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ON LOCATING A SINGLE PATH-LIKE FACILITY IN A GENERAL GRAPH (*)

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Abstract. – *The problem of locating a* path-like *facility of fixed length in a tree can be solved in polynomial time. However, this problem is NP-complete even on very sparse graphs such as outerplanar graphs. We show that if the length of the facility to be located is* small *then the corresponding path location problem can be solved efficiently. We suggest* $O(n^4 \log n)$ *and* $O(n^4)$ *algorithms to solve the problem of locating small paths with minmax and minsum criteria on a general graph. We also give a method to solve small path location problems with a geneal nonlinear objective function. If the path-like facility to be located is known to be contained in some path with at most r nodes then we show that, for fixed r, the path location problem is solvable in polynomial time.*

Keywords: Location, networks, computational complexity, polynomial algorithms.

Résumé. – *Le problème consistant à localiser un* simili-chemin *d'installations, de longueur fixée, sur un arbre peut être résolu en un temps polynomial. Cependant, ce problème est NP-complet, même sur un graphe très creux. Nous montrons que si la longueur des installations à localiser est* petite, *alors le problème de localisation de chemin peut être résolu efficacement. Nous suggérons des algorithmes en* $O(n^4 \log n)$ *et en* $O(n^4)$ *pour résoudre le problème consistant à localiser un chemin court avec des critères de « minmax » et « minisomme » sur un graphe général. Nous présentons aussi une méthode pour résoudre les petits problèmes de localisation de chemins avec une fonction objectif non-linéaire générale. Si le simili-chemin d'installations à localiser est réputé être contenu dans un chemin d'au plus r sommets, alors nous montrons que, pour r fixé, le problème de localisation du chemin peut être résolu en un temps polynomial.*

Mots clés : Localisation, réseaux, complexité de calcul, algorithmes polynomiaux.

1. INTRODUCTION

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. Let e_{rs} be the edge joining vertices r and s. For each $e_{rs} \in E(G)$, a positive weight c_{rs} is prescribed. Let $L > 0$ be a given real number and P_L be a path on the graph G of length L.

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It may be noted that the end points of P_L need not be vertices of G, but can be any point on the graph. Let $d_j(P_L)$ be the distance of vertex j from the nearest point in P_L and d_{ij} be the distance between vertices i and j. Then the *path location problem* considered in this paper is

$$\begin{aligned}\text{PLP}: \quad & \text{Minimize } f(d_1(P_L), d_2(P_L), \ldots, d_n(P_L)) \\ & \text{Subject to} \\ & \qquad P_L \in G,\end{aligned}$$

where $f : \mathfrak{R}^n \to \mathfrak{R}$ and $V(G) = \{1, 2, \ldots, n\}$. Location problems similar to PLP has been studied by several researchers in various contexts [4, 5, 10, 12-17]. In [10, 12, 14], it is pointed out that, problems of the type PLP arise in real life situations such as location of pipe lines, building express lanes in urban interstate highways, location of routes in automated guided vehicles, etc. Minieka [12] and Kincaid *et al.* [10] suggested polynomial time algorithms to solve PLP when the underlying graph is a tree and f is of the form $f(\cdot) = \max(\cdot)$ or $f(\cdot) = \Sigma(\cdot)$.

However, networks with cycles occur more frequently in practice than trees. Unfortunately, there is no exact or heuristic algorithm available to solve these problems on a general network. Recently, Richey [14] showed that PLP is NP-complete even on outerplanar graphs. Once a graph theoretic optimization problem is shown to be NP-complete, a natural attempt is to identify special classes of graphs on which it is solvable in polynomial time. The results of Richey [14] give some indication of the nonexistence (unless P = NP) of nontrivial classes of graphs with cycles on which PLP is solvable in polynomial time.

Another line of reasoning shows that it is not just the structure of the graph that is responsible for the NP-completeness of PLP. The length of the facility to be located also plays a role in this undesirable property. So, the question is: can we construct a polynomial time algorithm to solve PLP by restricting the length of P_L? The present work is an attempt to answer this question. We show that if L is "small" then PLP can be solved in polynomial time whenever the problem:

$$\begin{aligned}\text{RPLP}: \quad & \text{Min } f(a_1 x + b_1, a_2 x + b_2, \ldots, a_n x + b_n) \\ & \text{Subject to} \\ & \qquad a \le x \le b\end{aligned}$$

(where a_i, b_i, a, and b are real numbers) can be solved in polynomial time.

We also show that for "small" L, PLP can be solved in $O\left(n^4\right)$ time when $f(\cdot) = \Sigma(\cdot)$ and in $O\left(n^4 \log n\right)$ time when $f(\cdot) = \max(\cdot)$. Further, we observe that when $f(\cdot) = \Sigma(\cdot)$, there always exists an optimal solution of PLP in which one end of the optimal path coincides with a vertex. This can be viewed as an extension of Hakimi's node optimality theorem for point median problems. This result was independently obtained by Hakimi, Schmeichel and Labbe [4]. Further, we show that PLP can be solved in polynomial time whenever the optimal facility is known to contain at most r interior vertices, for some fixed r.

2. SOLUTION STRATEGY

We first assume that the length L of the path-like facility to be located is less than or equal to the smallest edge length of G *i.e.*

$$L \leq \operatorname{Min}\{c_{rs} : (r, s) \in E(G)\}. \tag{A1}$$

Even under assumption A1, we *cannot* conclude that an optimal facility will be contained in an edge of G. To observe this, consider the example in Figure 1. Let $L = 0.8$, $c_{rs} = c_{st} = 1$, and $f(\cdot) = \max(\cdot)$. Then an optimal path will be the one through s with 0.4 units on (r, s) and 0.4 units on (s, t).

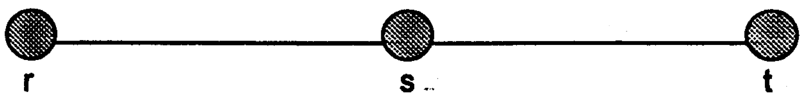

Figure 1.

A 1-path of a graph is a path joining two fixed vertices such that it contains exactly one interior vertex. Let (r, s, t) be the 1-path joining the vertices r and t with s as its interior vertex. The length of (r, s, t) is $c_{rs} + c_{st}$. The following theorem allows us to solve PLP efficiently whenever assumption A1 is satisfied.

THEOREM 1: *If A1 is satisfied, then there exists an optimal solution to PLP which is completely contained in some* 1*-path.*

Proof: If possible let an optimal path be not contained in any of the 1-paths of G. Then at least two nodes of the graph must be interior vertices of this path. Thus the length of the path must be greater than $\min\{c_{ij} : (i, j) \in E(G)\}$. This contradicts A1 and hence the result.

Let PLP (r, s, t) denote the problem PLP when P_L is restricted to the 1-path (r, s, t). The corresponding optimal solution is called a *local location on* (r, s, t) and the objective function value corresponding to a local location on (r, s, t) is called the *local distance on* (r, s, t).

The 1-path (r, s, t) which is of length $c_{rs} + c_{st}$ can be identified by the interval $[0, c_{rs} + c_{st}]$ with a distinguished point at a distance c_{rs} units from 0. Thus the point 0 represents r, c_{rs} represents s and $c_{rs} + c_{st}$ represents t. The path-like facility P_L restricted to (r, s, t) can be considered as the sub interval $[\alpha, \alpha + L]$ of $[0, c_{rs} + c_{st}]$ where $\alpha \leq c_{rs} + c_{st} - L$. Let $x = c_{rs} - L$, $y = c_{rs}$, and $z = c_{rs} + c_{st} - L$. Clearly $0 \leq x \leq y \leq z$. Now for each $j \in V(G)$, $d_j(P_L) \equiv d_j(\alpha)$, where $d_j : [0, z] \to \mathfrak{R}$ such that

$$d_j(\alpha) = \begin{cases} \min\{d_{rj} + \alpha,\ d_{sj} + c_{rs} - (\alpha + L)\} \\ \qquad \text{if} \quad 0 \leq \alpha \leq x \\ \min\{d_{rj} + \alpha,\ d_{sj},\ d_{tj} + c_{rs} + c_{st} - (\alpha + L)\} \\ \qquad \text{if} \quad x \leq \alpha \leq y \\ \min\{d_{sj} + \alpha - d_{rs},\ d_{tj} + c_{rs} + c_{st} - (\alpha + L)\} \\ \qquad \text{if} \quad y \leq \alpha \leq z. \end{cases} \tag{1}$$

Thus for $\alpha \in [0, z]$, $d_j(\alpha)$ is a continuous piecewise linear function. It can be seen that the linear pieces of $d_j(\alpha)$ have slopes $+1$, -1 or 0 and there are at most five such pieces for every j. Further, it is concave on $[0, x]$, $[x, y]$ and $[y, z]$ but not necessarily concave on $[0, z]$.

Let $\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jp(j)}$ be the break points (including end points 0 and z) of $d_j(\alpha)$, $\alpha \in [0, z]$. Note that $p(j) \leq 6$ for all $j \in V(G)$. Let, $0 = \beta_1 < \beta_2 \ldots < \ldots \beta_w = z$ be an ascending arrangement of distinct α_{ji}, $1 \leq i \leq p(j)$, $1 \leq j \leq n$. Clearly $w \leq 4n + 2$. It is easy to see that each of the $d_j(\alpha)$ is linear in $[\beta_i, \beta_{i+1}]$ and thus can be represented as $\gamma_{ij}\alpha + \delta_{ij}$ for some γ_{ij} and δ_{ij}. Consider the problem:

$$\text{RPLP}(i): \quad \text{Min } f(\gamma_{i1}\alpha + \delta_{i1}, \gamma_{i2}\alpha + \delta_{i2}, \ldots, \gamma_{in}\alpha + \delta_{in})$$
$$\text{Subject to}$$
$$\beta_i \leq \alpha \leq \beta_{i+1}$$

Let θ_i be the optimal objective function value of RPLP (i) and α_{i^*} be the corresponding optimal solution. Choose k such that $\theta_k = \text{Min}\{\theta_i : 1 \leq i \leq w - 1\}$. Then it can be seen that the path $[\alpha_{k^*}, \alpha_{k^*} + L]$ is an optimal solution of PLP (r, s, t).

Using equation (1), δ_{ij}'s and γ_{ij}'s can be obtained in $O(n)$ time. (We assume that the distance matrix $D = (d_{ij})$ is available). Let $Q(\beta_i) = \{j : \alpha_{jh} = \beta_i \text{ for some } h\}$. Then using $\gamma_{(i+1)j}$ and $\delta_{(i+1)j}$, $j \in Q(\beta_{i+1})$ the problem RPLP $(i+1)$ can be generated from RPLP (i) in $O(|Q(\beta_{i+1})|)$ time. Further, the computation of β_i's and its arrangement can be done in $O(n \log n)$ time [1] and $\sum_{i=1}^{w-1} |Q(\beta_i)| = O(n)$. Thus PLP (r, s, t) can be solved in $O(ng(n) + n \log n) = O(ng(n))$ time where $O(g(n))$ is the complexity of RPLP $(\cdot)$.

In a graph the number of 1-paths is $O(n^3)$. Now solving PLP (r, s, t) for each of these 1-paths and choosing that local location with smallest local distance gives an optimal solution of PLP. Thus in view of the above observations PLP can be solved in $O(n^4 g(n))$ time.

3. SPECIAL CASES

In this section we consider two special cases of PLP called the *path-centre problem* (PCP) (when $f(\cdot) = \max(\cdot)$) and the *path-median problem* (PMP) (when $f(\cdot) = \Sigma(\cdot)$). We show that these special cases can be solved more efficiently by exploiting the special nature of the function f.

(a) The path centre problem (PCP)

In this case RPLP (i) reduces to a minmax linear programming problem which can be solved in $O(n)$ time [11]. Thus the general algorithm given in section 2 can be used to solve PCP in $O(n^5)$ time. We now show that PCP can be solved in $O(n^4 \log n)$ time.

Let PCP (r, s, t) denote the restriction of PCP to the 1-path (r, s, t). The corresponding optimal solution is called a *local path centre on* (r, s, t) and its objective function value is called the *local path radius on* (r, s, t). PCP (r, s, t) can be decomposed into two subproblems:

$$\text{PCP1}(r, s, t): \ \text{Min Max}\{d_j(\alpha) : j \in V(G)\}$$
$$\text{Subject to}$$
$$\alpha \in [0, x] \cup [y, z]$$

and

$$\text{PCP2}(r,\ s,\ t):\ \text{Min Max}\{d_j(\alpha): j \in V(G)\}$$
$$\text{Subject to}$$
$$\alpha \in [x,\ y]$$

Now, for $\alpha \in [0,\ x]$ and for $\alpha \in [y,\ z]$, $d_j(\alpha)$ is a piecewise linear concave function with at most two linear pieces having slopes ± 1 only. Thus PCP1 can be solved using any algorithm to find a local absolute centre on an edge of a graph (with appropriate modifications). If we use the algorithm of Cunninghame-Green [2], or Kariv and Hakimi [9], $\text{PCP1}(r,\ s,\ t)$ can be solved in $O(n \log n)$ time.

The problem $\text{PCP2}(r,\ s,\ t)$ is essentially that of computing the lowest point on the upper envelope of linear segments with slope $+1$, -1, or 0. It can be solved in $O(n \log n)$ time with appropriate modifications of the methods discussed in [5, 8].

A local path centre on $(r,\ s,\ t)$ can be obtained from the optimal solutions of $\text{PCP1}(r,\ s,\ t)$ and $\text{PCP2}(r,\ s,\ t)$ whichever gives the least objective funciton value. Thus $\text{PCP}(r,\ s,\ t)$ can be solved in $O(n \log n)$ time. Repeating the same process for each 1-path and choosing the "best" local path centre obtained, gives an optimal solution to PCP. Thus we have an $O(n^4 \log n)$ algorithm to solve PCP.

As in the case of point location problems, the average performance of the algorithm can be improved by using some upper and lower bounds for the path radius (optimal function value). Since $d_j(\alpha)$ is concave in each of the interval $[0,\ x]$ and $[x,\ z]$,

$$\text{LB}(r,\ s,\ t) = \text{Max}\{\text{Min}(d_j(0),\ d_j(x),\ d_j(z)) :\ j \in V(G)\}$$

is a valid lower bound for the local path radius on $(r,\ s,\ t)$. If UB is an upper bound for path radius, then any 1-path in which $\text{LB}(r,\ s,\ t) > \text{UB}$ cannot contain a path centre of the network. Thus a local path centre on $(r,\ s,\ t)$ need not be computed for such 1-paths and can be discarded. The upper bound UB can be set initially as the local path radius of the first 1-path examined. Later, if we encounter a local path radius having lesser value than the current UB, the UB can be replaced by the new improved value.

(b) Path Median Problem (PMP)

PMP can be solved in $O(n^5)$ time using the algorithm in section 2. We now show that PMP can be solved in $O(n^4)$ time.

Let PMP (r, s, t) be the restriction of PMP to the 1-path (r, s, t). The corresponding optimal solution is called a *local path median on* (r, s, t) and its objective function value is called the *local path distance* on (r, s, t).

Since $d_j(\alpha)$ is concave in $[0, x]$ and $[x, z]$, $\sum_{j \in V(G)} d_j(\alpha)$ is also concave in these intervals. Thus

$$\operatorname*{Min}_{\alpha \in [0, z]} \sum_{j \in V(G)} d_j(\alpha) = \operatorname{Min}\{\sum_{j \in V(G)} d_j(0), \sum_{j \in V(G)} d_j(x), \sum_{j \in V(G)} d_j(z)\}.$$

Using the above expression, a local path median on (r, s, t) can be easily identified in $O(n)$ time. Repeating this for each 1-path, an optimal solution to PMP can be identified as the "best" local path median obtained. Hence PMP can be solved in $O(n^4)$ time.

The above discussion shows that there exists a local path median on (r, s, t) which will be of the form $[0, L]$, $[x, x+L]$, or $[z, z+L]$. Thus we have

THEOREM 2: *There exists an optimal path solution for PMP whose one end point is at a vertex.*

It may be noted that we have proved the above theorem under the assumption (A1). However, the theorem remains true even if we drop (A1) and allow L to be of arbitrary length. This result was obtained independently by Hakimi, Schmeichel, and Labbe [4]. Hooker *et al.* [8] presented a unifying approach for a class of point network location problems by introducing the concept of a finite dominating set (FDS) – a set of points on the network which contains an optimal solution. In the case of single facility point location problems, the identification of FDS is closely related to the identification of linear arc segments (also known as tree-like segments [7]). Our algorithm for path location problems may be considered as an extension of the ideas discussed in [7, 8] for point location problems to path location problems. For more details, we refer to [7, 8].

Suppose that in the path location problem, it is known *a priori* that an optimal facility will contain at most r interior vertices. (For example, when L is less than or equal to the length of the smallest $(r-1)$-path (path containing r interior vertices) this condition holds), then also PLP can be solved in polynomial time whenever RPLP can be solved in polynomial time. What is to be done in this case is to generate relevant k-paths for $k \leq r$ and solve PLP restricted to each of these paths to obtain "local" solutions. The "best" solution amongst these local solutions is an optimal to PLP. It can be verified that for fixed r, this method is polynomially bounded.

4. CONCLUSION

The main purpose of this paper is twofold. On one hand, we show that the complexity of the path location problem is not only dependent on the structure of the graph on which it is considered, but the length of the facility to be located also. On the other hand, we show how known methods from point location theory can be extended to path location problems. As a result, we have an $O\left(n^4 \log n\right)$ algorithm for the path centre problem and an $O\left(n^4\right)$ algorithm for the path median problem whenever the length of the path to be located is "small". It may be noted that for PCP, PCP1 (r, s, t) can be solved in $O(n)$ time by a modification of the algorithm of [9] for the unweighted absolute centre problem. So the bottleneck complexity is in solving $\mathrm{PCP2}(r, s, t)$. We believe that a careful modification of the algorithm of [9] will provided an $O(n)$ algorithm for $\mathrm{PCP2}(r, s, t)$ also and consequently we can have an $O\left(n^4\right)$ algorithm for PCP. It is also observed that if the facility to be located is known to be contained in some r-path, then for fixed r, PLP can be solved in polynomial time whenever RPLP can be solved in polynomial time. Another way to solve the general PLP is to restrict P_L to spanning trees of G. Since the number of spanning trees of G can be at most n^{n-2}, we get a finite algorithm to solve PLP whenever its restriction on trees can be solved by a finite algorithm. (Note that the number of feasible solutions of PLP are infinite). However these are just enumeration schemes and can take an enormous amount of computer time for larger values of n. There is no efficient algorithm available to solve PLP optimally even for specially structured f such as $f(\cdot) = \mathrm{Max}\,(\cdot)$ or $f(\cdot) = \Sigma\,(\cdot)$. We hope that the remarks made in this paper will be helpful in developing exact or heuristic algorithms to solve PLP without any restriction on L. In particular, for PMP with arbitrary L, the observation that one end of an optimal path will be at a vertex seems promising.

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