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ANALYZING DISCRETE-TIME BULK-SERVICE $Geo/Geo^b/m$ QUEUE

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Abstract. This paper analyzes a discrete-time multi-server queue in which service capacity of each server is a minimum of one and a maximum of b customers. The interarrival- and service-times are assumed to be independent and geometrically distributed. The queue is analyzed under the assumptions of early arrival system and late arrival system with delayed access. Besides, obtaining state probabilities at arbitrary and outside observer's observation epochs, some performance measures and waiting-time distribution in the queue have also been discussed. Finally, it is shown that in limiting case the results obtained in this paper tend to the continuous-time counterpart.

Keywords. Bulk-service, discrete-time, multi-server, queueing, waiting-time.

1. Introduction

In recent years discrete-time queues have been receiving increased attention due to their usefulness in performance analysis of communication systems. Their importance has further increased due to the emergence of the broadband integrated services digital network (B-ISDN) which can provide transfer of video, voice and data through high speed local area networks (LANs), on-demand video distribution, and video telephony, etc. The asynchronous transfer mode (ATM) is a network transfer technique and capable of supporting a wide variety of multimedia applications with diverse service and performance requirements. It transmits

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the information in small, fixed-length packets called "cells". Since the ATM is based on the packet-switching principle, the events (arrival of packets and their onward transmission) occur only at regularly spaced points of time. Thus the underlying transport mechanism is represented adequately by the discrete-time queueing system. A detailed discussion and applications of discrete-time queues can be found in books by Bruneel and Kim [2] and Woodward [20]. Besides many systems operate on discrete-time basis, *e.g.*, machine cycle of processor, synchronous communication channel (Slotted ALOHA multi-access channel) and traffic concentrators. In such queues arrivals and departures can occur simultaneously at boundary epoch of a slot. In case of simultaneity their order may be taken care of by either arrival-first (AF) or departure-first (DF) management policies. According to AF policy, arrivals take precedence over departures, while under DF policy the opposite effect is observed, Gravey and Hébuterne [12]. It may be remarked here that AF and DF policies also correspond to late arrival system with delayed access (LAS-DA) and early arrival system (EAS), respectively, Hunter [15].

In past, several authors have analyzed infinite- (finite-) buffer discrete-time multi-server queues with single (batch) arrival under EAS and LAS-DA. Earliest work was due to Chan and Maa [3] wherein they discussed the $GI/Geo/m$ queue with EAS and obtained the distribution of number of customers in the system at prearrival epoch. Further, Chaudhry and Gupta [7] have carried out a detailed analysis (including numerical aspects) of the same queueing model and obtained the state probabilities at prearrival, arbitrary and outside observer's observation epochs. In this connection see also paper by Gao *et al.* [11]. The performance analysis and optimal control of $Geo/Geo/c$ queue under LAS-DA has been discussed by Artalejo *et al.* [1]. The analysis of multi-server queue with batch arrivals: $Geo^X/Geo/c$ has been carried out by Rubin and Zhang [17]. Further, Chaudhry *et al.* [9] have discussed a more complex model: $GI^X/Geo/m$. In this connection see also papers by Chaudhry and Kim [8], Wittevrongel *et al.* [19]. The transient behaviour of the $Geo/Geo/m/m$ queue (discrete version of Erlang loss model) has been studied by Chaudhry and Gupta [5]. For the $GI/Geo/m/m$ queue, Chaudhry and Gupta [6] obtained the distribution of number of busy channels at various epochs under LAS-DA and EAS. Recently, Chaudhry *et al.* [10] have analyzed finite-buffer $GI/Geo/m/N$ queue with early arrival system and obtained the state probabilities at prearrival and arbitrary epochs. Further, Gupta *et al.* [14] have discussed the same queueing model for EAS and LAS-DA, and developed a recursive procedure, to obtain system length distributions at prearrival, arbitrary and outside observer's observation epochs. They have also obtained the distribution of the actual waiting time in the queue of a customer in both cases.

All the above studies on multi-server queue have been carried out under the assumption that server serves the customer one at a time. However, there are many instances where the services (transmission of packets) are carried out in batches of fixed (or variable) size to increase the service (transmission) rate. Bulk-service queues have applications in areas mentioned above and have been recently investigated by Gupta and Goswami [13], Chaudhry and Chang [4] in case of single server. It may be mentioned here that the modelling and analysis of discrete-time multi-server queue with bulk-service is more involved and quite different than the corresponding continuous-time counterpart.

In this paper, we consider a discrete-time multi-server queue with infinite-buffer in which interarrival- and service-times are independent and geometrically distributed. Further, each server performs services in batches of minimum of one and a maximum of b customers. We obtain steady-state queue-length distributions at arbitrary and outside observer's observation epochs for both early arrival system and late arrival system with delayed access. Some performance measures and waiting-time distribution in the queue have also been discussed. The results for continuous-time bulk-service $M/M^b/m$ queue is obtained as a limiting case of discrete-time $Geo/Geo^b/m$ queue. The advantage of analyzing discrete-time queue is that one can obtain continuous-time result from it as a limiting case but converse is not true.

This paper is organized as follows: Description of the queueing model, and analyzes of early arrival system and late arrival system with delayed access are given in Section 2. In Section 3, we discussed outside observer's distribution. The waiting-time analysis is carried out in Section 4. Finally, it has been shown in the appendix that in limiting case, the results tend to continuous-time $M/M^b/m$ queue as they should be.

2. Description of the model

We consider a discrete-time multi-server bulk-service infinite buffer queue in which interarrival times of customers are independent and geometrically distributed with probability mass function (p.m.f.) $a_n = (1-\lambda)^{n-1}\lambda,\ 0 < \lambda < 1,\ n \geq 1$. There are m servers and service times of batches (with a minimum of one and a maximum of b per server) are assumed to be independent and geometrically distributed with p.m.f. $s_n = (1-\mu)^{n-1}\mu,\ 0 < \mu < 1,\ n \geq 1$. At any time, each of the free servers are equally likely to take a batch for service. The traffic intensity is given by $\rho = \lambda/mb\mu$. Further, the probability that j batches complete service given that there are i busy servers is given by

$$c(j|i) = \binom{i}{j}\mu^j(1-\mu)^{i-j},\ \text{for } i = 0,1,2,\dots,m,\ j = 0,1,\dots,i;$$
$$c(j|i) = 0,\ \text{for } j > i \text{ or } j < 0.$$

Below we first discuss the early arrival system.

2.1. $Geo/Geo^b/m$ queue with EAS

Let us assume that the time axis is slotted into intervals of equal length with the length of a slot being unity. Further, let the time axis be marked as $0, 1, 2, \dots, t, \dots$ and assume that potential arrivals occur in the interval $(t, t+)$ and potential batch-departures occur in the interval $(t-, t)$. The various time epochs at which events occur are depicted in Figure 1.

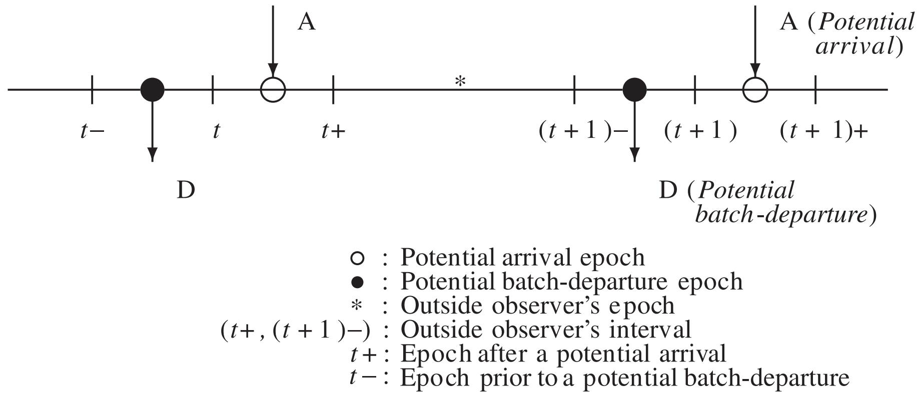

FIGURE 1. Various time epochs in early arrival system (EAS).

Let us define the joint probability as $Q_{n,i}(t) = P\{n$ customers in the queue excluding the batches in service and i servers are busy at time $t\}$, $n \geq 0,\ 0 \leq i \leq m$.

Relating the states of the system at two consecutive epochs t and $(t+1)$, we obtain

$$Q_{0,0}(t+1) = (1-\lambda)\sum_{j=0}^{m} c(j|j)Q_{0,j}(t) + \lambda \sum_{j=0}^{m-1} c(j+1|j+1)Q_{0,j}(t), \tag{1}$$

$$\begin{aligned} Q_{0,k}(t+1) = {} & (1-\lambda)\sum_{j=k}^{m} c(j-k|j)Q_{0,j}(t) \\ & +(1-\lambda)\sum_{j=m+1}^{m+k} c(j-k|m)\sum_{i=1}^{b} Q_{i+(j-m-1)b,m}(t) \\ & +\lambda\sum_{j=k}^{m} c(j-k|j)Q_{0,j-1}(t) \\ & +\lambda\sum_{j=m+1}^{m+k} c(j-k|m)\sum_{i=1}^{b} Q_{i+(j-m-1)b-1,m}(t), \\ & \qquad 1 \leq k \leq m, \end{aligned} \tag{2}$$

$$\begin{aligned} Q_{n,m}(t+1) = {} & (1-\lambda)\sum_{j=0}^{m} c(j|m)Q_{n+jb,m}(t) \\ & +\lambda\sum_{j=0}^{m} c(j|m)Q_{n+jb-1,m}(t), \qquad n \geq 1. \end{aligned} \tag{3}$$

Let us define in steady-state

$$Q_{n,i} = \lim_{t\to\infty} Q_{n,i}(t),\ n \geq 0,\ 0 \leq i \leq m.$$

In steady-state equations (1)–(3) reduce to

$$0 = -\lambda Q_{0,0} + (1-\lambda)\sum_{j=1}^{m} c(j|j)Q_{0,j} + \lambda \sum_{j=0}^{m-1} c(j+1|j+1)Q_{0,j}, \tag{4}$$

$$\begin{aligned} 0 = {} & [(1-\lambda)c(0|k) - 1]Q_{0,k} + (1-\lambda)\sum_{j=k+1}^{m} c(j-k|j)Q_{0,j} \\ & +\lambda \sum_{j=k}^{m} c(j-k|j)Q_{0,j-1} \\ & +(1-\lambda)\sum_{j=m+1}^{m+k} c(j-k|m)\sum_{i=1}^{b} Q_{i+(j-m-1)b,m} \\ & +\lambda \sum_{j=m+1}^{m+k} c(j-k|m)\sum_{i=1}^{b} Q_{i+(j-m-1)b-1,m}, \\ & \qquad 1 \leq k \leq m, \end{aligned} \tag{5}$$

$$\begin{aligned} 0 = {} & [(1-\lambda)c(0|m) - 1]Q_{n,m} + (1-\lambda)\sum_{j=1}^{m} c(j|m)Q_{n+jb,m} \\ & +\lambda \sum_{j=0}^{m} c(j|m)Q_{n+jb-1,m}, \quad n \geq 1. \end{aligned} \tag{6}$$

The solution of equations (4)–(6) will give the queue-length distribution $Q_{0,k}$, $(0 \leq k \leq m)$ and $Q_{n,m}$, $(n \geq 1)$. To get them, first we need to solve the difference equation (6). In order to do this we define the displacement operator E as $E^j Q_{n,m} = Q_{n+j,m}$, Spiegel [18], and rewrite the equation (6) as

$$\left[\left\{(1-\lambda)c(0|m) - 1\right\} + (1-\lambda)\sum_{j=1}^{m} c(j|m)E^{jb} + \lambda \sum_{j=0}^{m} c(j|m)E^{jb-1}\right]Q_{n,m} = 0. \tag{7}$$

The characteristic equation associated with (7) is

$$g(z) \equiv \left\{(1-\lambda)c(0|m) - 1\right\} + (1-\lambda)\sum_{j=1}^{m} c(j|m)z^{jb} + \lambda \sum_{j=0}^{m} c(j|m)z^{jb-1} = 0$$

which, after simplification, reduces to

$$g(z) \equiv \{(1-\lambda)z + \lambda\}\{\mu z^b + (1-\mu)\}^m - z = 0. \tag{8}$$

By using Rouché's theorem it can be shown that only one zero of $g(z)$ falls inside the unit circle and, this root is real and unique if and only if $\rho = \lambda/mb\mu < 1$. We denote this root by r $(0 < r < 1)$ and other mb roots by r_i $(i = 1, 2, \ldots, mb)$, $|r_i| \geq 1$. Thus, r satisfies the equation

$$\{(1-\lambda)r + \lambda\}\{\mu r^b + (1-\mu)\}^m - r = 0. \tag{9}$$

Now the solution of (6) can be written as

$$Q_{n,m} = A_0 r^n + \sum_{i=1}^{mb} A_i r_i^n, \qquad n \geq 0, \tag{10}$$

where A_i's are arbitrary constants.

Since

$$\sum_{k=0}^{m-1} Q_{0,k} + \sum_{n=0}^{\infty} Q_{n,m} = 1, \tag{11}$$

we must have $A_i = 0$, for all $i = 1, 2, \ldots, mb$, otherwise the left hand side of (11) diverges. Therefore, from (10) we get $Q_{n,m} = A_0 r^n$. Setting $n = 0$ yields $A_0 = Q_{0,m}$, and hence

$$Q_{n,m} = Q_{0,m} r^n, \qquad n \geq 1. \tag{12}$$

Now from equation (5), for $k = m$, we have

$$\begin{aligned} Q_{0,m-1} = \frac{1}{\lambda c(0|m)} \Bigg[& \{1 - (1-\lambda)c(0|m)\}Q_{0,m} \\ & -(1-\lambda)\sum_{j=m+1}^{2m} c(j-m|m) \sum_{i=1}^{b} Q_{i+(j-m-1)b,m} \\ & -\lambda \sum_{j=m+1}^{2m} c(j-m|m) \sum_{i=1}^{b} Q_{i+(j-m-1)b-1,m} \Bigg]. \end{aligned} \tag{13}$$

The terms $\sum_{i=1}^{b} Q_{i+(j-m-1)b,m}$ and $\sum_{i=1}^{b} Q_{i+(j-m-1)b-1,m}$ appear in the right hand side of (13) can be simplified using (12) and are given by

$$\begin{aligned} \sum_{i=1}^{b} Q_{i+(j-m-1)b,m} &= \frac{1-r^b}{1-r} r^{(j-m-1)b+1} Q_{0,m}, \\ \sum_{r=1}^{b} Q_{i+(j-m-1)b-1,m} &= \frac{1-r^b}{1-r} r^{(j-m-1)b} Q_{0,m}. \end{aligned}$$

Substituting the above expressions in (13), we obtain

$$\begin{aligned} Q_{0,m-1} &= \frac{Q_{0,m}}{\lambda c(0|m)}\Bigg[\Big\{1-(1-\lambda)c(0|m)\Big\} \\ &\quad -\Big\{(1-\lambda)r+\lambda\Big\}\frac{1-r^b}{1-r}\sum_{j=m+1}^{2m} c(j-m|m)r^{(j-m-1)b}\Bigg] \\ &= \frac{Q_{0,m}}{\lambda c(0|m)}\Bigg[\Big\{1-(1-\lambda)c(0|m)\Big\} \\ &\quad -\Big\{(1-\lambda)r+\lambda\Big\}\frac{1-r^b}{1-r}\sum_{j=1}^{m} c(j|m)r^{(j-1)b}\Bigg]. \end{aligned} \tag{14}$$

Using the relation (9) in the above equation, we obtain

$$Q_{0,m-1} = \frac{Q_{0,m}}{\lambda c(0|m)(1-r)}\Bigg[1-c(0|m)-\Big\{(1-\lambda)r+\lambda\Big\}\sum_{j=1}^{m} c(j|m)r^{(j-1)b}\Bigg]. \tag{15}$$

Again making use of (12), we get from (5), for $k = m-1, m-2, \dots, 2, 1$,

$$\begin{aligned} Q_{0,k-1} &= \frac{1}{\lambda c(0|k)}\Bigg[\Big\{1-(1-\lambda)c(0|k)\Big\}Q_{0,k} \\ &\quad -\sum_{j=k+1}^{m} c(j-k|j)\Big\{(1-\lambda)Q_{0,j}+\lambda Q_{0,j-1}\Big\} \\ &\quad -\Big\{(1-\lambda)r+\lambda\Big\}\frac{1-r^b}{1-r}\sum_{j=m+1}^{m+k} c(j-k|m)r^{(j-m-1)b}Q_{0,m}\Bigg]. \end{aligned} \tag{16}$$

From (15), it is clear that $Q_{0,m-1}$ is represented in terms of $Q_{0,m}$ and similarly, one can see from (16) that $Q_{0,k}$, for $k = m-2, m-3, \dots, 2, 1, 0$ can be recursively obtained in terms of $Q_{0,m}$. Therefore, the only unknown quantity $Q_{0,m}$ can be obtained using the normalization condition.

2.2. $Geo/Geo^b/m$ QUEUE WITH LAS-DA

As discussed earlier the two policies differ each other in the order of the arrival and departure of packets around a slot boundary. In LAS-DA, potential arrivals occur in the interval $(t-, t)$ and potential batch-departures occur in the interval $(t, t+)$. More specifically, various time epochs at which events occur are depicted in Figure 2.

Again, let us define the joint probability as $P_{n,i}(t-) = P\{n$ customers in the queue excluding the batches in service and i servers are busy at time $t-\}$, $n \geq 0,\ 0 \leq i \leq m$.

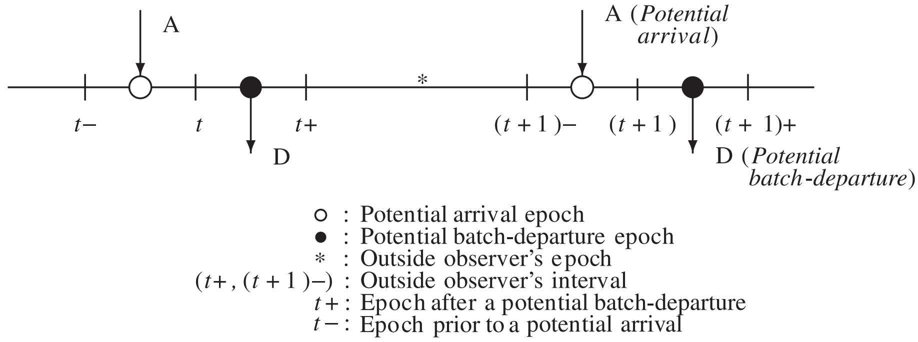

FIGURE 2. Various time epochs in late arrival system with delayed access (LAS-DA).

Observing the state of the system at two consecutive time epochs $t-$ and $(t+1)-$, we obtain following equations, where for the sake of simplicity we use symbol t instead of $t-$,

$$P_{0,0}(t+1) = (1-\lambda)\sum_{j=0}^{m} c(j|j)P_{0,j}(t), \tag{17}$$

$$\begin{aligned} P_{0,k}(t+1) = {} & (1-\lambda)\sum_{j=k}^{m} c(j-k|j)P_{0,j}(t) \\ & +(1-\lambda)\sum_{j=m+1}^{m+k} c(j-k|m)\sum_{i=1}^{b} P_{i+(j-m-1)b,m}(t) \\ & +\lambda\sum_{j=k}^{m} c(j-k|j-1)P_{0,j-1}(t) \\ & +\lambda\sum_{j=m+1}^{m+k} c(j-k|m)\sum_{i=1}^{b} P_{i+(j-m-1)b-1,m}(t), \\ & \qquad 1 \le k \le m, \end{aligned} \tag{18}$$

$$\begin{aligned} P_{n,m}(t+1) = {} & (1-\lambda)\sum_{j=0}^{m} c(j|m)P_{n+jb,m}(t) \\ & +\lambda\sum_{j=0}^{m} c(j|m)P_{n+jb-1,m}(t), \qquad n \ge 1. \end{aligned} \tag{19}$$

Let us define in steady-state

$$P_{n,i} = \lim_{t\longrightarrow\infty} P_{n,i}(t),\ n \ge 0,\ 0 \le i \le m.$$

In steady-state equations (17)–(19) reduce to

$$0 = -\lambda P_{0,0} + (1-\lambda)\sum_{j=1}^{m} c(j|j)P_{0,j}, \tag{20}$$

$$\begin{aligned}0 = {} & [(1-\lambda)c(0|k)-1]P_{0,k} + (1-\lambda)\sum_{j=k+1}^{m} c(j-k|j)P_{0,j} \\ & +\lambda\sum_{j=k}^{m} c(j-k|j-1)P_{0,j-1} \\ & +(1-\lambda)\sum_{j=m+1}^{m+k} c(j-k|m)\sum_{i=1}^{b} P_{i+(j-m-1)b,m} \\ & +\lambda\sum_{j=m+1}^{m+k} c(j-k|m)\sum_{i=1}^{b} P_{i+(j-m-1)b-1,m}, \qquad 1\le k\le m,\end{aligned} \tag{21}$$

$$\begin{aligned}0 = {} & [(1-\lambda)c(0|m)-1]P_{n,m} + (1-\lambda)\sum_{j=1}^{m} c(j|m)P_{n+jb,m} \\ & +\lambda\sum_{j=0}^{m} c(j|m)P_{n+jb-1,m}, \quad n\ge 1.\end{aligned} \tag{22}$$

It can be seen that equations (22) and (6) are same but others are different. Therefore, the characteristic equation and hence the value of r will be same in both (EAS and LAS-DA) cases. Following the procedure discussed for EAS we can obtain $P_{0,k}$ $(0\le k\le m)$ and $P_{n,m}$ $(n\ge 1)$. They are given as

$$\begin{aligned}P_{0,m-1} = {} & \frac{P_{0,m}}{\lambda c(0|m-1)}\Bigg[\Big\{1-(1-\lambda)c(0|m)\Big\} \\ & -\Big\{(1-\lambda)r+\lambda\Big\}\frac{1-r^b}{1-r}\sum_{j=1}^{m} c(j|m)r^{(j-1)b}\Bigg],\end{aligned} \tag{23}$$

$$\begin{aligned}P_{0,k-1} = {} & \frac{1}{\lambda c(0|k-1)}\Bigg[\Big\{1-(1-\lambda)c(0|k)\Big\}P_{0,k} \\ & -\sum_{j=k+1}^{m} c(j-k|j-1)\Big\{\frac{j}{k}(1-\lambda)(1-\mu)P_{0,j}+\lambda P_{0,j-1}\Big\} \\ & -\Big\{(1-\lambda)r+\lambda\Big\}\frac{1-r^b}{1-r}\sum_{j=m+1}^{m+k} c(j-k|m)r^{(j-m-1)b}P_{0,m}\Bigg],\end{aligned} \tag{24}$$

$$k = m-1, m-2, \ldots, 2, 1,$$

$$P_{n,m} = P_{0,m}r^n, \quad n\ge 1. \tag{25}$$

3. Outside observer's distribution

In EAS, since an outside observer's observation epoch falls in a time interval after a potential arrival and before a potential batch-departure, the probability $Q^o_{0,k}$ $(0 \le k \le m)$ and $Q^o_{n,m}$ $(n \ge 1)$ that the outside observer sees k servers busy (with no customers in the queue) and m servers busy (with n customers in the queue), respectively, can be obtained by observing arbitrary and outside observer's observation epochs in Figure 1. They are given by

$$Q_{0,0} = \sum_{j=0}^{m} c(j|j) Q^o_{0,j}, \tag{26}$$

$$Q_{0,k} = \sum_{j=k}^{m} c(j-k|j) Q^o_{0,j} + \sum_{j=m+1}^{m+k} c(j-k|m) \sum_{i=1}^{b} Q^o_{i+(j-m-1)b,m}, \quad 1 \le k \le m, \tag{27}$$

$$Q_{n,m} = \sum_{j=0}^{m} c(j|m) Q^o_{n+jb,m}, \qquad n \ge 1. \tag{28}$$

The solution of equations (26)–(28) will give the queue-length distribution $Q^o_{0,k}$, $(0 \le k \le m)$ and $Q^o_{n,m}$, $(n \ge 1)$. To get them, we first solve the difference equation (28) using $Q_{n,m} = Q_{0,m} r^n$ (Eq. (12)) and the displacement operator E defined by $E^j Q^o_{n,m} = Q^o_{n+j,m}$. Thus, we rewrite the equation (28) as

$$\sum_{j=0}^{m} c(j|m) E^{jb} Q^o_{n,m} = Q_{0,m} r^n, \quad n \ge 1. \tag{29}$$

The solution of homogeneous difference equation

$$\sum_{j=0}^{m} c(j|m) E^{jb} Q^o_{n,m} = 0 \tag{30}$$

of the non-homogeneous difference equation (29) is given by

$$Q^{o(h)}_{n,m} = \sum_{i=1}^{b} \left(\sum_{j=1}^{m} A_{ij} n^{j-1} \right) \gamma_i^n, \quad n \ge 1,$$

where A_{ij}'s are arbitrary constants and γ_i's (of multiplicity m) are the roots of the characteristic equation $(1-\mu+\mu z^b)^m = 0$ of the corresponding equation (30). The particular solution of (29) is given by

$$Q^{o(p)}_{n,m} = \frac{Q_{0,m} r^n}{(1-\mu+\mu r^b)^m}, \quad n \ge 1.$$

Thus, the general solution of (29) is given by

$$Q^o_{n,m} = \sum_{i=1}^{b}\left(\sum_{j=1}^{m} A_{ij}n^{j-1}\right)\gamma_i^n + \frac{Q_{0,m}r^n}{(1-\mu+\mu r^b)^m}, \quad n \geq 1. \tag{31}$$

The normalization condition $\sum_{j=0}^{m} Q^o_{0,j} + \sum_{n=1}^{\infty} Q^o_{n,m} = 1$ will be satisfied only if the equation $f(z) \equiv 1-\mu+\mu z^b = 0$ has any real root γ_i between 0 and 1. Since $f(0) = 1-\mu > 0,\ f(1) = 1 > 0$ and $f'(z) = \mu b z^{b-1} > 0$, for $z > 0$, *i.e.*, $f(z)$ is a monotonic increasing function in $0 < z < 1$, therefore $f(z) = 0$ has no real root γ_i between 0 and 1. Thus, all A_{ij} must be zero and hence (31) becomes

$$Q^o_{n,m} = \frac{Q_{0,m}r^n}{(1-\mu+\mu r^b)^m}, \quad n \geq 1. \tag{32}$$

Using (32) in equation (27), after simplification, we obtain

$$Q_{0,k} = \sum_{j=k}^{m} c(j-k|j)Q^o_{0,j} + \frac{Q_{0,m}r(1-r^b)}{(1-r)(1-\mu+\mu r^b)^m}\sum_{j=m+1}^{m+k} c(j-k|m)r^{(j-m-1)b}$$
$$1 \leq k \leq m. \tag{33}$$

By considering $k = m$ in (33), we obtain

$$Q^o_{0,m} = \frac{1}{c(0|m)}\left\{Q_{0,m} - \frac{Q_{0,m}r(1-r^b)}{(1-r)(1-\mu+\mu r^b)^m}\sum_{j=m+1}^{2m} c(j-m|m)r^{(j-m-1)b}\right\}. \tag{34}$$

Again from (33) for $k = m-1, m-2, \ldots, 3, 2, 1$, we obtain

$$Q^o_{0,k} = \frac{1}{c(0|k)}\left\{Q_{0,k} - \sum_{j=k+1}^{m} c(j-k|j)Q^o_{0,j} - \frac{Q_{0,m}r(1-r^b)}{(1-r)(1-\mu+\mu r^b)^m}\sum_{j=m+1}^{m+k} c(j-k|m)r^{(j-m-1)b}\right\}. \tag{35}$$

Finally, $Q^o_{0,0}$ is obtained from (26) and given by

$$Q^o_{0,0} = Q_{0,0} - \sum_{j=1}^{m} c(j|j)Q^o_{0,j}. \tag{36}$$

In LAS-DA, since an outside observer's observation epoch falls in a time interval after a potential batch-departure and before a potential arrival, the probability $P^o_{0,k}$ $(0 \leq k \leq m)$ and $P^o_{n,m}$ $(n \geq 1)$ that outside observer sees k servers busy (no customer in the queue) and m servers busy (n customers in the queue) is the same as $P_{0,k}$ and $P_{n,m}$, respectively. Hence $P^o_{0,k} = P_{0,k}$ and $P^o_{n,m} = P_{n,m}$.

4. Waiting-time distribution

In this section, we obtain actual waiting-time (in queue) distribution (measured in slots) of an arrival customer under the FCFS discipline for both EAS and LAS-DA. Let us define the random variable T_q as "time spent waiting in the queue" of an arrival with the corresponding p.m.f. $w_k = P(T_q = k),\ k \geq 0$. Further, let $\tilde{P}_i(T_q > k)$ be the conditional probability of the event $T_q > k$, given that an arrival finds i $(i \geq 0)$ customers in the queue.

4.1. Waiting-time in EAS system

An arriving customer may observe the system in any one of the following two cases.

Case 1. $w_0 = P(T_q = 0)$.
This happens, if prior to an arrival, there are no customers in the queue and 'i' $(0 \leq i \leq m-1)$ servers are busy. Therefore, the probability that an arriving customer does not wait is given by

$$P(T_q = 0) = \sum_{i=0}^{m-1} Q_{0,i}. \tag{37}$$

Case 2. $w_k = P(T_q = k),\ k \geq 1$.
This occurs, if prior to an arrival, there are '$jb + n$' $(j \geq 0;\ 0 \leq n \leq b-1)$ customers in the queue and m servers are busy. Therefore, the probability that an arriving customer will have to wait greater than k $(k \geq 0)$ slots is

$$\begin{aligned} P(T_q > k) &= \sum_{j=0}^{\infty}\sum_{n=0}^{b-1} Q_{jb+n,m}\tilde{P}_{jb+n}(T_q > k) \\ &= \frac{1-r^b}{1-r} Q_{0,m} \sum_{j=0}^{\infty} r^{jb} \sum_{i=0}^{j} \binom{mk}{i} \mu^i (1-\mu)^{mk-i} \\ &= \frac{Q_{0,m}(1-\mu+\mu r^b)^{mk}}{1-r}, \end{aligned}$$

where

$$\tilde{P}_{jb+n}(T_q > k) = \sum_{i=0}^{j} \binom{mk}{i} \mu^i (1-\mu)^{mk-i}$$

is the probability that during k slots after the arrival of a customer there will not occur more than 'j' batch-departures from the system. Then, consequently, we obtain

$$\begin{aligned} w_0 &= P(T_q = 0) = 1 - P(T_q > 0), \quad \text{where } P(T_q > 0) = \frac{Q_{0,m}}{1-r}, \qquad (38) \\ w_k &= P(T_q > k-1) - P(T_q > k) \\ &= \frac{Q_{0,m}}{1-r}\left[(1-\mu+\mu r^b)^{m(k-1)} - (1-\mu+\mu r^b)^{mk}\right] \\ &= \frac{Q_{0,m}}{1-r}(1-\mu+\mu r^b)^{m(k-1)}\left[1-(1-\mu+\mu r^b)^m\right] \quad k \geq 1. \qquad (39) \end{aligned}$$

The average waiting-time in the queue ($W_q = \sum_{k=1}^{\infty} kw_k$) is given by

$$W_q = \frac{Q_{0,m}}{(1-r)[1-(1-\mu+\mu r^b)^m]}.$$

Remark 1. It may be noted here that w_0 can be obtained either using (37) or (38). This has also been checked numerically.

Remark 2. The average queue-length (L_q^o) at outside observer's observation epoch is given by

$$\begin{aligned} L_q^o &= \sum_{n=1}^{\infty} nQ_{n,m}^o = \frac{Q_{0,m} r}{(1-r)^2(1-\mu+\mu r^b)^m} \\ &= \frac{r[1-(1-\mu+\mu r^b)^m]}{(1-r)(1-\mu+\mu r^b)^m} \cdot \frac{Q_{0,m}}{(1-r)[1-(1-\mu+\mu r^b)^m]} \\ &= \lambda W_q, \end{aligned}$$

where

$$\lambda = \frac{r[1-(1-\mu+\mu r^b)^m]}{(1-r)(1-\mu+\mu r^b)^m} \qquad (40)$$

is obtained from equation (9). Thus, the Little's formula $L_q^o = \lambda W_q$ is verified.

4.2. Waiting-time in LAS-DA system

Here also an arriving customer may observe the system in any one of the following two cases.

Case 1. $w_0 = P(T_q = 0)$.
This happens, if prior to an arrival, there are no customers in the queue and 'i' ($0 \leq i \leq m-1$) servers are busy or if there are '$jb+n$' ($0 \leq j \leq m-1$; $0 \leq n \leq b-1$) customers present in the queue and m servers are busy such that out of m servers there are at least $(j+1)$ batches about to departs, so that service of the new arrival starts immediately. Therefore, the probability that an arriving customer

does not wait is given by

$$P(T_q = 0) = \sum_{i=0}^{m-1} P_{0,i} + \sum_{j=0}^{m-1}\sum_{n=0}^{b-1} P_{jb+n,m} \sum_{i=j+1}^{m} \binom{m}{i} \mu^i (1-\mu)^{m-i}.$$

Case 2. $w_k = P(T_q = k),\ k \geq 1$.
This occurs, if prior to an arrival, there are '$jb+n$' ($j \geq 0;\ 0 \leq n \leq b-1$) customers in the queue, m servers busy and 'i' ($0 \leq i \leq j$) batches out of m servers are about to depart. Therefore, the probability that an arriving customer will have to wait greater than k ($k \geq 0$) slots is

$$\begin{aligned} P(T_q > k) &= \sum_{j=0}^{\infty}\sum_{n=0}^{b-1} P_{jb+n,m} \tilde{P}_{jb+n}(T_q > k) \\ &= \frac{1-r^b}{1-r} P_{0,m} \sum_{j=0}^{\infty} r^{jb} \sum_{i=0}^{j} \binom{m(k+1)}{i} \mu^i (1-\mu)^{m(k+1)-i} \\ &= P_{0,m} \frac{(1-\mu+\mu r^b)^{m(k+1)}}{1-r}, \end{aligned}$$

where

$$\tilde{P}_{jb+n}(T_q > k) = \sum_{i=0}^{j} \binom{m(k+1)}{i} \mu^i (1-\mu)^{m(k+1)-i}$$

is the probability that during $(k+1)$ slots after the arrival of a customer there will not occur more than 'j' batch-departures from the system. Then, consequently, we obtain

$$\begin{aligned} w_0 &= P(T_q = 0) = 1 - P(T_q > 0), \quad \text{where } P(T_q > 0) = \frac{P_{0,m}(1-\mu+\mu r^b)^m}{1-r}, \\ w_k &= P(T_q > k-1) - P(T_q > k) \\ &= \frac{P_{0,m}}{1-r}\left[(1-\mu+\mu r^b)^{mk} - (1-\mu+\mu r^b)^{m(k+1)}\right] \\ &= \frac{P_{0,m}}{1-r}(1-\mu+\mu r^b)^{mk}\left[1-(1-\mu+\mu r^b)^m\right] \quad k \geq 1. \end{aligned}$$

The average waiting-time in the queue is given by

$$W_q = \frac{P_{0,m}(1-\mu+\mu r^b)^m}{(1-r)[1-(1-\mu+\mu r^b)^m]}.$$

Remark 3. The average queue-length (L_q^o) at outside observer's observation epoch is given by

$$\begin{aligned} L_q^o &= \sum_{n=1}^{\infty} nP_{n,m}^o = \frac{P_{0,m}r}{(1-r)^2} \\ &= \frac{r[1-(1-\mu+\mu r^b)^m]}{(1-r)(1-\mu+\mu r^b)^m} \cdot \frac{P_{0,m}(1-\mu+\mu r^b)^m}{(1-r)[1-(1-\mu+\mu r^b)^m]} \\ &= \lambda W_q. \end{aligned}$$

Appendix

Here we study the relationship between the discrete-time $Geo/Geo^b/m$ queue and its continuous-time counterpart. For the continuous-time multi-server bulk-service $M/M^b/m$ queue, we assume that the customers arrive according to a Poisson process with rate α and are served in batches (with a minimum of one and a maximum of b customers per server). Further, we assume that the service time distribution of each batch is exponential with mean $1/\beta$ and is independent of batch size. Until now we have only considered slots of unit length. Now we assume that the time be slotted into intervals of equal length $\Delta > 0$, so that $\lambda = \alpha\Delta$ and $\mu = \beta\Delta$, where Δ is sufficiently small. One may note that by substituting $\lambda = \alpha\Delta$ and $\mu = \beta\Delta$ in $\lambda < mb\mu$, we get $\alpha < mb\beta$, *i.e.*, both positive recurrence conditions (for the discrete- and continuous-time systems) are consistent. Now, equation (9) can be written as

$$\Big\{r+\lambda(1-r)\Big\}\left\{(1-\mu)^m + \binom{m}{1}(\mu r^b)(1-\mu)^{m-1} + \cdots + (\mu r^b)^m\right\} - r = 0,$$

or,

$$\Big\{r+\lambda(1-r)\Big\}\left[\left\{1-\binom{m}{1}\mu + \cdots + (-1)^m\mu^m\right\} + m(\mu r^b)\left\{1-\binom{m-1}{1}\mu + \cdots + (-1)^{m-1}\mu^{m-1}\right\} + \cdots + (\mu r^b)^m\right] - r = 0.$$

Putting $\lambda = \alpha\Delta$, $\mu = \beta\Delta$ in the above and taking the limit as $\Delta \to 0$ yields

$$m\beta r^{b+1} - (\alpha + m\beta)r + \alpha = 0. \tag{41}$$

The equation (15) can be written as

$$\lambda(1-\mu)^m(1-r)Q_{0,m-1} = \Bigg[1-(1-\mu)^m - \{(1-\lambda)r+\lambda\}m\mu(1-\mu)^{m-1} - \Big\{(1-\lambda)r+\lambda\Big\}\sum_{j=2}^{m}\binom{m}{j}\mu^j(1-\mu)^{m-j}r^{(j-1)b}\Bigg]Q_{0,m},$$

$$\text{or,}\quad \lambda\left[1-\binom{m}{1}\mu+\cdots+(-1)^m\mu^m\right](1-r)Q_{0,m-1}=$$
$$\Bigg[\left\{m\mu-\binom{m}{2}\mu^2+\cdots+(-1)^{m-1}\mu^m\right\}$$
$$-\{r+\lambda(1-r)\}\bigg\{m\mu\Big\{1-(m-1)\mu+\cdots+(-1)^{m-1}\mu^{m-1}\Big\}$$
$$+\sum_{j=2}^{m}\binom{m}{j}\mu^j(1-\mu)^{m-j}r^{(j-1)b}\bigg\}\Bigg]Q_{0,m}.$$

Using $\lambda=\alpha\Delta$, $\mu=\beta\Delta$ in the above and taking the limit as $\Delta\to 0$, we get

$$Q_{0,m-1}=\frac{m\beta}{\alpha}Q_{0,m}. \tag{42}$$

After rearranging the equation (16), we have

$$\begin{aligned}\lambda(1-\mu)^kQ_{0,k-1} = &\Bigg[\Big\{1-(1-\lambda)(1-\mu)^k\Big\}Q_{0,k}\\ &-\binom{k+1}{1}\mu(1-\mu)^k\Big\{(1-\lambda)Q_{0,k+1}+\lambda Q_{0,k}\Big\}\\ &-\sum_{j=k+2}^{m}\binom{j}{j-k}\mu^{j-k}(1-\mu)^k\Big\{(1-\lambda)Q_{0,j}+\lambda Q_{0,j-1}\Big\}\\ &-\Big\{(1-\lambda)r+\lambda\Big\}\frac{1-r^b}{1-r}\\ &\sum_{j=m+1}^{m+k}\binom{m}{j-k}\mu^{j-k}(1-\mu)^{m-j+k}r^{(j-m-1)b}Q_{0,m}\Bigg],\\ &\qquad k=m-1,m-2\ldots,2,1.\end{aligned}$$

Substituting $\lambda=\alpha\Delta$, $\mu=\beta\Delta$ in the above and taking the limit as $\Delta\to 0$ yields

$$\alpha Q_{0,k-1}=(\alpha+k\beta)Q_{0,k}-(k+1)\beta Q_{0,k+1},\qquad k=m-1,\ldots,2,1. \tag{43}$$

Using $Q_{0,m-1}=\frac{m\beta}{\alpha}Q_{0,m}$ in (43) and after repeated substitution, we obtain

$$Q_{0,k}=\left(\frac{\alpha}{\beta}\right)^k\frac{1}{k!}Q_{0,0},\qquad 1\le k\le m. \tag{44}$$

Finally, the equation (12) can be written as

$$Q_{n,m}=\left(\frac{\alpha}{\beta}\right)^m\frac{1}{m!}r^nQ_{0,0},\qquad n\ge 1. \tag{45}$$

The above results are the same as those given in Medhi [16], p. 219, for the continuous-time case. Similarly, in limiting case we can obtain equivalent results from the LAS-DA. This leads to the conclusion that, in continuous-time, results for both LAS-DA and EAS queues tend to same as it should be.

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