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OPTIMAL REPLACEMENT POLICIES FOR A TWO-UNIT SYSTEM WITH FAILURE INTERACTIONS (*)

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Abstract. – *This paper considers a two-unit system with the following two failure interactions: When unit 1 fails,* (i) *unit 2 fails with probability* α_j *and* (ii) *unit 1 causes damage with distribution* $G(z)$ *to unit 2. Expected cost rates of two models are derived when the system is replaced at failure of unit 2 or at N-th failure of unit 1. Optimal replacement numbers* N^* *to minimize expected costs are discussed. Finally, an extended model of model 2 is introduced.*

Keywords: Two units, failure interaction, replacement, expected cost, optimal policy.

Résumé. – *Nous considérons dans cet article un système à deux unités avec les deux interactions suivantes en cas de panne : dans l'unité 1 tombe en panne, (I) : l'unité 2 (II) tombe en panne avec la probabilité* α_j*; l'unité 1 cause des dommages avec une distribution* $G(z)$ *à l'unité 2. Nous donnons les formules de l'espérance du coût lorsque le système est remplacé en cas de panne de l'unité 2 ou à la N-ième panne de l'unité 1. Les nombres* N^* *de renouvellement pour minimiser les coûts moyens sont examinés. Finalement, nous introduisons une extension au modèle 2.*

Mots clés : Deux unités, interaction de pannes, renouvellement, espérance du coût, politique optimale.

1. INTRODUCTION

Preventive maintenance policies for a single unit system have received a lot of attention and a variety of policies have been formulated and studied [1]. If failures of different units for multi-unit systems are statistically independent and maintenances of each unit can be done separately, then the analysis is straight forward as each unit is effectively a single unit system. However, failures of units offer the opportunity to replace one or more of non-failed

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units at a cheaper cost. Such policies are called opportunistic replacement policies and have been studied [6, 7, 9].

In a multi-unit system, often the failure times of different units are statistically correlated [8]. In other instances, the failures of units can affect one or more of the remaining units. Such types of interactions between units have been termed as *failure interaction* by Murthy and Nguyen [2]. They defined two types of failure interactions – *induced failure* and *shock damage*. Further, Murthy and Casey [3] considered preventive maintenances of a two-unit system with shock damage interaction.

In this paper, we consider a system with unit 1 and unit 2. If unit 1 failes then it undergoes only minimal repair, and unit failures occur at non-homogeneous Poisson processes with an intensity function $r(t)$, where $r(t)$ is increasing in t.

Further, when unit 1 fails, we indicate the following two failure interactions between two units:

(i) Induced failure; unit 2 fails with probability α_j at the j-th time of unit 1 failure.

(ii) Shock damage; unit 1 causes damage with distribution $G(z)$ to unit 2.

Suppose that the system is replaced at failure of unit 2 or a N-th failure of unit 1, whichever occurs first. Expected cost rate of two models are derived and optimal replacement numbers N^* to minimize them are discussed. Finally, we introduce an extended model of model 2 where the system is also replaced at time T.

2. MODEL 1: INDUCED FAILURE

Whenever unit 1 fails, it acts as a shock to induce an instantaneous failure of unit 2 with a certain probability. Let α_j denote the probability that unit 2 fails at j-th failure of unit 1. It is assumed that $0<\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_j\leq\cdots$. The system is replaced at failure of unit 2 or N-th failure of unit 1.

The probability that the system is replaced at N-th failure of unit 1 is

$$(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{N-1}). \tag{1}$$

The probability that the system is replaced at failure of unit 2 is

$$\sum_{j=1}^{N-1}(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{j-1})\alpha_j, \tag{2}$$

where $\alpha_0 \equiv 0$. It is evident that (1)+(2)=1.

The mean time to replacement is

$$(1-\alpha_1)\cdots(1-\alpha_{N-1})\int_0^\infty t\,\frac{[R(t)]^{N-1}}{(N-1)!}\,e^{-R(t)}\,r(t)\,dt$$
$$+\sum_{j=1}^{N-1}(1-\alpha_1)\cdots(1-\alpha_{j-1})\,\alpha_j\int_0^\infty t\,\frac{[R(t)]^{j-1}}{(j-1)!}\,e^{-R(t)}\,r(t)\,dt$$
$$=\sum_{j=0}^{N-1}(1-\alpha_1)\cdots(1-\alpha_j)\int_0^\infty \frac{[R(t)]^j}{j!}\,e^{-R(t)}\,dt, \quad (3)$$

since

$$\int_0^\infty t\,\frac{[R(t)]^{j-1}}{(j-1)!}\,e^{-R(t)}\,r(t)\,dt=\int_0^\infty te^{-R(t)}\,d\left[\frac{[R(t)]^j}{j!}\right]$$
$$=\int_0^\infty t\,\frac{[R(t)]^j}{j!}\,e^{-R(t)}\,r(t)\,dt-\int_0^\infty \frac{[R(t)]^j}{j!}\,e^{-R(t)}\,dt,$$

were $R(t) \equiv \int_0^t r(u)\,du$.

The expected number of unit 1 failures until replacement is

$$(N-1)(1-\alpha_1)\cdots(1-\alpha_{N-1})$$
$$+\sum_{j=1}^{N-1}(j-1)(1-\alpha_1)\cdots(1-\alpha_{j-1})\,\alpha_j$$
$$=\sum_{j=1}^{N-1}(1-\alpha_1)\cdots(1-\alpha_j). \quad (4)$$

Note that we do not include the number of j-th failure when the system is replaced at j-th failure of unit 1 ($j=1, 2, \cdots, N$).

Therefore, the expected cost rate is

$$C_1(N)=\frac{c_1\sum_{j=1}^{N-1}A_j+c_3-(c_3-c_2)\,A_{N-1}}{\sum_{j=0}^{N-1}A_j\int_0^\infty p_j(t)\,dt}, \quad (5)$$

where $A_j \equiv (1-\alpha_1) \cdots (1-\alpha_j)\ (j=1, 2, \cdots)$, $A_0 \equiv 1$, $p_j(t) \equiv ([R(t)]^j/j!)\, e^{-R(t)}$ $(j=0, 1, 2, \cdots)$ which is used for the other models throughout the paper, $c_1=$ cost of one unit 1 failure, $c_2=$replacement cost at N-th failure of unit 1, and $c_3=$replacement cost at failure of unit 2 with $c_3 \geq c_2$.

We seek an optimal number N^* which minimizes $C_1(N)$. From the inequality $C_1(N+1) \geq C_1(N)$, we have

$$c_1 \left\{ \frac{\sum_{j=0}^{N-1} A_j \int_0^\infty p_j(t)\, dt}{\int_0^\infty p_N(t)\, dt} - \sum_{j=1}^{N-1} A_j \right\}$$

$$+(c_3 - c_2) \left\{ \frac{A_{N-1} - A_N}{A_N \int_0^\infty p_N(t)\, dt} \sum_{j=0}^{N-1} A_j \int_0^\infty p_j(t)\, dt + A_{N-1} \right\} \geq c_3. \quad (6)$$

Denote the left side of (6) by $L_1(N)$.

$$L_1(N+1) - L_1(N)$$
$$= \sum_{j=0}^{N} A_j \int_0^\infty p_j(t)\, dt \left\{ c_1 \left[\frac{1}{\int_0^\infty p_{N+1}(t)\, dt} - \frac{1}{\int_0^\infty p_N(t)\, dt} \right] \right.$$
$$\left. +(c_3 - c_2) \left[\frac{A_N - A_{N+1}}{A_{N+1} \int_0^\infty p_{N+1}(t)\, dt} - \frac{A_{N-1} - A_N}{A_N \int_0^\infty p_N(t)\, dt} \right] \right\}.$$

Suppose that either of α_j or $r(t)$ is strictly increasing. Then, from [5], if $r(t)$ is strictly increasing then $\int_0^\infty p_j(t)\, dt$ is strictly decreasing in j and if α_j is strictly increasing then $[A_N - A_{N+1}]/A_{N+1}$ is also strictly increasing. Then, $L_1(N)$ is strictly increasing in N, and hence, an optimal number N^* is given by a unique minimum such that $L_1(N) \geq c_3$.

Consider two particular cases:

(i) Suppose that α_j is constant, *i.e.*, $\alpha_j \equiv \alpha$ and $A_j \equiv (1-\alpha)^j\ (j=0, 1, 2, \cdots)$. Then, equation (6) is rewritten as

$$\frac{\sum_{j=0}^{N-1} (1-\alpha)^j \int_0^\infty p_j(t)\, dt}{\int_0^\infty p_N(t)\, dt} - \frac{1-\alpha-(1-\alpha)^N}{\alpha}$$
$$\geq \frac{c_2}{c_1 + (\alpha/(1-\alpha))(c_3 - c_2)}. \quad (7)$$

If $r(t)$ is strictly increasing then the left side $L_1(N)$ of (7) is strictly increasing and

$$\lim_{N \to \infty} L_1(N) = r(\infty) \int_0^{\infty} e^{-\alpha R(t)}\, dt - \frac{1-\alpha}{\alpha}.$$

Thus, if

$$r(\infty) \int_0^{\infty} e^{-\alpha R(t)}\, dt > \frac{1-\alpha}{\alpha} \frac{c_1 + (\alpha/(1-\alpha))\, c_3}{c_1 + (\alpha/(1-\alpha))\, (c_3 - c_2)},$$

then a finite N^* is given by a unique minimum which satisfies (7).

Note that the expected number of unit 1 failures until unit 2 failure is

$$m \equiv \sum_{j=1}^{\infty} (1-\alpha)^j = \frac{1-\alpha}{\alpha}.$$

Then,

$$\frac{1-\alpha}{\alpha} \frac{c_1 + (\alpha/(1-\alpha))\, c_3}{c_1 + (\alpha/(1-\alpha))\, (c_3 - c_2)} = \frac{(c_1 m + c_3)\, m}{c_1\, m + c_3 - c_2},$$

is increasing in m from 0 to ∞. Thus, a finite N^* exists, when m is smaller.

EXAMPLE. Suppose that $p_j(t) = [(t^2)^j / j!]\, e^{-t^2}$. Then, since $r(t) = 2t$ is strictly increasing to ∞, there exists a unique minimum which satisfies (7). Table 1 shows the optimal numbers N^* for $(c_3 - c_2)/c_1 = 1, 2, 5, 10, 20, 50$ and $c_2/c_1 = 2, 3, 5, 10, 20, 50$ when $\alpha = 0.1$.

(ii) Suppose that $r(t) = \lambda$, *i.e.*, unit 1 failure occurs at a Poisson process with rate λ.

Then, equation (6) is

$$\frac{\alpha_N}{1-\alpha_N} \sum_{j=0}^{N-1} A_j + A_{N-1} \geq \frac{c_3 - c_1}{c_3 - c_2}. \tag{8}$$

If α_j is strictly increasing in j then the left side $L_1(N)$ of (8) is strictly increasing and

$$\lim_{N \to \infty} L_1(N) = \frac{\alpha_\infty}{1-\alpha_\infty} \sum_{j=0}^{N-1} A_j,$$

TABLE 1
Optimal numbers N^* to minimize $C_1(N)$ when
$p_j(t) = [(t^2)^j/j!]\, e^{-t^2}$ *and* $\alpha_j = 0.1$.

$(c_3-c_2)/c_1$	c_2/c_1					
	2	3	5	10	20	50
1	2	2	2	4	6	11
2	1	2	2	4	6	11
5	1	2	2	3	5	9
10	1	1	1	3	4	7
20	1	1	1	2	3	5
50	1	1	1	1	2	3

where $\alpha_\infty \equiv \lim_{j \to \infty} \alpha_j$. Thus, if

$$\frac{\alpha_\infty}{1-\alpha_\infty} \sum_{j=0}^{\infty} A_j \geq \frac{c_3 - c_1}{c_3 - c_2},$$

then a finite N^* is a unique minimum which satisfies (8).

EXAMPLE. Suppose that $r(t) = \lambda$ and $\alpha_j = 1 - \beta^j$. Then, from the results (ii), a finite n^* exists and is given by a unique minimum which satisfies

$$\frac{1-\beta^N}{\beta^N} \sum_{j=0}^{N} \beta^{j(j+1)/2} + \beta^{(N-1)N/2} \geq \frac{c_3 - c_1}{c_3 - c_2}.$$

Table 2 shows the optimal number N^* for $(c_3 - c_2)/c_1 = 1$, 2, 5, 10, 20, 50 and $c_2/c_1 = 2$, 3, 5, 10, 20, 50 when $\beta = 0.9$.

3. MODEL 2: SHOCK DAMAGE

Whenever unit 1 fails, it acts as a shock to unit 2 and causes damage with distribution $G(x)$ to unit 2. The damage is cumulative and unit 2 fails whenever the total damages exceed a failure level Z. The system is replaced at failure of unit 2 or N-th failure of unit 1.

TABLE 2

Optimal numbers N^* to minimize $C_1(N)$ when $p_j(t) = [(\lambda t)^j/j!]\, e^{-\lambda t}$ and $\alpha_j = 1 - (0.9)^j$.

$(c_3 - c_2)/c_1$	c_2/c_1					
	2	3	5	10	20	50
1	1	4	7	12	17	25
2	1	1	4	8	12	19
5	1	1	1	4	7	12
10	1	1	1	1	4	8
20	1	1	1	1	1	5
50	1	1	1	1	1	1

The probability that the system is replaced at N-th failure of unit 1 is

$$G^{(N)}(Z), \tag{9}$$

where $G^{(j)}(x)$ is the j-th convolution of $G(x)$ with itself.

The mean time to replacement is

$$\sum_{j=1}^{N-1} G^{(j)}(Z) \int_0^\infty p_j(t)\, dt. \tag{10}$$

The expected number of unit 1 failures until replacement is

$$(N-1)\, G^{(N)}(Z) + \sum_{j=1}^{N-1} (j-1)\,[G^{(j-1)}(Z) - G^{(j)}(Z)] = \sum_{j=1}^{N-1} G^{(j)}(Z), \tag{11}$$

where $G^{(0)}(x) \equiv 1$ for $x \geq 0$.

Therefore, the expected cost rate is

$$C_2(N) = \frac{c_1 \sum_{j=1}^{N-1} G^{(j)}(Z) + c_3 - (c_3 - c_2)\, G^{(N)}(Z)}{\sum_{j=0}^{N-1} G^{(j)}(Z) \int_0^\infty p_j(t)\, dt}, \tag{12}$$

where c_1 = cost of one unit 1 failure, c_2 = replacement cost at N-th failure of unit 1, and c_3 = replacement cost at failure of unit 2 with $c_3 \geq c_2$.

In particular, when Z goes to infinity, $C_2(N)$ is

$$C_2(N) = \frac{c_1(N-1) + c_2}{\sum_{j=0}^{N-1} \int_0^\infty p_j(t)\,dt}, \tag{13}$$

which corresponds to (11) of [5].

We seek an optimal number N^* which minimizes $C_2(N)$ in (12). From the inequality $C_2(N+1) \geq C_2(N)$, we have

$$c_1 \left\{ \frac{1}{\int_0^\infty p_N(t)\,dt} \sum_{j=0}^{N-1} G^{(j)}(Z) \int_0^\infty p_j(t)\,dt - \sum_{j=1}^{N-1} G^{(j)}(Z) \right\} + (c_3 - c_2) \left\{ \frac{G^{(N)}(Z) - G^{(N+1)}(Z)}{G^{(N)}(Z) \int_0^\infty p_N(t)\,dt} \sum_{j=0}^{N-1} G^{(j)}(Z) \int_0^\infty p_j(t)\,dt + G^{(N)}(Z) \right\} \geq c_3. \tag{14}$$

Denote the left side of (14) by $L_2(N)$.

$$\begin{aligned} &L_2(N+1) - L_2(N) \\ &= \sum_{j=0}^{N-1} G^{(j)}(Z) \int_0^\infty p_j(t)\,dt \left\{ c_1 \left[\frac{1}{\int_0^\infty p_{N+1}(t)\,dt} - \frac{1}{\int_0^\infty p_N(t)\,dt} \right] \right. \\ &\left. + (c_3 - c_2) \left[\frac{G^{(N+1)}(Z) - G^{(N+2)}(Z)}{G^{(N+1)}(Z) \int_0^\infty p_{N+1}(t)\,dt} - \frac{G^{(N)}(Z) - G^{(N+1)}(Z)}{G^{(N)}(Z) \int_0^\infty p_N(t)\,dt} \right] \right\}. \end{aligned}$$

Suppose that either of $[1 - G^{(N+1)}(Z)/G^{(N)}(Z)]$ or $r(t)$ is strictly increasing. Then, $L_2(N)$ is also strictly increasing in N, and hence, an optimal number N^* is given by a unique minimum which satisfies (14).

In particular, suppose that $G(x) = 1 - e^{-\mu x}$. Then,

$$\frac{G^{(N+1)}(x)}{G^{(N)}(x)} = \frac{\sum_{j=N+1}^{\infty} (\mu x)^j / j!}{\sum_{j=N}^{\infty} (\mu x)^j / j!},$$

is decreasing in N. Further,

$$\lim_{N \to \infty} L_2(N) = c_1 \left\{ r(\infty) \sum_{j=0}^{\infty} G^{(j)}(Z) \int_0^{\infty} p_j(t)\, dt - \mu Z \right\} + (c_3 - c_2) \left\{ r(\infty) \sum_{j=0}^{\infty} G^{(j)}(Z) \int_0^{\infty} p_j(t)\, dt \right\},$$

where $G^{(j)}(Z) = \sum_{i=j}^{\infty} \frac{(\mu Z)^i}{i!} e^{-\mu Z}$. Thus, if

$$r(\infty) \sum_{j=0}^{\infty} G^{(j)}(Z) \int_0^{\infty} p_j(t)\, dt > \frac{c_3 + c_1 \mu Z}{c_3 - c_2 + c_1},$$

then a finite N^* is given by a unique minimum which satisfies (14). Further, when $r(t) = \lambda$, if $1 + \mu Z > (c_3 - c_1)/(c_3 - c_2)$ then a finite N^* exists uniquely, since $r(\infty) \int_0^{\infty} p_j(t)\, dt = 1$ and $\sum_{j=0}^{\infty} G^{(j)}(Z) = 1 + \mu Z$.

4. EXTENDED MODEL

In model 2, suppose that the system is replaced at time T, at failure of unit 2 or N-th failure of unit 1, whichever occurs first.

The probability that the system is replaced at time T is

$$\sum_{j=0}^{N-1} p_j(T)\, G^{(j)}(Z). \tag{15}$$

The probability that the system is replaced at N-th failure of unit 1 is

$$\sum_{j=N}^{\infty} p_j(T)\, G^{(N)}(Z). \tag{16}$$

The probability that the system is replaced at failure of unit 2 is

$$\sum_{j=0}^{N-1} p_j(T)\,[1 - G^{(j)}(Z)] + \sum_{j=N}^{\infty} p_j(T)\,[1 - G^{(N)}(Z)] = \sum_{j=1}^{N} [G^{(j-1)}(Z) - G^{(j)}(Z)] \sum_{i=j}^{\infty} p_i(T). \tag{17}$$

It is evident that (15)+(16)+(17)=1.

The mean time to replacement is

$$T \sum_{j=0}^{N-1} p_j(T)\, G^{(j)}(Z) + G^{(N)}(Z) \int_0^T t p_{N-1}(t)\, r(t)\, dt + \sum_{j=1}^{N} [G^{(j-1)}(Z) - G^{(j)}(Z)] \int_0^T t p_{j-1}(t)\, r(t)\, dt = \sum_{j=0}^{N-1} G^{(j)}(Z) \int_0^T p_j(t)\, dt. \quad (18)$$

The expected number of unit 1 failures until replacement is

$$\sum_{j=0}^{N-1} j p_j(T)\, G^{(j)}(Z) + (N-1) \sum_{j=N}^{\infty} p_j(T)\, G^{(N)}(Z) + \sum_{j=1}^{N-1} (j-1)\, [G^{(j-1)}(Z) - G^{(j)}(Z)] \sum_{i=j}^{\infty} p_i(T) = \sum_{j=1}^{N-1} G^{(j)}(Z) \sum_{i=j}^{\infty} p_i(T). \quad (19)$$

Therefore, the expected cost rate is

$$C(T, N) = \frac{\left(\begin{array}{l} c_1 \sum_{j=1}^{N-1} G^{(j)}(Z) \sum_{i=j}^{\infty} p_i(T) + c_2\, G^{(N)}(Z) \sum_{j=N}^{\infty} p_j(T) \\ \quad + c_3 \sum_{j=1}^{N} [G^{(j-1)}(Z) - G^{(j)}(Z)] \sum_{i=j}^{\infty} p_i(T) \\ \quad\quad + c_4 \sum_{j=0}^{N-1} G^{(j)}(Z)\, p_j(T) \end{array} \right)}{\sum_{j=0}^{N-1} G^{(j)}(Z) \int_0^T p_j(t)\, dt}, \quad (20)$$

where c_1 = cost of one unit failure c_2 = replacement cost at N-th failure of unit 1, c_3 = replacement cost at failure of unit 2, and c_4 = replacement cost at time T.

In particular, when T goes to infinity, $C(T, N)$ agrees with $C_2(N)$ in (12). Further, when both N and Z go to infinity, (20) is

$$C(T)=\frac{c_1 R(T)+c_4}{T}, \tag{21}$$

which agrees with [1] of periodic replacement.

5. CONCLUSIONS

We considered two types of failure interactions and discussed the optimal replacement policies.

The above two models characterize many real systems. The following is an illustrative example from chemical industry. The system consists of a metal container (unit 2) in which chemical reactions take place and the temperature of the container is controlled by cold water pumped through a pneumatic pump (unit 1). Consider the case where the pump fails and as a result the pressure inside can build up to lead to an explosion if the quantity of reacting fluid is high. This situation is modelled by model 1 with $\alpha_j=\alpha$ for all j and α is the probability that the volume of fluid in the container is high. A different scenario is the following. Whenever the pump fails, the temperature of the tank rises and the container surface gets corroded. As a consequence, the thickness of the container decreases. The damage is the reduction in the wall thickness and it is additive. The container fails when the total reduction in the wall thickness exceeds some specified limit. This situation is modelled by model 2. Note that without unit 1 failure, there is no damage to unit 2 and hence it does not fail. If the container is preventively maintained at time T before failure and is like new, the system corresponds to an extended model.

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