1)\tau}{(k + 1)\tau} & \text{for } (k - 1)\tau \leq T_S \leq \vartheta \\ \exp\left\{\frac{k\tau}{2(T_S - (k - 1)\tau)}\right\} & \text{for } T_S \geq \vartheta \end{cases}$$

This can easily be shown through Monte Carlo simulations.

### 4.3. Results

At this point we have all the necessary elements to complete the analysis of section 2 and calculate the probability of  $n$  jobs in the system at the reduced time  $\vartheta_S = \mu T_S$  when breakdowns are possible. To this end, let  $\vartheta_j^* = \mu\tau_j$  and

$$\bar{M}(n, j; \vartheta_S) = \sum_{l=0}^{\infty} p(l, 0) \sum_{i=0}^n p(i, \vartheta_S - \vartheta_j^* | l, 0) \frac{(\rho\vartheta_j^*)^{n-i}}{(n - i)!} e^{-\rho\vartheta_j^*} \quad (7)$$

Then, as an extension of eq. (2), we can write that

$$p^*(n, \vartheta_S) = Q_0(\vartheta_S) \bar{p}(n, \vartheta_S) + \sum_{j=1}^{\infty} Q_j(\vartheta_S) \bar{M}(n, j; \vartheta_S) \quad (8)$$

which is the probability we were looking for. Note that  $\bar{p}(n, \vartheta_S)$  is given by eq. (2). Thus, instead of eq. (3) we will have that

$$p^*(n, \vartheta_S + \vartheta_M) = \sum_{i=0}^n p^*(i, \vartheta_S) \frac{(\rho\vartheta_M)^{n-i}}{(n - i)!} e^{-\rho\vartheta_M} \quad (9)$$

Substitution of (8) into (9), yields the equivalent of eq. (4), that we obtain when only ordinary maintenance is considered.

## 5. ANALYSIS OF THE APPROXIMATIONS

As will be noted, eq. (7) assumes that the total time lost to repair breakdowns may be simply added to  $\vartheta_M$ . As a consequence, in case of  $j$  breakdowns, the total service time reduces from  $\vartheta_S$  to  $\vartheta_S - \vartheta_j^*$ . Needless to say the procedure is approximate as in the interval  $(0, \vartheta_S)$  the distribution  $p(n, \vartheta)$  is not constant and, in particular, it changes the probability of empty queue when a breakdown occurs. (Note that a breakdown does not produce queue effects as long as there is no unit in the system). The entity of the approximation deriving from the use of eq. (8) can be assessed if we have recourse to the opposite extreme case of placing the cumulated repair time at the beginning of the service time. In other words, assume now that there is no service in  $(0, \vartheta_j^*)$ , there is service in  $(\vartheta_j^*, \vartheta_S)$  and, finally, that there is an ordinary maintenance in  $(\vartheta_S, \vartheta_S + \vartheta_M)$ . In this case, instead of (7) we have that

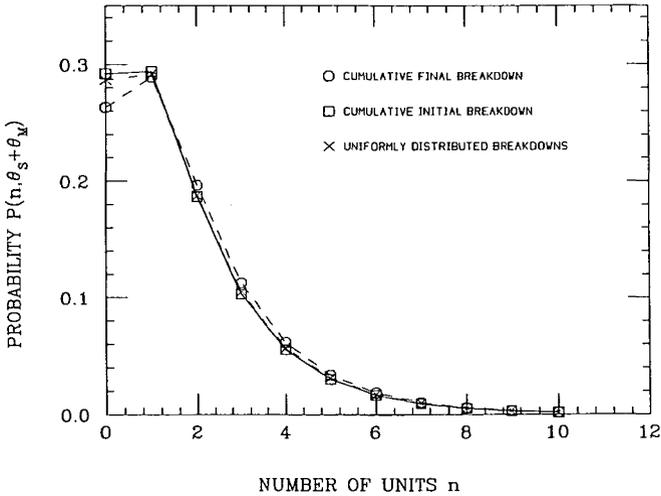
$$\bar{M}(n, j; \vartheta_S) = \sum_{l=0}^{\infty} \left[ \sum_{k=0}^l p(k, 0) \frac{(\rho \vartheta_j^*)^{l-k}}{(l-k)!} e^{-\rho \vartheta_j^*} \right] p(n, \vartheta_S - \vartheta_j^* | l, 0) \quad (10)$$

where it is taken into account that

$$p(n, \vartheta_S | l, \vartheta_j^*) = p(n, \vartheta_S - \vartheta_j^* | l, 0)$$

As regards the equations for  $p^*(n, \vartheta_S)$  and  $p^*(n, \vartheta_S + \vartheta_M)$ , they preserve the same forms (8) and (9). In fact, the only variation concerns the transition matrix  $\bar{M}(n, j; \vartheta_S)$  which must be calculated from (10) instead of (7).

At this point, it becomes interesting to analyze the differences which follow from the use of eq. (10) instead of eq. (7) in the calculation of  $p(n, \vartheta_S + \vartheta_M) = p^*(n, \vartheta_S + \vartheta_M)$ . To this end, in figure 4 we report a comparison between the distributions obtained in two cases, for  $\rho=0.5$ ,  $\vartheta_S = 10$  and  $\vartheta_M = 1$ . Concerning the breakdowns, it is assumed that  $\mu\tau = \vartheta_M/2 = 0.5$ . The mean time between extraordinary interruptions is given the value  $\mu/\gamma = 2\vartheta_S = 20$ , *i. e.*, on the average, we have one breakdown in  $2\vartheta_S$ . As one can see, the differences are generally small and often do not seem to justify the recourse to still more complex formulations, even if they can improve the effect of the interruptions caused by breakdowns. These differences are still reduced for lower values of  $\mu/\gamma$  or  $\mu\tau$ . For instance,



**Figure 4. – Probability  $P(n, \vartheta_S + \vartheta_M)$  as obtained for  $\rho=0.5$ ,  $\vartheta_S = 10$ ,  $\vartheta_M = 1$ . As regards breakdowns,  $\mu\tau=0.5$  (i. e., breakdown repair time of  $\vartheta_M/2$ ) and  $\gamma/\mu=0.05$  (i. e., on the average, one breakdown in  $2\vartheta_S$ ) with various distributions of the cumulative interruption.**

under the conditions of figure 4 but with  $\mu\tau=0.1$  the discrepancies disappear. But there is another interesting observation. Up to this point the cumulated interruption  $T_R$  produced by breakdowns has been placed at the end or at the beginning of the service time. As a third case we can distribute the waste of time  $T_R$  uniformly in  $(0, \vartheta_S)$ . This is equivalent to reduce the value of  $\mu$  to a new value  $\mu^*$  in such a way that the following condition

$$\mu(T_S - T_R) = \mu^* T_S \quad \Rightarrow \quad \mu^* = \mu[1 - (T_R/T_S)]$$

is satisfied. Thus, this case is equivalent to consider a conventional  $M/M/1$  model without breakdowns with service rate  $\mu^*$ . This means that we have only to substitute

$$\vartheta^* = \frac{\mu^*}{\mu} \vartheta \quad \text{and} \quad \rho^* = \frac{\mu^*}{\mu} \rho$$

to  $\vartheta$  and  $\rho$  in eq. (3), or eq. (4), to obtain the distribution in the presence of breakdowns uniformly distributed over the service time. As will be noted, in

Figure 4 the distribution relevant to this case is also reported. The result of the comparison is particularly interesting as it reveals how this distribution is not mid way between the two corresponding distributions obtained with cumulated interruption paced at the beginning or at the end of the service interval  $(0, \vartheta_S)$ . In fact, *distributing the waste of time uniformly does not appear so different from placing it at the beginning*. But this can be justified. At time  $\vartheta = \vartheta_S$  the probability of an empty queue is higher than that at any other time and, in particular, at  $\vartheta = 0$ . (See, for instance, fig. 2). An interruption at the end of the service interval drastically decreases the probability of an empty queue at the end of the maintenance period while the probability of  $n > 1$  units is increased. However, the effect of interruptions is almost negligible, under our conditions, if the cumulated interruption is placed at the beginning of the service interval or is uniformly distributed during the service. The reason is that, at any time, the probability for the queue to be empty is not zero. Thus, for a certain fraction of time the system is not busy. In a sense, in the absence of breakdowns this time is lost. The presence of interruptions caused by breakdowns tends to reduce the probability of empty queue and is in competition with the activity of the system which, on the contrary, tends to empty the queue. If we permit the system to serve after an interruption which has increased the queue, most probably it will be working at full rate and will restore soon the normal situation. In fact, under our conditions, the presence of breakdowns only reduces the time of absence of units in the system. It does not matter very much when the interruptions occur, at least if they do not occur too close to the end of service. In this case, as mentioned, no time is left to the system to reduce (if not to empty) the queue and a substantial lowering of  $p(0, \vartheta_S)$  and  $p(0, \vartheta_S + \vartheta_M)$  will be observed.

To complete our analysis, a Monte Carlo simulation assessed the validity of the distributions  $p(n, \vartheta)$  in the presence of breakdowns. As an example, in Figure 5 we report a comparison between Monte Carlo and theoretical distributions  $p(n, \vartheta_S + \vartheta_M)$  as obtained for  $\rho=0.5$ ,  $\vartheta_S = 10$  and  $\vartheta_M = 1$ , and  $10^5$  samples. The two distributions assume absence of breakdowns [see eq. (4)] and cumulative breakdown at the end of the service interval [see eqs. (7) to (9)], respectively. As one can see and already mentioned, the agreement is quite good, which confirms that even in the presence of breakdowns our iterative process converges to the correct solution. Of course, this also confirms the validity of our approximations. But the same agreement is also observed for different distributions of the cumulative breakdown interruption. However, for the sake of clarity, in figure 5 we have avoided to report further points as nothing would be added to the conclusion.

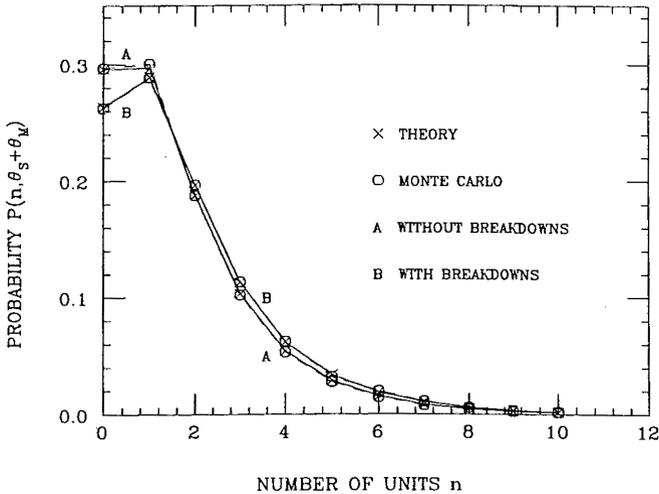


Figure 5. – Comparison between Monte Carlo and theoretical distributions  $p(n, \vartheta_S + \vartheta_M)$  for  $\rho=0.5$ ,  $\vartheta_S = 10$  and  $\vartheta_M = 1$ . As regards breakdowns, we have assumed an interruption of  $\vartheta_M/2$  per single breakdown and a rate of one breakdown per  $2\vartheta_S$ .

## 6. EXTENSION TO THE MODIFIED M/G/1 MODEL

To this point our considerations have been restricted to an  $M/M/1$  model. This has been done first for the sake of simplicity. Secondly, our model permits a complete analytical treatment of the problem and to give *explicit* forms to results (e. g., probability distributions) which hardly could be written under general conditions. However, the conclusions of our analysis, as reported in section 5, when considered *a posteriori*, turn out to have a much more general validity than spelled out so far. In particular, the interpretative considerations of section 5 relevant to the advantages to cumulate the time lost in breakdowns at the beginning or at the end of the service period are clearly independent of the particular service law for any given rate  $\mu$ . In other words, using a simple  $M/M/1$  model we really have been able to obtain results that certainly can be applied to  $M/G/1$  models properly modified to take account of breakdowns.

## 7. CONCLUSIVE REMARKS

The study of an  $M/M/1$  system with ordinary maintenance and including the possibility of service interruptions (extraordinary maintenance) caused by breakdowns, can be based on the (numerical) iterative solution of an

appropriate equation for the probability  $p(n, t)$  that  $n$  units are in the system at the generic instant of time  $t$ . The equation for  $p(n, t)$ , in turn, can be obtained from the fundamental solution of the conventional  $M/M/1$  model, that is the probability  $p(n, t|k, 0)$  of  $n$  costumers at time  $t$  conditioned to the assumption that the population is  $k$  at the initial instant of time. Comparisons with Monte Carlo simulations confirm that the iterations really converge to the correct solutions.

As regards interruptions of service, while there is no difficulty to take account of the ordinary maintenace, the rigorous treatment of the waste of time caused by breakdowns can make the theory much more complex. In part, the analysis becomes simpler if the reparation times are assumed to be constant. In this case there is no particular difficulty even to assume that a number of interruptions are possible during the service time. Of some interest are the conclusions that are obtained when some limiting cases are considered for the temporal distribution of breakdowns. In particular, it is observed that the differences between the distributions  $p(n, t)$  obtained under the assumptions that all the breakdowns occur at the beginning or at the end of the service period, are generally small but can become significant under certain conditions. On the contrary, the differences between the  $p(n, t)$ 's obtained under the assumption that all the breakdowns occur at the beginning of the service interval or are uniformly distributed over the service time, always remain small. The analysis suggests a number of approximations that, in practice, can simplify the rigorous but more complex treatment of the problem of breakdowns (even for modified  $M/G/1$  models) and also throws light on their limits. The validity of the approximations is also assessed by means of Monte Carlo simulations.

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