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NEW CONVERGENCE RESULTS ON AN ALGORITHM FOR NORM CONSTRAINED REGULARIZATION AND RELATED PROBLEMS (*)

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Abstract. – *The constrained least-squares regularization of nonlinear ill-posed problems is a nonlinear programming problem for which trust-region methods have been developed. In this paper we complement the convergence theory of one of those methods showing that, under suitable hypotheses, local (superlinear or quadratic) convergence holds and every accumulation point is second-order stationary.*

Keywords: Trust-region methods, Regularization, Ill Conditioning, Ill-Posed Problems, Constrained Minimization, Fixed-Point Quasi-Newton methods.

Résumé. – *La régularisation, sous forme de moindres carrés contraints, de problèmes non-linéaires mal posés est un problème de programmation non-linéaire, pour lequel ont été proposées des méthodes de régions de confiance (trust-région). Nous complétons dans cet article la théorie de la convergence de l'une de ces méthodes en montrant que, sous des hypothèses appropriées, il y a convergence locale (superlinéaire ou quadratique), tandis que tout point d'accumulation est stationnaire du second ordre.*

Mots clés : Méthodes de région de confiance, régularisation, mauvais conditionnement, problèmes mal posés, minimisation contrainte, point fixe, méthodes quasi-newtoniennes.

1. INTRODUCTION

Many practical problems in applied sciences and engineering give rise to ill-conditioned (linear or nonlinear) systems

$$F(x) = y \tag{1}$$

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where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Neither "exact solutions" of (1) (when they exist), nor global minimizers of $\|F(x) - y\|$ have physical meaning since they are, to a great extent, contaminated by the influence of measuring and rounding errors and, perhaps, uncertainty in the model formulation. From the numerical point of view, this inadequacy usually produces "unreasonably large solutions" x, for some problem-dependent vectorial norm. On the other hand, problems like (1) usually come from discretization of ill-posed infinite dimensional problems, for which bounds on the function or derivatives are generally known.

The most popular way to deal with these problems is through Tikhonov regularization [23]. This amounts to consider, instead of (1), the regularized problem

$$\text{Minimize } \|F(x) - y\|^2 + \mu|x|^2 \tag{2}$$

where $|\cdot|$ is an appropriate (problem-dependent) norm and $\mu > 0$ is a regularization parameter. However, for very ill-conditioned problems, an extremely small value of μ produces a very small norm of $x(\mu)$ (the solution of (2)) and, so, useful characteristics of the estimator x can be lost by the effort of regularization. As a simple example, consider the system

$$x_1 + x_2 = 1, \quad (1 + 10^{-6})x_1 + x_2 = 1 - 10^{-6} - 10^{-2} \tag{3}$$

which was obtained as a perturbation of

$$x_1 + x_2 = 1, \quad (1 + 10^{-6})x_1 + x_2 = 1 - 10^{-6}. \tag{4}$$

The exact solution of (4) is $(-1, 2)$, while the exact solution of (3), which coincides with the solution of (2) for $\mu = 0$, is $\approx (-10001.0, 10002.0)$. However, for all $\mu \in [10^{-7}, 10^{-2}]$ the solution of (2) is $\approx (0.5, 0.5)$, and $\|x(\mu)\|_2$ decreases monotonically for $\mu > 10^{-2}$.

This phenomenon motivated some authors to develop regularization procedures where the norm of the solution is controlled directly, and not through the regularization parameter. *See* [24, 10]. With this approach, instead of (2), the following problem can be considered:

$$\text{Minimize } \|F(x) - y\|^2 \text{ subject to } |x| \leq \theta, \tag{5}$$

where, generally, $\|\cdot\|$ is the Euclidian norm and $|\cdot|$ depends on the problem and, frequently, reflects some tolerance for the variation of the unknown on the considered domain. Vogel and Heinkenschloss used trust-region methods for solving (5). The feasible region of (5) is, generally, an ellipsoid (which can be reduced to an Euclidian ball by a change of variables). Clearly, the amount of structure of an ellipsoidal constraint is too much appealing to be ignored by a linearization. So, in the above mentioned works, trust-region methods were used, keeping the feasible region in its original form. Consequently, the subproblems to be solved consist of the minimization of a quadratic on the intersection of two Euclidian balls. In [24] and [10] only convex quadratic models are considered, so that the subproblem of minimizing the quadratic in the two-ball intersection is not hard. However, when $F(x)$ is nonlinear, the Hessian of the objective function of (5) can have negative eigenvalues and, so, it becomes desirable to consider more general quadratic models. The subproblem of minimizing an arbitrary quadratic in the intersection of two balls turned out to be tractable only after the characterization of local-nonglobal minimizers of quadratics on spheres, given independently in [15] and [13]. Using this characterization, a suitable algorithm for solving the subproblem was proposed by the authors in [16]. In that work, it was also developed a global convergence theory for a trust region algorithm with approximate solutions of the subproblems. Moreover, the theory of [16] is not restricted to ball domains and can be applied to general closed feasible regions, although, of course, its applicability is restricted to the case in which the subproblems are solvable, at least approximately.

One of the main motivations for developing the theory in a general setting is the consideration of problems where the domain is the intersection of the level sets of two (or more) quadratics which, in the regularization framework, can represent bounds on two (or more) different norms of derivatives of the unknown. Recent research on the minimization of quadratics on the intersection of quadratic domains (cf. [18]) indicate that subproblems like that will be probably solved in a satisfactory way, from the computational point of view, in the near future. *See* [21, 25]. Other applications of this subproblem can be found in [20, 4, 6].

The present research complements the convergence results of [16]. In fact, in [16] a global convergence theory was developed, but nothing was said about local speed of convergence or convergence to second order stationary points. The main objective of this paper is to fill those gaps. We assume that, at the final stages of the trust-region algorithm developed in [16] the active constraints at the solution are identified (this was proved, under suitable hypotheses, by Bitar and Friedlander [2]), so that, in the end, the algorithm becomes a trust-region algorithm for equality constrained optimization. Studying the algorithm under this point of view, we give sufficient conditions for local superlinear and quadratic convergence and we prove that stationary points satisfy second-order stationary conditions.

Although the main practical application of our algorithms corresponds to the case where the domain is a ball (ultimately, a sphere), we have strong reasons for developing the theory in a more general context. In fact, as we mentioned above, we have in mind regularizing domains formed by (one or more) quadratic constraints and we are optimistic with practical progress on the resolution of the corresponding subproblems. Moreover, in these cases, nonregular points (points where the gradients of the active constraints are linearly dependent) can appear and, so, we wish to develop a theory that is not based on the usual regularity assumption as a constraint qualification for optimality. This is the main reason for not supporting our proofs on local coordinates, or related differential geometry arguments.

The organization of the paper is as follows: in Section 2 we describe a *Local Algorithm* for solving the Equality Constrained Minimization Problem. The local algorithm is well defined in a neighborhood of a point that satisfies the second-order sufficient conditions for local minimizer. We prove local convergence and superlinear convergence, if the Hessian approximations satisfy a Dennis-Moré condition. Under the Dennis-Moré hypothesis, we also prove that the iterations of the local algorithm produce sufficient descent of the objective function. The main ingredient for the proofs on this section is the theory of Fixed-Point Quasi-Newton methods [14]. In Section 3, we describe the trust-region method as a general algorithm for equality constrained minimization. Global convergence to first-order stationary points follows from the results of [16]. Here we prove that, if we use true Hessian matrices, every accumulation point must be second-order stationary. Finally, we prove that, in a neighborhood of a point that satisfies second-order sufficient conditions, the local algorithm and the trust-region algorithm coincide, so the trust-region algorithm also has local convergence properties. In Section 4, we show some numerical examples concerning the regularization problem. Conclusions are given in Section 5.

2. THE LOCAL METHOD

In this section we define a *local algorithm* for solving the Equality Constrained Minimization Problem. By this we mean that we introduce a method that is well defined in a neighborhood of an appropriate solution, we prove convergence of the method if the initial point is close enough to this solution, and we give conditions for superlinear convergence. Let us define the *Equality Constrained Minimization Problem* as follows:

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & h(x) = 0, \end{array} \tag{6}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$, f, $h \in C^2(\mathbb{R}^n)$. We denote by $h'(x)$ the Jacobian matrix of $h(x)$ and we define $\mathcal{S} = \{x \in \mathbb{R}^n \mid h(x) = 0\}$. From now on, $\| \cdot \|$ will denote an arbitrary norm on $\mathbb{R}^n$.

The "local" method for solving (6) is defined by Algorithm 2.1 below.

ALGORITHM 2.1: Let $x_0 \in \mathbb{R}^n$ be a given initial approximation to the solution of (6). Given $x_k \in \mathbb{R}^n$, B_k a symmetric $n \times n$ matrix, we compute x_{k+1} as the solution y of

$$\begin{array}{ll} \text{Minimize} & \frac{1}{2}(y - x_k)^T B_k (y - x_k) + g_k^T (y - x_k) \\ \text{subject to} & h(y) = 0, \end{array} \tag{7}$$

where $g_k \equiv g(x_k)$ and $g \equiv \nabla f$.

The solution of (7) exists and is unique only under special circumstances, which we will study later. Algorithm 2.1 may be interpreted as a Fixed-Point Quasi-Newton method in the sense of [14]. Given $x \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times n}$ symmetric, we define $\Phi(x, B)$ as the solution of

$$\begin{array}{ll} \text{Minimize} & \frac{1}{2}(y - x)^T B (y - x) + g(x)^T (y - x) \\ \text{subject to} & h(y) = 0 \, . \end{array} \tag{8}$$

So, Algorithm 2.1 may be written as

$$x_{k+1} = \Phi(x_k, B_k).$$

As in [14], we denote $\Phi'(x, B)$ the Jacobian matrix with respect to x. In the following lemma, we compute this Jacobian.

LEMMA 2.1: *Assume that for some $x \in \mathbb{R}^n$, $B = B^T$, (8) has a unique solution y, where rank $h'(y) = m$, $\mu \in \mathbb{R}^m$ is the corresponding vector of Lagrange multipliers, and*

$$z^T \left(B + \sum_{i=1}^{m} \mu_i \nabla^2 h_i(y) \right) z > 0 \tag{9}$$

for all $z \in \mathcal{N}(h'(y))$ *(the null-space of* $h'(y)$*),* $z \neq 0$*. Then*

$$\Phi'(x,\ B) = P\left[P^T\left(B + \sum_{i=1}^{m} \mu_i \nabla^2 h_i(y)\right)P\right]^{-1} P^T(B - \nabla^2 f(x)), \quad (10)$$

where $P \in \mathbb{R}^{n\times(n-m)}$ *is a matrix whose columns form a basis of* $\mathcal{N}(h'(y))$.

Proof: If $y \in \mathbb{R}^n$ is a solution of (8), by the Lagrange optimality conditions, we have that

$$\begin{aligned} B(y-x) + g(x) + h'(y)^T\mu = 0 \\ h(y) = 0\,. \end{aligned} \quad (11)$$

This is a system of $n + m$ nonlinear equations with variables x, y, B and μ. Since rank $h'(y) = m$, and by (9), we have that the matrix $\begin{pmatrix} B + \sum_{i=1}^{m} \mu_i \nabla^2 h_i(y) & h'(y)^T \\ h'(y) & 0 \end{pmatrix}$ is nonsingular. So, we can apply the Implicit Function Theorem on (11), which, by derivation with respect to x, gives

$$\begin{pmatrix} B + \sum_{i=1}^{m} \mu_i \nabla^2 h_i(y) & h'(y)^T \\ h'(y) & 0 \end{pmatrix} \begin{pmatrix} \Phi'(x,\ B) \\ C \end{pmatrix} = \begin{pmatrix} B - \nabla^2 f(x) \\ 0 \end{pmatrix}$$

where C is the matrix of derivatives of μ with respect to x. So,

$$\left[B + \sum_{i=1}^{m} \mu_i \nabla^2 h_i(y)\right] \Phi'(x,\ B) + h'(y)^T C = B - \nabla^2 f(x) \quad (12)$$

and

$$h'(y)\Phi'(x,\ B) = 0. \quad (13)$$

By (13), there exists $M \in \mathbb{R}^{(n-m)\times n}$ such that

$$\Phi'(x,\ B) = PM. \quad (14)$$

Replacing (14) in (12), and pre-multiplying by P^T, we obtain

$$P^T\left[B + \sum_{i=1}^{m} \mu_i \nabla^2 h_i(y)\right] PM = P^T(B - \nabla^2 f(x)).$$

So,

$$M = \left\{ P^T \left[B + \sum_{i=1}^{m} \mu_i \nabla^2 h_i(y) \right] P \right\}^{-1} P^T (B - \nabla^2 f(x)). \tag{15}$$

Therefore, (10) follows from (14) and (15). □

General Local Assumptions. Let us assume now that $x_* \in \mathbb{R}^n$ is a solution of (6) where $h'(x_*)$ has full rank and the second-order sufficient conditions for local minimizer hold. That is

$$z^T G_* z > 0 \tag{16}$$

for all $z \in \mathcal{N}(h'(x_*)), z \neq 0$, where $G_* = \nabla^2 f(x_*) + \sum_{i=1}^{m} \mu_i^* \nabla^2 h_i(x_*)$ and $\mu^* \in \mathbb{R}^m$ is the vector of Lagrange multipliers associated to (6) and x_*.

By the Implicit Function Theorem, these assumptions guarantee that $\Phi(x, B)$ and $\Phi'(x, B)$ exist in a neighborhood $\Omega \times D$ of $(x_*, \nabla^2 f(x_*))$. Moreover, we can assume that $x_* = \Phi(x_*, B)$ for all $B \in D$ and so, by (10),

$$\Phi'(x_*, B) = P_* \left[P_*^T \left(B + \sum_{i=1}^{m} \mu_i^*(B) \nabla^2 h_i(x_*) \right) P_* \right]^{-1} P_*^T (B - \nabla^2 f(x_*)).$$

The continuity of $\Phi'(x_*, B)$ with respect to B in D is guaranteed by elementary arguments, with a possible restriction of D. We also assume that there exist $L, p > 0$, such that

$$\|\Phi'(x, B) - \Phi'(x_*, B)\| \leq L \|x - x_*\|^p \tag{17}$$

for all $x \in \Omega$, $B \in D$. Clearly

$$\Phi'(x_*, \nabla^2 f(x_*)) = 0\,. \tag{18}$$

The discussion above allows us to prove the following local convergence theorem.

THEOREM 2.2: *Suppose that f, h, x_* satisfy the General Local Assumptions. Let $r \in (0, 1)$. Then there exist $\varepsilon = \varepsilon(r)$, $\delta = \delta(r)$ such that, if $\|x - x_*\| \leq \varepsilon$, and $\|B - \nabla^2 f(x_*)\| \leq \delta$, we have*

$$\|\Phi(x, B) - x_*\| \leq r \|x - x_*\|. \tag{19}$$

Moreover, if $\|x_0 - x_*\| \leq \varepsilon$, *and* $\|B_k - \nabla^2 f(x_*)\| \leq \delta$ *for all* $k = 0, 1, 2, \ldots$ *the sequence generated by Algorithm 2.1 is well defined, converges to* x_*, *and satisfies*

$$\|x_{k+1} - x_*\| \leq r\|x_k - x_*\|$$

for all $k = 0, 1, 2, \ldots$.

Proof: The result follows from (17), (18) and (10) as a consequence of Theorem 3.1 of [14]. □

LEMMA 2.3: *Assume the hypotheses of Theorem 2.2. If* $\mu^k \in \mathbb{R}^m$ *is the vector of Lagrange multipliers associated to (7) then there exist* $c_1, c_2 > 0$, $k_0 \in \mathbb{N}$ *such that* $\|\mu^k\| \leq c_1$ *and*

$$\frac{(x_{k+1} - x_k)^T (B_k + \sum_{i=1}^m \mu_i^k \nabla^2 h_i(x_k))(x_{k+1} - x_k)}{\|x_{k+1} - x_k\|^2} \geq c_2$$

for all $k \geq k_0$.

Proof: It results from Theorem 2.2, the continuity of the Lagrange multipliers, (16) and the fact that $h(x_k) = 0$ for all $k \in \mathbb{N}$. □

The following theorem gives a Dennis-Moré type condition for the superlinear convergence of a sequence generated by Algorithm 2.1. The Dennis - Moré type condition associated to superlinear convergence of SQP algorithms [3] involves the effect of the approximation of the Hessian of de Lagrangian on the increment. It is interesting to observe that, when we do not approximate the constraints by their linear model, the condition for superlinear convergence is associated with approximations of the Hessian of the objective function.

THEOREM 2.4: *Assume the hypotheses of Theorem 2.2. Suppose that*

$$\lim_{k \to \infty} \frac{\|[B_k - \nabla^2 f(x_*)](x_{k+1} - x_k)\|}{\|x_{k+1} - x_k\|} = 0. \tag{20}$$

Then

$$\lim_{k \to \infty} \frac{\|x_{k+1} - x_*\|}{\|x_k - x_*\|} = 0. \tag{21}$$

Proof: By elementary continuity arguments, (20) and (10) imply that

$$\lim_{k \to \infty} \frac{\|[\Phi'(x_*, B_k) - \Phi'(x_*, \nabla^2 f(x_*))](x_{k+1} - x_k)\|}{\|x_{k+1} - x_k\|} = 0.$$

Therefore, (21) follows from Theorem 4.2 of [14]. □

The following theorem states the order of convergence of the Newton version of Algorithm 2.1.

THEOREM 2.5: *Assume the hypotheses of Theorem 2.2. Suppose that, for all* $k = 0, 1, 2, \ldots, B_k = \nabla^2 f(x_k)$. *Then, there exists* $c > 0$ *such that*

$$\|x_{k+1} - x_*\| \leq c\|x_k - x_*\|^{p+1}. \tag{22}$$

Proof: The desired result follows from Theorem 4.3 of [14]. □

The final result of this section is very important to support global convergence properties of the method. Briefly, it states that, in an appropriate neighborhood of x_*, when the Dennis-Moré condition holds, a sufficient descent property takes place.

THEOREM 2.6: *Suppose that the General Local Assumptions hold,* $f, h \in C^2(\mathbb{R}^n)$, $\alpha \in (0,1)$. *Suppose that* $\{x_k\}$ *is an arbitrary sequence of points that satifies the constraints of (6) and converges to* x_* *and that* $\{B_k\}$ *is a sequence of matrices such that* $\Phi(x_k, B_k)$ *is well defined for all* $k \in \mathbb{N}$ *and*

$$\lim_{k\to\infty} \frac{\|[B_k - \nabla^2 f(x_*)]s_k\|}{\|s_k\|} = 0. \tag{23}$$

Then, there exists $k_0 \in \mathbb{N}$ *such that, for all* $k \geq k_0$,

$$f(x_k + s_k) \leq f(x_k) + \alpha\psi_k(s_k),$$

where $s_k = \Phi(x_k, B_k) - x_k$ *and* $\psi_k(s) = g_k^T s + \frac{1}{2}s^T B_k s$ *for all* $s \in \mathbb{R}^n$.

Proof: By the first order optimality conditions of (7), we have that there exists $\mu^k \in \mathbb{R}^m$ such that

$$\begin{aligned} B_k s_k + g_k + h'(y_k)^T \mu^k &= 0 \\ h(y_k) &= 0 \end{aligned} \tag{24}$$

for all $k \in \mathbb{N}$, where $y_k = x_k + s_k$. By (24),

$$g_k^T s_k = -(\mu^k)^T h'(y_k)s_k - s_k^T B_k s_k. \tag{25}$$

By Taylor's formula, we have, for $i = 1, \ldots, m$,

$$h_i(x_k) = h_i(y_k) - h_i'(y_k)s_k + \frac{1}{2}s_k^T \nabla^2 h_i(y_k)s_k + o(\|s_k\|^2)\ . \tag{26}$$

Since $h_i(x_k) = h_i(y_k) = 0$, (26) implies that

$$(\mu^k)^T h'(y_k)s_k = \frac{1}{2}s_k^T\left(\sum_{i=1}^{m}\mu_i^k\nabla^2 h_i(y_k)\right)s_k + (\mu^k)^T o(\|s_k\|^2). \tag{27}$$

By (25) and (27), we have that

$$g_k^T s_k = -s_k^T\left[B_k + \frac{1}{2}\sum_{i=1}^{m}\mu_i^k\nabla^2 h_i(y_k)\right]s_k - (\mu^k)^T o(\|s_k\|^2). \tag{28}$$

Now, by Taylor's formula, we have

$$f(y_k) = f(x_k) + g_k^T s_k + \frac{1}{2}s_k^T\nabla^2 f(x_k)s_k + o(\|s_k\|^2).$$

So, by (28), and the boundedness of $\|\mu^k\|$,

$$\begin{aligned} f(y_k) &= f(x_k) \\ &\quad - \frac{1}{2}s_k^T\left[2B_k - \nabla^2 f(x_k) + \sum_{i=1}^{m}\mu_i^k\nabla^2 h_i(y_k)\right]s_k + o(\|s_k\|^2)\,. \end{aligned} \tag{29}$$

But, by the Dennis-Moré condition (23), $\|[B_k - \nabla^2 f(x_k)]s_k\| = o(\|s_k\|)$. Thus, by (29),

$$\begin{aligned} f(y_k) &= f(x_k) - \frac{1}{2}s_k^T\left[B_k + \sum_{i=1}^{m}\mu_i^k\nabla^2 h_i(y_k)\right]s_k + o(\|s_k\|^2) \\ &= f(x_k) - \frac{\overline{\alpha}}{2}s_k^T\left[B_k + \sum_{i=1}^{m}\mu_i^k\nabla^2 h_i(x_k)\right]s_k \\ &\quad - \frac{(1-\overline{\alpha})}{2}s_k^T\left[B_k + \sum_{i=1}^{m}\mu_i^k\nabla^2 h_i(x_k)\right]s_k + o(\|s_k\|^2), \end{aligned}$$

where $\overline{\alpha} \in (\alpha,\ 1)$.

By Lemma 2.3, there exist $c_2 > 0$ and $k_0 \in \mathbb{N}$ such that

$$\frac{(1-\overline{\alpha})}{2}s_k^T\left[B_k + \sum_{i=1}^{m}\mu_i^k\nabla^2 h_i(x_k)\right]s_k \geq c_2\|s_k\|^2$$

for all $k \geq k_0$. Since $-c_2\|s_k\|^2 + o(\|s_k\|^2) < 0$ for large enough k, we conclude that, for k large enough,

$$f(y_k) \leq f(x_k) - \frac{\overline{\alpha}}{2} s_k^T \left[B_k + \sum_{i=1}^{m} \mu_i^k \nabla^2 h_i(x_k) \right] s_k. \tag{30}$$

Hence, by (28) and (30),

$$\begin{aligned} f(y_k) \leq f(x_k) &- \frac{\alpha}{2} s_k^T \left[B_k + \sum_{i=1}^{m} \mu_i^k \nabla^2 h_i(x_k) \right] s_k \\ &- \frac{\overline{\alpha} - \alpha}{2} s_k^T \left[B_k + \sum_{i=1}^{m} \mu_i^k \nabla^2 h_i(x_k) \right] s_k \\ &= f(x_k) + \alpha \left[g_k^T s_k + \frac{1}{2} s_k^T B_k s_k + o(\|s_k\|^2) \right] \\ &- \frac{\overline{\alpha} - \alpha}{2} s_k^T \left[B_k + \sum_{i=1}^{m} \mu_i^k \nabla^2 h_i(x_k) \right] s_k \\ &- f(x_k) + \alpha \left[g_k^T s_k + \frac{1}{2} s_k^T B_k s_k \right] \\ &- \frac{\overline{\alpha} - \alpha}{2} s_k^T \left[B_k + \sum_{i=1}^{m} \mu_i^k \nabla^2 h_i(x_k) \right] s_k + o(\|s_k\|^2) \end{aligned}$$

and the desired result follows from Lemma 2.3. □

3. THE TRUST-REGION METHOD

In this section we introduce a trust-region algorithm for solving the Equality Constrained Minimization Problem (6). Throughout this section we assume that $f, h \in C^3(\mathbb{R}^n)$. We can think the method as an independent one, or just as representing the final stages of a trust-region algorithm for general constrained optimization of the type considered in [16], when the active constraints are identified.

ALGORITHM 3.1: Let $x_0 \in \mathbb{R}^n$ be an initial approximation, $h(x_0) = 0$. Let σ_1, σ_2, α, γ, $\Delta_{\min}$, Δ^0 be such that $0 < \sigma_1 < \sigma_2 < 1$, $\alpha \in (0,1)$, $\Delta_{\min} > 0$, $\Delta^0 \geq \Delta_{\min}$. Given $x_k \in \mathbb{R}^n$ such that $h(x_k) = 0$, $\Delta^k \geq \Delta_{\min}$, B_k a symmetric $n \times n$ matrix, the steps for obtaining x_{k+1} are:

STEP 1: $\Delta \leftarrow \Delta^k$.

STEP 2: Compute $\overline{s}_k(\Delta)$, a global solution of

$$\begin{aligned} \text{Minimize} \quad & \psi_k(s) \equiv \frac{1}{2} s^T B_k s + g_k^T s \\ \text{subject to} \quad & h(x_k + s) = 0, \\ & \|s\| \leq \Delta \end{aligned} \tag{31}$$

If $\psi_k(\overline{s}_k(\Delta)) = 0$, stop.

STEP 3: If

$$f(x_k + \overline{s}_k(\Delta)) \leq f(x_k) + \alpha \psi_k(\overline{s}_k(\Delta)), \tag{32}$$

define $x_{k+1} = x_k + \overline{s}_k(\Delta)$, $\Delta_k = \Delta$.

Otherwise, choose $\Delta \leftarrow \Delta_{new} \in [\sigma_1 \|\overline{s}_k(\Delta)\|, \sigma_2 \Delta]$, and go to Step 2.

Notice that the first trust-region radius Δ^k tried at each iteration is not smaller than a fixed parameter $\Delta_{min} > 0$. This requirement allows us to take large steps far from the solution, eliminating artificially small trial steps inherited from previous iterations. More subtle motivations for the introduction of the algorithmic bound Δ_{min} come from convergence proofs to first-order stationary points of trust-region algorithms with approximate solution of subproblems. In fact, in [16] (*see* also [8]) first-order stationarity is obtained under a condition that, essentially, corresponds to uniform continuity of ∇f on the domain under consideration. Other first-order convergence proofs for constrained trust-region methods (*see*, for example, [5]) use existence and boundedness of second derivatives. A careful analysis of the proofs reveals that, in fact, the stronger assumption on f can be avoided in [16] and [8] due to the introduction of Δ_{min}, which forces the existence of infinitely many rejected steps when, for some subsequence, $\Delta_k \to 0$.

The rest of this section is dedicated to prove that every limit point of a sequence generated by Algorithm 3.1 satisfies optimality conditions. Since we are potentially interested in domains where nonregular points appear naturally (for example, intersection of level sets of quadratic functions), our arguments must be general enough to cope with that type of points. By this reason, we decided to rely on more general constraint qualifications and optimality conditions than the usual ones in nonlinear programming. Arguments based on feasible arcs will provide adequate tools for our objectives.

DEFINITION 3.2: *Given $x \in \mathbb{R}^n$ such that $h(x) = 0$, $b > 0$, we say that $\gamma : [-b, b] \to \mathbb{R}^n$ is a feasible arc that passes through x if*

(a) $h(\gamma(t)) = 0$ *for all* $t \in [-b,\ b]$;
(b) $\gamma \in C^3([-b,\ b]),\ \gamma'(0) \neq 0$;
(c) $\gamma(0) = x$.

THEOREM 3.3: *If* x_* *is a local minimizer of (6), then for all feasible arc* γ *that pass through* x_*, *we have that*

$$g(x_*)^T \gamma'(0) \equiv (f \circ \gamma)'(0) = 0 \tag{33}$$

and

$$(f \circ \gamma)''(0) \geq 0. \tag{34}$$

Proof: Trivial, considering that 0 is a local minimizer of $f \circ \gamma : [-b,\ b] \to \mathbb{R}$. □

Theorem 3.3 motivates the following definition.

DEFINITION 3.4: *We say that* $x_* \in \mathcal{S}$ *is a* second-order stationary point *of (2.1) if for all feasible arc* γ *that passes through* x_*, *(33) and (34) are satisfied.*

In Theorem 3.5 we establish that Algorithm 3.1 can stop only at a second-order stationary point.

THEOREM 3.5: *If* $B_k = \nabla^2 f(x_k)$ *and Algorithm 3.1 stops at Step 2 (so* $\psi_k(\bar{s}_k(\Delta)) = 0$*), then* x_k *is a second-order stationary point of (6).*

Proof: Let γ be a feasible arc that passes through x_k. Since $\psi_k(0) = 0 = \psi_k(\bar{s}_k(\Delta))$, we have that 0 is a solution of (31). Since 0 is an interior point of the feasible region of (31), we have that $(\psi_k \circ \gamma)'(0) = 0$ and $(\psi_k \circ \gamma)''(0) \geq 0$. It is easy to see that these two conditions imply (33) and (34). □

The following theorem states that, if Algorithm 3.1 does not stop at Step 2, then the k-th iteration terminates in finite time. Observe that we do not assume that x_k is a regular point of the feasible region (gradient of the constraints linearly independent). Of course, when the feasible set is a sphere, all its points are regular, but this is not the case when the domain is the intersection of the level sets of two quadratics. As it is well known, defining iterations of algorithms that linearize the constraints is very troublesome if the gradients are not linearly independent.

THEOREM 3.6: *If x_k is not a second-order stationary point of (6) and $B_k = \nabla^2 f(x_k)$, then x_{k+1} is well defined by Algorithm 3.1.*

Before proving Theorem 3.6, we need to introduce a definition and a technical lemma.

DEFINITION 3.7: *Given γ, a feasible arc that passes through x, we define, for $\Delta \geq 0$*

$$\tau_+(\gamma, \Delta) = \min \{t \in [0, b] \mid \|\gamma(t) - \gamma(0)\| = \Delta\},$$
$$\tau_-(\gamma, \Delta) = \max \{t \in [-b, 0] \mid \|\gamma(t) - \gamma(0)\| = \Delta\}.$$

LEMMA 3.8: *Assume that $\gamma_k : [-b, b] \to \mathbb{R}^n$, $\gamma : [-b, b] \to \mathbb{R}^n$, $b > 0$, γ_k, $\gamma \in C^3([-b, b])$ for all $k \in \mathbb{N}$, $\gamma'(0) \neq 0$, and*

$$\lim_{k \to \infty} \|\gamma_k - \gamma\|_3 = 0$$

where $\|\beta\|_3 \equiv \max \{\|\beta(t)\|, \|\beta'(t)\|, \|\beta''(t)\|, \|\beta'''(t)\| \mid t \in [-b, b]\}$. Then there exist c_3, c_4, $\overline{\Delta} > 0$, $k_0 \in \mathbb{N}$ such that $\tau_+(\gamma_k, \Delta)$, $\tau_-(\gamma_k, \Delta)$, $\tau_+(\gamma, \Delta)$ and $\tau_-(\gamma, \Delta)$ are well defined and

$$\begin{aligned} c_3\Delta &\leq \tau_+(\gamma_k, \Delta) &&\leq c_4\Delta \\ c_3\Delta &\leq |\tau_-(\gamma_k, \Delta)| &&\leq c_4\Delta \\ c_3\Delta &\leq \tau_+(\gamma, \Delta) &&\leq c_4\Delta \\ c_3\Delta &\leq |\tau_-(\gamma, \Delta)| &&\leq c_4\Delta \end{aligned} \tag{35}$$

for all $\Delta \in [0, \overline{\Delta}]$, $k \geq k_0$.

Proof: The result follows from a slight adaptation of Lemma 2.1 of [16]. □

Proof of Theorem 3.6: Since x_k is not a second-order stationary point, there exists a feasible arc $\gamma : [-b, b] \to \mathcal{S}$ passing through x_k such that either

$$(f \circ \gamma)'(0) = g(x_k)^T \gamma'(0) < 0 \tag{36}$$

$$\text{or} \quad (f \circ \gamma)'(0) = 0, \quad (f \circ \gamma)''(0) < 0\,. \tag{37}$$

If (36) occurs, the result is proved in the same way of Theorem 2.3 of [16]. It remains to consider the possibility (37). Thus, we have

$$(f \circ \gamma)''(0) = \gamma'(0)^T \nabla^2 f(x_k) \gamma'(0) + g(x_k)^T \gamma''(0) \equiv a < 0\,. \tag{38}$$

Let $\overline{\Delta} > 0$ be such that $\tau_+(\Delta) \equiv \tau_+(\gamma, \Delta)$ and $\tau_-(\Delta) \equiv \tau_-(\gamma, \Delta)$ are well defined and let $c_3, c_4 > 0$ be such that (35) holds for all $\Delta \in [0, \overline{\Delta}]$. So, if $\Delta \in [0, \overline{\Delta}]$, from Step 2 of Algorithm 3.1 we have

$$\begin{aligned}\psi_k(\overline{s}_k(\Delta)) &\leq \psi_k(\gamma(t) - x_k)\\ &= \frac{1}{2}(\gamma(t) - x_k)^T \nabla^2 f(x_k)(\gamma(t) - x_k) + g(x_k)^T(\gamma(t) - x_k)\end{aligned}$$

where $t = \tau_+(\Delta)$ or $t = \tau_-(\Delta)$.

Now, $\gamma(t) = \gamma(0) + t\gamma'(0) + \dfrac{t^2}{2}\gamma''(0) + o(t^2)$, so,

$$\begin{aligned}\psi_k(\overline{s}_k(\Delta)) \leq \frac{t^2}{2}&(\gamma'(0)^T \nabla^2 f(x_k)\gamma'(0)\\ &+ g(x_k)^T \gamma''(0)) + tg(x_k)^T\gamma'(0) + o(t^2)\,.\end{aligned}$$

But, by (37), $g(x_k)^T\gamma'(0) = 0$, and from (35) we have

$$\begin{aligned}\frac{\psi_k(\overline{s}_k(\Delta))}{\Delta^2} &\leq c_4^2 \frac{\psi_k(\overline{s}_k(\Delta))}{t^2}\\ &\leq \frac{c_4^2}{2}(\gamma'(0)^T\nabla^2 f(x_k)\gamma'(0) + g(x_k)^T\gamma''(0)) + \frac{o(t^2)}{t^2}\,.\end{aligned}$$

From (38) it follows that

$$\frac{\psi_k(\overline{s}_k(\Delta))}{\Delta^2} \leq \frac{ac_4^2}{2} + \frac{o(t^2)}{t^2}\,. \tag{39}$$

Thus, from (39) we have

$$\limsup_{\Delta\to 0} \frac{\psi_k(\overline{s}_k(\Delta))}{\Delta^2} \leq \frac{ac_4^2}{2} < 0\,.$$

Therefore, there exists $\overline{\overline{\Delta}} > 0$ such that

$$\frac{\psi_k(\overline{s}_k(\Delta))}{\Delta^2} \leq c_5 \equiv \frac{ac_4^2}{4} < 0 \tag{40}$$

for all $\Delta \in (0, \overline{\overline{\Delta}}]$.

Define, for $\Delta > 0$,

$$\rho(\Delta) = \frac{f(x_k + \overline{s}_k(\Delta)) - f(x_k)}{\psi_k(\overline{s}_k(\Delta))}\,. \tag{41}$$

Then, if $\Delta \in (0, \overline{\overline{\Delta}}]$ we have by (40) and (41) that

$$|\rho(\Delta) - 1| = \left| \frac{f(x_k + \bar{s}_k(\Delta)) - f(x_k) - \psi_k(\bar{s}_k(\Delta))}{\psi_k(\bar{s}_k(\Delta))} \right|$$
$$= \frac{o(\|\bar{s}_k(\Delta)\|^2)}{|\psi_k(\bar{s}_k(\Delta))|} = \frac{o(\Delta^2)}{\Delta^2} \frac{\Delta^2}{|\psi_k(\bar{s}_k(\Delta))|} \leq \frac{1}{|c_5|} \frac{o(\Delta^2)}{\Delta^2}.$$

So,

$$\lim_{\Delta \to 0} \rho(\Delta) = 1,$$

which implies that after a finite number of reductions in the trust-region radius, the condition (32) is verified. As a result, x_{k+1} is well defined. □

Before establishing the global convergence result of Algorithm 3.1 we define a weak regularity assumption that suits the level of generality intended at this section.

DEFINITION 3.9: *We say that $x \in S$ is* weakly regular *if for all feasible arc $\gamma : [-b, b] \to S$ that passes through x and for every sequence $\{x_k\}_{k=1}^{\infty} \subset S$ converging to x there exist $b_1 \in (0, b)$ and $\gamma_k : [-b_1, b_1] \to S$ ($k \in \mathbb{N}$) a sequence of feasible arcs that pass through x_k such that*

$$\lim_{k \to \infty} \|\gamma_k - \gamma\|_3 = 0, \tag{42}$$

where $\|\beta\|_3 \equiv \max \{\|\beta(t)\|, \|\beta'(t)\|, \|\beta''(t)\|, \|\beta'''(t)\| \mid t \in [-b_1, b_1]\}$.

A direct consequence of Theorem 3.1 of [16] is that every regular point in the usual sense of Nonlinear Programming (rank $h'(x) = m$, *cf.* [7, 12]) is weakly regular. The converse is not true. Consider, for example, the set $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 = 0\}$. Clearly all points in S are weakly regular but not regular. Less trivial examples include intersections of tangent cylinders or ellipsoids in $\mathbb{R}^n$. The key point is that weak regularity is a completely geometric concept that does not depend on the algebraic representation of the surface.

The following is the main global convergence result of the paper, that complements the first-order global convergence theorem of [16]. We prove that, if a limit point of a sequence generated by the algorithm with true Hessians is weakly regular, then it is stationary, in the "second-order" sense given by Definition 3.4.

THEOREM 3.10: *Assume that the sequence $\{x_k\}$ is generated by Algorithm 3.1 with $B_k \equiv \nabla^2 f(x_k)$, $x_* \in S$ is weakly regular and*

$\lim_{k\in\mathbb{K}_1} x_k = x_*$, *where* $\mathbb{K}_1$ *is an infinite subset of* $\mathbb{N}$. *Then* x_* *is a second-order stationary point of problem (6).*

Proof: We consider two possibilities:

$$\inf_{k\in\mathbb{K}_1} \Delta_k = 0 \tag{43}$$

and

$$\inf_{k\in\mathbb{K}_1} \Delta_k > 0. \tag{44}$$

Assume first that (43) holds. Then there exists $\mathbb{K}_2$, an infinite subset of $\mathbb{K}_1$ such that

$$\lim_{k\in\mathbb{K}_2} \Delta_k = 0\ . \tag{45}$$

So, there exists $k_2 \in \mathbb{N}$ such that $\Delta_k < \Delta_{min}$ for all $k \geq k_2$, $k \in \mathbb{K}_2$. But, at each iteration k we try first the radius $\Delta^k \geq \Delta_{min}$. Thus, for all $k \in \mathbb{K}_3 \equiv \{k \in \mathbb{K}_2 \mid k \geq k_2\}$ there exist $\overline{\Delta}_k$ and $\overline{s}_k(\overline{\Delta}_k)$ such that $\overline{s}_k(\overline{\Delta}_k)$ is a global solution of

$$\begin{array}{rl} \text{Minimize} & \psi_k(s) \\ \text{subject to} & h(x_k + s) = 0 \\ & \|s\| \leq \overline{\Delta}_k \end{array} \tag{46}$$

and

$$f(x_k + \overline{s}_k(\overline{\Delta}_k)) > f(x_k) + \alpha\psi_k(\overline{s}_k(\overline{\Delta}_k)). \tag{47}$$

By the trust-region radius updating in Algorithm 3.1, for $k \in \mathbb{K}_3$, we have

$$\Delta_k \geq \sigma_1 \|\overline{s}_k(\overline{\Delta}_k)\|. \tag{48}$$

Therefore, by (45) and (48),

$$\lim_{k\in\mathbb{K}_3} \|\overline{s}_k(\overline{\Delta}_k)\| = 0. \tag{49}$$

Suppose that x_* is not second-order stationary. Then, there exist $b > 0$, $\gamma : [-b, b] \to S$ a feasible arc passing through x_*, such that either

$$(f \circ \gamma)'(0) = g(x_*)^T \gamma'(0) < 0 \tag{50}$$

or

$$(f \circ \gamma)'(0) = g(x_*)^T \gamma'(0) = 0 \tag{51}$$

and

$$(f \circ \gamma)''(0) \equiv \gamma'(0)^T \nabla^2 f(x_*) \gamma'(0) + g(x_*)^T \gamma''(0) \equiv a_* < 0. \tag{52}$$

If (50) takes place, the proof follows the same structure of Theorem 3.2 of [16], where first-order stationary conditions were considered. So, we have to focus on (51) and (52).

Since x_* is weakly regular and $\lim_{k \in \mathbb{K}_3} x_k = x_*$, there exist $b_1 \in (0, b)$, $\gamma_k : [-b_1, b_1] \to \mathcal{S}$, $(k \in \mathbb{K}_3)$, a sequence of feasible arcs passing through x_k, such that

$$\lim_{k \in \mathbb{K}_3} \|\gamma_k - \gamma\|_3 = 0. \tag{53}$$

By (53) and Lemma 3.8, there exist $k_3 \in \mathbb{N}$ and $\overline{\Delta} > 0$ such that $\tau_+(\gamma_k, \Delta)$, $\tau_-(\gamma_k, \Delta)$, $\tau_+(\gamma, \Delta)$ and $\tau_-(\gamma, \Delta)$ are well defined for all $k \in \mathbb{K}_4 \equiv \{k \in \mathbb{K}_3 \mid k \geq k_3\}$, $\Delta \in [0, \overline{\Delta}]$. Moreover, (35) holds for all $k \in \mathbb{K}_4$, $\Delta \in [0, \overline{\Delta}]$. Let $k_4 \in \mathbb{N}$ be such that

$$\|\bar{s}_k(\overline{\Delta}_k)\| \leq \overline{\Delta}$$

for all $k \in \mathbb{K}_5 \equiv \{k \in \mathbb{K}_4 \mid k \geq k_4\}$. There are two possibilities for defining t_k:

$$t_k = \tau_+(\gamma_k, \|\bar{s}_k(\overline{\Delta}_k)\|) \qquad \text{or} \qquad t_k = \tau_-(\gamma_k, \|\bar{s}_k(\overline{\Delta}_k)\|).$$

The convenient choice will be made below. Anyway, by Lemma 3.8,

$$c_3 \|\bar{s}_k(\overline{\Delta}_k)\| \leq |t_k| \leq c_4 \|\bar{s}_k(\overline{\Delta}_k)\| \tag{54}$$

for all $k \in \mathbb{K}_5$.

Now, since $\bar{s}_k(\overline{\Delta}_k)$ is the global minimizer of (46),

$$\begin{aligned} \psi_k(\bar{s}_k(\overline{\Delta}_k)) &\leq \psi_k(\gamma_k(t_k) - \gamma_k(0)) \\ &= \frac{t_k^2}{2} \left(\frac{\gamma_k(t_k) - \gamma_k(0)}{t_k} \right)^T \nabla^2 f(x_k) \left(\frac{\gamma_k(t_k) - \gamma_k(0)}{t_k} \right) \\ &\quad + t_k g(x_k)^T \left(\frac{\gamma_k(t_k) - \gamma_k(0)}{t_k} \right). \end{aligned} \tag{55}$$

But, by Taylor's theorem, since, by (53) the third derivatives of γ_k are bounded,

$$\frac{\gamma_k(t_k) - \gamma_k(0)}{t_k} = \gamma_k'(0) + \frac{t_k}{2} \gamma_k''(0) + \frac{o(t_k^2)}{t_k}. \tag{56}$$

From (55) and (56) we have

$$\psi_k(\overline{s}_k(\overline{\Delta}_k)) \leq \frac{t_k^2}{2}(\gamma_k'(0)^T \nabla^2 f(x_k)\gamma_k'(0) + g(x_k)^T \gamma_k''(0)) + t_k g(x_k)^T \gamma_k'(0) + o(t_k^2). \tag{57}$$

Exchanging t_k by σt_k, where $\sigma = \pm 1$ is chosen such that

$$\sigma t_k g(x_k)^T \gamma_k'(0) \leq 0, \tag{58}$$

it follows from (54), (57) and (58) that

$$\begin{aligned}\frac{\psi_k(\overline{s}_k(\overline{\Delta}_k))}{\|\overline{s}_k(\overline{\Delta}_k)\|^2} &\leq c_4^2 \frac{\psi_k(\overline{s}_k(\overline{\Delta}_k))}{t_k^2} \\ &\leq \frac{c_4^2}{2}(\gamma_k'(0)^T \nabla^2 f(x_k)\gamma_k'(0) + g(x_k)^T \gamma_k''(0)) + \frac{o(t_k^2)}{t_k^2}.\end{aligned}$$

Therefore, by (49), (52) and (53),

$$\begin{aligned}\liminf_{k \in \mathbb{K}_5} \frac{\psi_k(\overline{s}_k(\overline{\Delta}_k))}{\|\overline{s}_k(\overline{\Delta}_k)\|^2} &\leq \frac{c_4^2}{2}(\gamma'(0)^T \nabla^2 f(x_*)\gamma'(0) + g(x_*)^T \gamma''(0)) \\ &= \frac{a_* c_4^2}{2} < 0.\end{aligned}$$

So, there exists $k_5 \in \mathbb{N}$ such for all $k \in \mathbb{K}_6 \equiv \{k \in \mathbb{K}_5 \mid k \geq k_5\}$ we have

$$\frac{\psi_k(\overline{s}_k(\overline{\Delta}_k))}{\|(\overline{s}_k(\overline{\Delta}_k)\|^2} \leq c_6 \equiv \frac{a_* c_4^2}{4} < 0. \tag{59}$$

Define, for $k \in \mathbb{K}_6$,

$$\overline{\rho}_k = \frac{f(x_k + \overline{s}_k(\overline{\Delta}_k)) - f(x_k)}{\psi_k(\overline{s}_k(\overline{\Delta}_k))}.$$

Then, by (59)

$$\begin{aligned}|\overline{\rho}_k - 1| &= \left|\frac{f(x_k + \overline{s}_k(\overline{\Delta}_k)) - f(x_k) - \psi_k(\overline{s}_k(\overline{\Delta}_k))}{\psi_k(\overline{s}_k(\overline{\Delta}_k))}\right| \\ &= \frac{o(\|\overline{s}_k(\overline{\Delta}_k)\|^2)}{|\psi_k(\overline{s}_k(\overline{\Delta}_k))|} = \frac{o(\|\overline{s}_k(\overline{\Delta}_k)\|^2)}{\|\overline{s}_k(\overline{\Delta}_k)\|^2} \frac{\|\overline{s}_k(\overline{\Delta}_k)\|^2}{|\psi_k(\overline{s}_k(\overline{\Delta}_k))|} \\ &\leq \frac{1}{|c_6|} \frac{o(\|\overline{s}_k(\overline{\Delta}_k)\|^2)}{\|\overline{s}_k(\overline{\Delta}_k)\|^2}.\end{aligned}$$

Thus,

$$\lim_{k \in \mathbb{K}_6} \overline{\rho}_k = 1. \tag{60}$$

As (60) contradicts (47), x_* is second-order stationary in this case.

Assume now that (44) holds. Since $\lim_{k \in \mathbb{K}_1} x_k = x_*$ and $f(x_k)$ is monotonically decreasing, we have that

$$\lim_{k \in \mathbb{K}_1} (f(x_{k+1}) - f(x_k)) = 0. \tag{61}$$

But, by (32),

$$f(x_{k+1}) \leq f(x_k) + \alpha \psi_k(\overline{s}_k(\Delta_k)). \tag{62}$$

So, from (61), and (62), it follows that

$$\lim_{k \in \mathbb{K}_1} \psi_k(\overline{s}_k(\Delta_k)) = 0. \tag{63}$$

Define $\underline{\Delta} = \inf_{k \in \mathbb{K}_1} \Delta_k > 0$ and let s_* be a global solution of

$$\begin{array}{ll} \text{Minimize} & \frac{1}{2} s^T \nabla^2 f(x_*) s + g(x_*)^T s \\ \text{subject to} & h(x_* + s) = 0 \\ & \|s\| \leq \underline{\Delta}/2 \end{array} \tag{64}$$

Let $k_6 \in \mathbb{K}_1$ be such that

$$\|x_k - x_*\| \leq \underline{\Delta}/2 \tag{65}$$

for all $k \in \mathbb{K}_7 = \{k \in \mathbb{K}_1 \mid k \geq k_6\}$.

Define, for $k \in \mathbb{K}_7$

$$\widehat{s}_k = x_* + s_* - x_k. \tag{66}$$

By (64) and (65) we have that

$$\|\widehat{s}_k\| \leq \underline{\Delta} \leq \Delta_k \tag{67}$$

for all $k \in \mathbb{K}_7$. Moreover,

$$x_k + \widehat{s}_k = x_* + s_* \in \mathcal{S}. \tag{68}$$

By (67), (68) and (31) we have that

$$\psi_k(\overline{s}_k(\Delta_k)) \le \psi_k(\widehat{s}_k) \tag{69}$$

for all $k \in \mathbb{K}_7$. So, by (63), (66) and (69),

$$\begin{aligned}\frac{1}{2}s_*^T\nabla^2 f(x_*)s_* + g(x_*)^T s_* &= \lim_{k\in\mathbb{K}_7}[\frac{1}{2}\widehat{s}_k^T\nabla^2 f(x_k)\widehat{s}_k + g(x_k)^T\widehat{s}_k] \\ &\ge \lim_{k\in\mathbb{K}_7}\psi_k(\overline{s}_k(\Delta_k)) = 0.\end{aligned}$$

Therefore, 0 is a minimizer of (64). This implies that x_* is second-order stationary of (6) and the proof is complete. □

THEOREM 3.11: *Assume the hypotheses of Theorem 3.10. Suppose that x_* is a limit point of $\{x_k\}$ that satisfies the General Local Assumptions of Section 2. Then, the whole sequence $\{x_k\}$ converges to x_* and there exists $c > 0$ such that (22) holds.*

Proof: Since x_* satisfies the sufficient conditions for a strict local minimizer, there exists $\varepsilon_1 > 0$ such that x_* is the only limit point of $\{x_k\}$ in the set $\{x \in \mathcal{S} \mid \|x - x_*\| \le \varepsilon_1\}$. Let $\varepsilon_2 \in (0, \varepsilon_1)$. By (19), there exists $\varepsilon_3 \in (0, \varepsilon_2)$ such that

$$\|\Phi(x, \nabla^2 f(x)) - x\| < \varepsilon_1 - \varepsilon_2 \tag{70}$$

whenever $\|x - x_*\| \le \varepsilon_3$. Define $\underline{m} = \min\{f(x) \mid x \in \mathcal{S},\ \varepsilon_3 \le \|x - x_*\| \le \varepsilon_1\}$ and $\mathcal{U} = \{x \in \mathcal{S} \mid \|x - x_*\| < \varepsilon_1 \text{ and } f(x) < \underline{m}\}$. Clearly, $\mathcal{U}$ is an open set, $x_* \in \mathcal{U}$, and $\|x - x_*\| < \varepsilon_3$ for all $x \in \mathcal{U}$. Since x_* is a limit point of $\{x_k\}$, there exists $k_0 \in \mathbb{N}$ such that $x_{k_0} \in \mathcal{U}$. Now, by (70) and the definition of Algorithm 3.1,

$$\|x_{k_0+1} - x_{k_0}\| \le \|\Phi(x_{k_0}, \nabla^2 f(x_{k_0})) - x_{k_0}\| \le \varepsilon_1 - \varepsilon_2.$$

Therefore, $\|x_{k_0+1} - x_*\| \le \|x_{k_0} - x_*\| + \|x_{k_0+1} - x_{k_0}\| < \varepsilon_1$. By the definition of the algorithm, $f(x_{k_0+1}) < \underline{m}$, so $x_{k_0+1} \in \mathcal{U}$. By an inductive argument we can prove that $x_k \in \mathcal{U}$ for all $k \ge k_0$. So, the sequence converges to x_*. Now, by (19),

$$\lim_{k\to\infty} \|\Phi(x_k, \nabla^2 f(x_k)) - x_k\| = 0.$$

So, there exists $k_1 \in \mathbb{N}$ such that $\|\Phi(x_k, \nabla^2 f(x_k)) - x_k\| < \Delta_{min}$ for all $k \ge k_1$. Therefore, for $k \ge k_1$, the first trial point $\overline{s}_k(\Delta)$ at Step 3 of

Algorithm 3.1 is $\|\Phi(x_k, \nabla^2 f(x_k)) - x_k\|$. But, by Theorem 2.6, there exists $k_2 > k_1$ such that this trial increment satisfies (32) for all $k \geq k_2$. This means that Algorithm 3.1 coincides with the Local Algorithm for all $k \geq k_2$. So, the desired result follows from Theorem 2.5. □

4. NUMERICAL EXPERIMENTS

We used the Algorithm 3.1, with $B_k = \nabla^2 f(x_k)$ for solving problems of the type (6), where

$$h(x) = \|Ax\|^2 - \theta^2 , \tag{71}$$

A is a nonsingular matrix and $\|\cdot\|$ is the Euclidian norm.

The test problems were generated as follows (cf. [24]). We considered the integral equation

$$F(\mathrm{x})(t) \equiv \int_0^1 \log\left[\frac{(t-\tau)^2 + 0.04}{(t-\tau)^2 + (0.2 - \mathrm{x}(\tau))^2}\right] d\tau = y(t) \tag{72}$$

with the boundary conditions $\mathrm{x}(0) = \mathrm{x}(1) = 0$. Given y, the problem of finding $\mathrm{x}(t)$ that satisfies approximately (72) is ill-posed, so for solving it we need regularization (*see* [23]). The regularization approach used by Vogel for solving (72) is to replace this equation by

$$\begin{array}{ll} \text{Minimize} & |||F(\mathrm{x}) - y|||^2 \\ \text{subject to} & |\mathrm{x}|^2 \leq \beta^2 \end{array} \tag{73}$$

where $|||y|||^2 = \int_0^1 |y(t)|^2 dt$, $|\mathrm{x}|^2 = \int_0^1 \mathrm{x}'(t)^2 dt$, with indicating the derivative with respect to t. We are interested in solutions of (73) that belong to the boundary, so that problem (73) is equivalent to

$$\begin{array}{ll} \text{Minimize} & |||F(\mathrm{x}) - y|||^2 \\ \text{subject to} & |\mathrm{x}|^2 = \beta^2. \end{array} \tag{74}$$

The resolution of (73) using trust-region methods was considered in [16]. Since the solution of (73) is on the boundary for all the relevant cases, the restriction to (74) is natural. After discretization, (74) becomes a finite dimensional problem of type (6), where h is given by (71), with

$$A = (n+1)\begin{pmatrix} -1 & 1 & & 0 \\ 0 & -1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 0 & -1 \end{pmatrix} \in \mathbb{R}^{n\times n} \qquad \text{and} \qquad \theta^2 = \beta^2(n+1).$$

Moreover, using the change of variables $\overline{x} = Ax$, it can be transformed onto a problem of type

$$\begin{array}{ll} \text{Minimize} & f(\overline{x}) \\ \text{subject to} & \|\overline{x}\|^2 = \theta^2. \end{array} \tag{75}$$

We use Algorithm 3.1 for solving (75). In the implementation of this algorithm we need to solve problem (31), for the special case where the feasible region is the intersection of a (trust-region) ball and a sphere. Observe that the quadratic objective function is not necessarily convex, as in the approach of Vogel. The global solution of (31) can be a local minimizer of $\psi_k(s)$ on the sphere, or a global minimizer of $\psi_k(s)$ on the intersection of the sphere with the boundary of the ball. This intersection is a sphere of lower dimension, so the global minimizer on it can be found using a classical characterization ([9, 22, 19, 17]). A global minimizer on the original sphere can also be found using the same techniques, and the local-nonglobal minimizer can be found, if it exists, using the algorithm given by Martínez (*cf.* [15]). Therefore, we are able to solve the subproblem in a completely satisfactory way, for a general nonconvex quadratic objective function.

We choose $\mathrm{x}_*(\tau)$, a solution to (72), given by

$$\mathrm{x}_*(\tau) = c_1 \exp(d_1(\tau - p_1)^2) + c_2 \exp(d_2(\tau - p_2)^2) + c_3\tau + c_4$$

where $c_1 = -0.1$, $c_2 = -0.075$, $d_1 = -40$, $d_2 = -60$, $p_1 = 0.4$, $p_2 = 0.67$ and c_3, c_4 are chosen so that $\mathrm{x}_*(0) = \mathrm{x}_*(1) = 0$. Consequently, we define $y_* = F(\mathrm{x}_*)$. The data y_i used in the discretization of (74) are

$$y_i = y_*(t_i) + \varepsilon_i,$$

where $t_i = i/(m+1)$, $i = 1, \ldots, m$. In the experiments we used $m = 30$. The "errors" ε_i were generated randomly with normal distribution with mean 0 and standard deviation 0.002 $\|F(\mathrm{x}_*)\|$. The solution x_* satisfies $|\mathrm{x}_*|^2 = (0.277)^2$.

All computations were carried out on a Sun Sparc-Station 2, using Fortran 77. We solved ten sequences of finite dimensional problems (75) with increasing $\beta \in \{0.2, 0.25, 0.275, 0.3, 0.325, 0.4, 0.5\}$ generated with ten different seeds for perturbing the data y_i. The initial feasible point was $\overline{x}_0 = \frac{\beta\sqrt{26}}{5}(1, 1, \ldots, 1)^T \in \mathbb{R}^{25}$ and the maximum number of iterations performed was 30, never reached in the tests. The average results are presented in Table 1 where IT and FE denote, respectively, the number of iterations and the number of function evaluations performed by Algorithm 3.1. We also present comparative results using the Gauss-Newton approximation for the Hessian, which corresponds to Vogel's choice. We should point out that the results in [24] are presented just by means of graphs, so we cannot make a direct quantitative comparison with his approach. However, by plotting the curves corresponding to the approximate solutions obtained by our algorithm with true Hessians we observe that our results are visually similar to the ones obtained by Vogel. We also emphasize that Table 1 is different from Table 7 in [16] because here all iterates are feasible with respect to the regularizing sphere, which does not necessarily happens in [16].

TABLE 1
Average comparative results

	β	0.200	0.250	0.275	0.300	0.325	0.400	0.500
True Hessians	IT	5.7	8.0	8.6	9.3	10.3	11.3	17.4
	FE	6.8	9.0	10.1	10.9	11.4	15.2	23.0
Gauss-Newton	IT	16	17.1	17.6	18	18.1	18.5	19.5
	FE	17	18.1	18.6	19	19.1	19.5	20.5

5. CONCLUSIONS

In this paper we have introduced a trust-region method for equality constrained problems, where the constraints are not approximated by linear functions. The main application of our techniques is the solution of constrained least-squares regularization of nonlinear ill-posed problems using the trust-region approach. Our approach for this problem differs from Vogel's one [24] in that we admit nonconvex quadratic functions in the subproblem.

This work is in continuation of a previous paper where we analyzed the trust-region algorithm with arbitrary constraints, and we proved first-order convergence results. For equality constrained problems, we proved in this paper second-order global convergence results, and local convergence results, using the theory of Fixed-Point Quasi-Newton methods. The scope of problems to which the new approach is presently applicable is limited because of the difficulty of the subproblems. However, we expect that in the next few years more complicated subproblems will be solved with ad hoc efficient methods, so that the general approach presented here should be widely applicable. In particular, regularization techniques can be incorporated to take into account limitations of several derivatives of the solution of an ill-posed problem. For that type of problems, the development of quadratic minimizers with general quadratic constraints becomes particularly relevant in order to efficiently solve trust-region subproblems.

Future research includes the application of the techniques introduced in [16] and improved in this paper to prove theoretical properties of nonlinear programming algorithms that follow closely the feasible region, as it is the case of classical GRG techniques ([1, 11]).

REFERENCES

1. J. Abadie and J. Carpentier, Generalization of the Wolfe reduced-gradient method to the case of nonlinear constraints, in *Optimization*, R. Fletcher (Ed.), Academic Press, London, 1969.
2. S. D. B. Bitar and A. Friedlander, On the identification properties of a trust-region algorithm on domains given by nonlinear inequalities, Relatório Técnico, Instituto de Matemática, Universidade Estadual de Campinas, Brazil 1995.
3. P. T. Boggs, J. W. Tolle and P. Wang, On the local convergence of quasi-Newton methods for constrained optimization, *SIAM Journal on Control and Optimization*, 1982, *20*, pp. 161-171.
4. M. R. Celis, J. E. Dennis and R. A. Tapia, A trust region strategy for nonlinear equality constrained optimization, in *Numerical Optimization*, (P. T. Boggs, R. Byrd and R. Schnabel, eds.), SIAM, Philadelphia, 1984, pp. 71-82.
5. A. R. Conn, N. I. M. Gould and Ph. L. Toint, Global convergence of a class of trust region algorithms for optimization with simple bounds, *SIAM Journal on Numerical Analysis*, 1988, *25*, pp. 433-460. See also *SIAM Journal on Numerical Analysis*, 1989, *26*, pp. 764-767.
6. M. M. El-Alem, A global convergence theory for the Celis-Dennis-Tapia trust region algorithm for constrained optimization, *SIAM Journal on Numerical Analysis*, 1991, *28*, pp. 266-290.
7. R. Fletcher, *Practical Methods of Optimization*, (2nd edition), John Wiley and Sons, Chichester, New York, Brisbane, Toronto and Singapore, 1987.
8. A. Friedlander, J. M. Martínez and S. A. Santos, A new algorithm for bound constrained minimization, *Journal of Applied Mathematics and Optimization*, 1994, *30*, pp. 235-266.
9. D. M. Gay, Computing optimal locally constrained steps, *SIAM J. Sci. Stat. Comput.*, 1981, *2*, pp. 186-197.
10. M. Heinkenschloss, Mesh independence for nonlinear least squares problems with norm constraints, *SIAM Journal on Optimization*, 1993, *3*, pp. 81-117.
11. L. S. Lasdon, Reduced gradient methods, in *Nonlinear Optimization 1981*, 1982, edited by M. J. D. Powell, Academic Press, New York, pp. 235-242.
12. D. Luenberger, *Linear and Nonlinear Programming*, Addison Wesley, 1984.
13. D. Lyle and M. Szularz, Local minima of the trust-region problem, *Journal of Optimization Theory an Applications*, 1994, *80*, pp. 117-134.
14. J. M. Martínez, Fixed-point quasi-Newton methods, *SIAM Journal on Numerical Analysis*, 1992, *5*, pp. 1413-1434.

15. J. M. MARTÍNEZ, Local minimizers of quadratic functions on Euclidean balls and spheres, *SIAM Journal on Optimization*, 1994, *4*, pp. 159-176.
16. J. M. MARTÍNEZ and S. A. SANTOS, A trust-region strategy for minimization on arbitrary domains, *Mathematical Programming*, 1995, *68*, pp. 267-301.
17. J. J. MORÉ, Recent developments in algorithms and software for trust region methods, in *Mathematical Programming Bonn 1982. The State of Art*, A. Bachem, M Grötschel and B. Korte, eds., Springer-Verlag, 1983.
18. J. J. MORÉ, Generalizations of the trust-region Problem, *Optimization Methods and Software*, 1993, *2*, pp. 189-209.
19. J. J. MORÉ and D. C. SORENSEN, Computing a trust region step, *SIAM Journal on Scientific and Statistical Computing*, 1983, *4*, pp. 553-572.
20. M. J. D. POWELL and Y. YUAN, A trust region algorithm for equality constrained optimization, *Mathematical Programming*, 1991, *49*, pp. 189-211.
21. R. J. STERN and H. WOLKOWICZ, Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations, Technical Report SOR 93-1, School of Engineering and Applied Science, Department of Civil Engineering and Operations Research, Princeton University, 1993.
22. D. C. SORENSEN, Newton's method with a model trust region modification, *SIAM Journal on Numerical Analysis*, 1982, *19*, pp. 409-426.
23. A. TIKHONOV and V. ARSENIN, *Solutions of ill-posed problems*, John Wiley and Sons, New York, Toronto, London, 1977.
24. C. R. VOGEL, A constrained least-squares regularization method for nonlinear ill-posed problems, *SIAM Journal on Control and Optimization*, 1990, *28*, pp. 34-49.
25. H. WOLKOWICZ, On the resolution of the trust region problem, Communication at the NATO-ASI Meeting on Continuous Optimization, Il Ciocco, Italy, September 1993, 1993.