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TECHNICAL NOTE ON DUALITY IN LINEAR VECTOR MAXIMIZATION (*)

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Abstract. – *A dual to the linear vector maximization problem is proposed. This new version sharpens Isermann's dual. We also give necessary and sufficient conditions for non emptiness of the optimal set.*

Keywords: *Vector maximization, efficient points, duality.*

Résumé. – *Un dual au problème de maximisation vectorielle linéaire est proposé. Cette nouvelle version améliore la version d'Isermann. Nous donnons aussi des conditions nécessaires et suffisantes pour que l'ensemble optimal soit non vide.*

Mots clés : *Maximisation vectorielle, points efficaces, dualité.*

1. INTRODUCTION

Let the usual order relations on \mathbb{R} “greater than or equal to” and “strictly greater than” be respectively noted \geq and $>$. On \mathbb{R}^n the relations \geq , \geq and $>$ are defined as follow:

$$x \geq y \Leftrightarrow x_i \geq y_i \quad \text{for all } i.$$

$$x \geq y \Leftrightarrow x_i \geq y_i \quad \text{for all } i \text{ and there exists an } i_0 \\ \text{such that (s.t.) } x_{i_0} > y_{i_0}.$$

$$x > y \Leftrightarrow x_i > y_i \quad \text{for all } i.$$

Although these non standard notations are used in vector inequalities (*see* [5]) and vector-optimization, quite often in the literature, our inequality $x \geq y$ (resp. $x \geq y$) is replaced by the standard $x \geq y$ (resp. $x \geq y, x \neq y$).

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If S is any subset of \mathbb{R}^n , then

$$\begin{aligned} \text{MAX } S &= \{x \in S \mid \text{there is no } y \in S \text{ with } y \geq x\} \\ \text{MIN } S &= \{x \in S \mid \text{there is no } y \in S \text{ with } y \leq x\}, \end{aligned}$$

e will always designate a real vector the components of which are all unity. The scalar product of two real vectors x and y is noted xy and the product of x and a real matrix C is noted Cx or xC depending on the one allowed.

Let A and M be given $m \times n$ and $k \times n$ real matrices and b a constant real m -vector.

We suppose that $k \geq 2$ and that $X = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\} \neq \emptyset$.

The linear vector maximization problem is to characterize the set $\text{MAX } \{Mx \mid x \in X\}$. This problem called primal problem is noted

$$(P) \quad \text{MAX } F, F = \{Mx \mid Ax = b, x \geq 0\}$$

We define the dual of (P) as

$$(D) \quad \text{MIN } F^*, F^* = \{y \in \mathbb{R}^k \mid \exists U, (M - UA)x + z(y - Ub) \geq 0 \\ \text{for no } z \in \mathbb{R}, x \in \mathbb{R}^n, x \geq 0\}$$

in which U is always a real $k \times m$ -matrix.

In the literature on vector optimization, a point $x \in \text{MAX } F$ is said to be an efficient or Pareto-optimal solution for (P) (see for instance [1], [2] and [3]). We mean here the strong efficiency defined through our inequality \geq instead of the weak efficiency defined through $>$. In addition, as far as the linear case is concerned, efficiency coincides with proper efficiency which matters in non linear cases.

In [4], Isermann defined the dual of (P) as

$$(D_I) \quad \text{MIN } F_I^*, F_I^* = \{Ub \mid (M - UA)x \geq 0 \text{ for no } x \in \mathbb{R}^n, x \geq 0\},$$

where U is as above.

2. THE MAIN RESULTS

For a first difference between (D) and (D_I) , let us observe that $F_I^* \subset F^*$. Further, let $y \in F^*$. Then there exists U s.t. $(M - UA)x + z(y - Ub) \geq 0$ for no $z \in \mathbb{R}$ and $x \in \mathbb{R}^n$ with $x \geq 0$. Thus for any such U , taking $x = 0$, we see that we can neither have $y - Ub \geq 0$ nor $y - Ub \leq 0$ and that does not necessarily mean that $y = Ub$. Consequently the inclusion above cannot be replaced by an equality.

Isermann's main duality result is that $\text{MIN } F_I^* \subset \text{MAX } F$.

We will show that $\text{MIN } F^* = \text{MAX } F$ so that conditions necessary or sufficient or both for $\text{MIN } F^* = \emptyset$ (resp. $y^0 \in \text{MIN } F^*$) are the same for $\text{MAX } F \neq \emptyset$ (resp. $y^0 \in \text{MAX } F$). For any $y \in \mathbb{R}^k$ given, consider the two dual scalar linear programs

$$LP(y) : \max \{es \mid Ax = b, Mx - s = y, s \in \mathbb{R}^k, x \in \mathbb{R}^n, s \geq 0, x \geq 0\}$$

$$LP^*(y) : \min \{bv - py \mid vA \geq pM, v \in \mathbb{R}^n, p \geq e\}.$$

For any $x^0 \in X$, from [3], $Mx^0 \in \text{MAX } F$ iff the optimal objective value to $LP(Mx^0)$ is 0. From [2], any \bar{x} optimal solution to $LP(Mx^0)$ is *s.t.*

$M\bar{x} \in \text{MAX } F$ and $LP(Mx^0)$ does not have a finite optimal objective value, in other words $LP^*(Mx^0)$ is infeasible (since $LP(Mx^0)$ is feasible) iff $\text{MAX } F = \emptyset$. From this latter necessary and sufficient condition for non emptiness of $\text{MAX } F$, the following result becomes obvious.

1. THEOREM: $\text{MAX } F \neq \emptyset$ iff one of the following equivalent conditions holds:

- 1) The system $Ax = 0, Mx \geq 0, x \geq 0$ has no solution $x \in \mathbb{R}^n$.
- 2) The system $vA - aM \geq 0, a > 0$ has a solution $(v, a) \in \mathbb{R}^{m+k}$.
- 3) There exists a real $k \times m$ -matrix U for which the system $a(M - UA) \leq 0, a > 0$ has a solution $a \in \mathbb{R}^k$.
- 4) There exists a real $k \times m$ -matrix U for which the system $(M - UA)x \geq 0$ has no solution $x \in \mathbb{R}^n$.

Proof: As for any $v \in \mathbb{R}^m$ and $a \in \mathbb{R}^k$ with $a > 0$ there exists a $k \times m$ real matrix U *s.t.* $v = aU$, the equivalences follow from judicious applications of the Tucker's theorem of the alternative (see [5]). 2) holds iff for any $x \in X$, $LP^*(Mx)$ is infeasible, that is $\text{MAX } F \neq \emptyset$. \square

For proving that we always have $\text{MAX } F = \text{MIN } F^*$, we will need the following lemma which has also the advantage of telling us a bit more on the structure of F^* , the feasible set of (D).

2. LEMMA: $F^* = \{y \in \mathbb{R}^k \mid ay = bv, vA \geq aM, \text{ for some } a \in \mathbb{R}^k, a > 0, v \in \mathbb{R}^m\}$ where we may take every $a \geq e$ without any loss. \square

Proof: For a given real $k \times m$ -matrix U , the system

$$(M - UA)x + z(y - Ub) \geq 0, \quad x \geq 0$$

has no solution $x \in \mathbb{R}^n, z \in \mathbb{R}$ iff there exists $a \in \mathbb{R}^k$ with $a > 0$ *s.t.* $a(M - UA) \leq 0, a(y - Ub) = 0$ according to Tucker's theorem of the

alternative, in which case we set $v = aU$. On the other hand, given $v \in \mathbb{R}^m$ and $a \in \mathbb{R}^k$ with $a > 0$, there exists a real $k \times m$ -matrix U s.t. $v = aU$ and the lemma follows. \square

Let us recall that it is well known that $Mx^0 \in \text{MAX } F$ iff there exists $a \in \mathbb{R}^k$ with $a > 0$ (we may take $a \geq e$) s.t. aMx^0 is the optimal objective value of the scalar linear program $P_l(a) = \max \{aMx \mid Ax = b, x \geq 0\}$. The dual of $P_l(a)$ being $D_l(a) : \min \{bv \mid vA \geq aM\}$, it follows that $\text{MAX } F = \{y \in \mathbb{R}^k \mid ay = \max [aMx \mid Ax = b, x \geq 0], a \in \mathbb{R}^k, a > 0\} = \{y \in \mathbb{R}^k \mid ay = \min [bv \mid vA \geq aM, v \in \mathbb{R}^m], a \in \mathbb{R}^k, a > 0\}$.

Let $y \in F^*$. Then there exist $v \in \mathbb{R}^m$, $a \in \mathbb{R}^k$ s.t. $a > 0$, $ay = bv$ and $vA \geq aM$.

If there exists $u \in \mathbb{R}^m$ s.t. $bv > bu$ and $uA \geq aM$, then taking $\bar{y} \in \mathbb{R}^k$ s.t. $\bar{y}_i = y_i - (1/ea)(ay - bu)$, we have $\bar{y} \in F^*$ with $\bar{y} < y$ and consequently $\text{MIN } F^* \subset \text{MAX } F$. This inclusion is showed slightly differently in the next result.

3. THEOREM (Strong Duality): $\text{MAX } F = \text{MIN } F^*$. \square

Proof: Let $y^0 \in \text{MIN } F^*$. Then from the lemma there exist $a^0 \in \mathbb{R}^k$, $a^0 \geq e$ and $v^0 \in \mathbb{R}^m$ s.t. $a^0 y^0 = bv^0$, $v^0 A \geq a^0 M$. Consider the linear program $LP^*(y^0)$ taking $v = v^0$ and $p = a^0$, we obtain a feasible solution with 0 objective value. Suppose \bar{v} and \bar{p} feasible with $b\bar{v} - \bar{p}y^0 < 0$. Taking $\bar{y} \in \mathbb{R}^k$ s.t. $\bar{y}_i = y_i^0 + (1/e\bar{p})(b\bar{v} - \bar{p}y^0)$, we have $\bar{p}\bar{y} = b\bar{v}$, $\bar{v}A \geq \bar{p}M$, thus $\bar{y} \in F^*$ according to the lemma, but $\bar{y} < y^0$ contradicting $y^0 \in \text{MIN } F^*$. We therefore have $b\bar{v} - \bar{p}y^0 = 0$ and the optimal objective value to $LP^*(y^0)$ is 0. The dual $LP(y^0)$ of $LP^*(y^0)$ has 0 optimal objective value for $s = 0$ and some $x = x^0$ and we have $y^0 = Mx^0$. It is an easy matter to show that the optimal solution (v^0, p^0) to $LP^*(y^0)$ is s.t. $p^0 Mx^0$ is the optimal objective value of the problem $P_l(p_0)$ and consequently $y^0 = Mx^0 \in \text{MAX } F$.

Conversely let $Mx^0 \in \text{MAX } F$. This is the case iff $LP(Mx^0)$ and $LP^*(Mx^0)$ have optimal objective values equal to 0 (see [2] or [3]). Now since $Mx^0 \in \text{MAX } F$ there exists $a^0 \in \mathbb{R}^k$, $a^0 > 0$ s.t. $a^0 Mx^0$ is the optimal objective value to $P_l(a^0)$. Hence there exists v^0 optimal to $D_l(a^0)$ with $a^0 Mx^0 = bv^0$, $v^0 A \geq a^0 M$ and we have $Mx^0 \in F^*$ according to the lemma. If $Mx^0 \notin \text{MIN } F^*$, then there exists $\bar{y} \in F^*$ s.t. $\bar{y} \leq Mx^0$, in other words, there exist $\bar{a} \in \mathbb{R}^k$ with $\bar{a} \geq e$ and $\bar{v} \in \mathbb{R}^m$ s.t. $\bar{a} Mx^0 > \bar{a}\bar{y} = b\bar{v}$ and $\bar{v}A \geq \bar{a}M$. This is in contradiction with the fact that the optimal objective value to $LP^*(Mx^0)$ is 0 and consequently $Mx^0 \in \text{MIN } F^*$. \square

It follows from this theorem and the lines preceding it that

$$\begin{aligned} \text{MIN } F^* &= \{y \in \mathbb{R}^k \mid ay = \min [bv \mid v A \geq a M, v \in \mathbb{R}^m], a \in \mathbb{R}^k, a > 0\} \\ &= \{y \in \mathbb{R}^k \mid ay = \max [a M x \mid A x = b, x \geq 0], a \in \mathbb{R}^k, a > 0\} \\ &= \text{MAX } F \end{aligned}$$

as should be expected.

CONCLUSION

We have proposed a dual to the linear vector maximization problem which completes Isermann's dual. We also gave necessary and sufficient conditions for non emptiness of the efficient set.

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ERRATA

In *RAIRO-Operations Research* Vol. 27, n°4, 1993, the following papers were respectively communicated by :

- Pierre TOLLA (paper of Benchakroun *et al.*)
- Bernard LEMAIRE (paper of Pellaumail and Lebah)
- Pierre TOLLA (paper of Casas and Pola)
- Naoto KAIO (paper of Nakagawa and Murthy)

The title of the paper of A. Issoufou Kouada (p. 77) was : Sur la propriété de domination et l'existence de points Pareto-efficaces en optimisation vectorielle convexe.