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ON COMPROMISE SOLUTIONS IN MULTIPLE OBJECTIVE PROGRAMMING

Majid Soleimani–damaneh[1,*] and Moslem Zamani[1]

Abstract. Compromise solutions, as feasible points as close as possible to the ideal (utopia) point, are important solutions in multiple objective programming. It is known in the literature that each compromise solution is a properly efficient solution if the sum of the image set and conical ordering cone is closed. In this paper, we prove the same result in a general setting without any assumption.

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1. Introduction and preliminaries

We consider the following multiple objective optimization problem (MOP):

$$\min_{\mathbf{x}\in X} f(\mathbf{x}) = \Big(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})\Big). \qquad (MOP)$$

The set of feasible solutions of this problem is $X \subseteq \mathbb{R}^n$ and the vector-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the objective function.

For two vectors $\mathbf{y}^1, \mathbf{y}^2 \in \mathbb{R}^m$ (with $m \geq 2$), the vector inequality $\mathbf{y}^1 \leqq$ (resp. $<$) $\mathbf{y}^2$ means $y_j^1 \leq$ (resp. $<$) y_j^2 for each j. The vector inequalities $\geqq$ and $>$ are defined analogously.

For two sets $A, B \subseteq \mathbb{R}^n$,

$$A \pm B := \{\mathbf{x} \pm \mathbf{y} \ : \ \mathbf{x} \in A, \ \mathbf{y} \in B\}.$$

For the sake of simplicity, we use $\mathbf{x} \pm A$ instead of $\{\mathbf{x}\} \pm A$. The positive hull of $A \subseteq \mathbb{R}^n$, denoted by $Pos(A)$, is defined as

$$Pos(A) := \left\{\mathbf{x} \in \mathbb{R}^n \ : \exists m \in \mathbb{N}; \ \ \mathbf{x} = \sum_{i=1}^{m} \lambda_i \mathbf{x}_i, \ \ \lambda_i \geq 0, \ \ \mathbf{x}_i \in A, \ i = 1, 2, \ldots, m\right\}.$$

For ordering the criterion space $\mathbb{R}^m$, we use the natural (conical) ordering cone

$$\mathbb{R}^m_{\geqq} := \{\mathbf{x} \in \mathbb{R}^m \ : \ x_j \geq 0, \ j = 1, 2, \ldots, m\}.$$

Keywords. Multiple objective programming, compromise solution, properly efficient solution.

[1] School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Enghelab Avenue, Tehran, Iran.
[*] Corresponding author: soleimani@khayam.ut.ac.ir

Utilizing this ordering cone, a feasible solution $\hat{\mathbf{x}} \in X$ is called an efficient solution of (MOP) if

$$\left(f(\hat{\mathbf{x}}) - \mathbb{R}^m_{\geqq}\right) \bigcap f(X) = \{f(\hat{\mathbf{x}})\}. \tag{1.1}$$

From a practical standpoint, a feasible solution $\hat{\mathbf{x}} \in X$ is efficient if improvement of some objective function at $\hat{\mathbf{x}}$ without deterioration of at least one other objective function is not possible. From a mathematical view, $\hat{\mathbf{x}} \in X$ is efficient if and only if $f(\mathbf{x}) \leqq f(\hat{\mathbf{x}}),\ \mathbf{x} \in X$ imply $f(\mathbf{x}) = f(\hat{\mathbf{x}})$. The Relation (1.1) means that if one translates the convex cone $-\mathbb{R}^m_{\geqq}$ by $f(\hat{\mathbf{x}})$, and then intersects the obtained set with the image set $f(X)$, then the intersection will be the singleton $\{f(\hat{\mathbf{x}})\}$. See Figure 1a.

A strict version of $\mathbb{R}^m_{\geqq}$, denoted by $\mathbb{R}^m_{>}$, is

$$\mathbb{R}^m_{>} := \{\mathbf{x} \in \mathbb{R}^m \ : \ x_j > 0,\ j = 1, 2, \ldots, m\}.$$

Given $A \subseteq \mathbb{R}^n$, two notations $\operatorname{int}(A)$ and ∂A stand for the interior and the boundary of A, respectively. The tangent cone to $A \subseteq \mathbb{R}^n$ at $\bar{\mathbf{x}} \in A$, denoted by $T[A; \bar{\mathbf{x}}]$, is defined as

$$T[A; \bar{\mathbf{x}}] := \left\{ \mathbf{d} \in \mathbb{R}^n \ : \exists \Big(\{\mathbf{x}_\nu\} \subseteq A,\ t_\nu \downarrow 0 \Big) \ \text{ s.t. } \frac{\mathbf{x}_\nu - \bar{\mathbf{x}}}{t_\nu} \to \mathbf{d} \text{ as } \nu \to \infty \right\}.$$

One of the solution concepts which plays an important role in multiple objective programming, from both theoretical and practical points of view, is the proper efficiency notion [1, 3–5, 7, 8]. As mentioned above, at an efficient solution, improving some objective function can only be obtained at the expense of the deterioration of at least one other objective function [1]. This property leads to a quantity called trade-off among objectives when moving from considered efficient solution to another feasible solution; See [6]. Geoffrion [3] first realized that such trade-offs can be unbounded at some efficient solutions. Due to this, proper efficiency has been introduced to eliminate the efficient solutions with unbounded trade-offs [3, 5].

Definition 1.1 [3]**.** A feasible solution $\hat{\mathbf{x}} \in X$ is called a properly efficient solution for (MOP) in the sense of *Geoffrion*, if it is efficient and there is a real number $M > 0$ such that for all $i \in \{1, 2, \ldots, m\}$ and $\mathbf{x} \in X$ satisfying $f_i(\mathbf{x}) < f_i(\hat{\mathbf{x}})$ there exists an index $j \in \{1, 2, \ldots, m\}$ such that $f_j(\mathbf{x}) > f_j(\hat{\mathbf{x}})$ and

$$\frac{f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\hat{\mathbf{x}})} \leq M.$$

Although Geoffrion's definition enjoys nice economical interpretations and it is useful for numerical purposes, there is another proper efficiency definition, by Henig [4], which is a geometrical concept and makes checking proper efficiency easier.

Definition 1.2 [4]**.** A feasible solution $\hat{\mathbf{x}} \in X$ is called a properly efficient solution for (MOP) in the sense of *Henig* if $(f(\hat{\mathbf{x}}) - C) \bigcap f(X) = \{f(\hat{\mathbf{x}})\}$, for some convex pointed cone C satisfying $\mathbb{R}^m_{\geqq} \backslash \{\mathbf{0}\} \subseteq \operatorname{int}(C)$.

Due to Definition 1.2, an efficient solution $\hat{\mathbf{x}} \in X$ is properly efficient if it remains efficient after making some small perturbations in the conical ordering cone such that the new cone contains the conical ordering cone (except for the origin) in its interior. This solution concept is illustrated in Figure 1a. In this figure, the hatched area is $\bar{\mathbf{y}} - \mathbb{R}^2_{\geqq}$. Furthermore, the shaded area represents a translated ordering cone containing $\bar{\mathbf{y}} - \mathbb{R}^2_{\geqq} \backslash \{\mathbf{0}\}$ in its interior. It is seen that $\bar{\mathbf{y}}$ is still efficient after perturbing the conical ordering cone.

Since we are using natural ordering cone, the above two definitions are equivalent [4, 7]. Hereafter, the set of efficient solutions and the set of properly efficient solutions are denoted by X_E and X_{PE}, respectively. Also, by setting $Y := f(X)$, the set of nondominated points, denoted by Y_N, is defined as $Y_N := f(X_E)$; and the set of properly nondominated points, denoted by Y_{PN}, is defined as $Y_{PN} := f(X_{PE})$. A set $Y \subseteq \mathbb{R}^m$ is called $\mathbb{R}^m_{\geqq}-$closed if $Y + \mathbb{R}^m_{\geqq}$ is closed.

Definition 1.3 [1]**.** The point $\mathbf{y}^I = (y_1^I, y_2^I, \ldots, y_m^I) \in \mathbb{R}^m$ in which $y_i^I = \min_{\mathbf{x}\in X} f_i(\mathbf{x}),\ i = 1,2,\ldots,m$, is said the ideal point of (MOP). The point $\mathbf{y^U} \in \mathbb{R}^m$ in which $y_i^U = y_i^I - \alpha_i,\ i = 1,2,\ldots,m$ for some $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) > \mathbf{0}$, is called a utopia point for (MOP).

In fact, the i-th component of the ideal point represents the optimal value of the i-th objective function over the feasible set (See Fig. 1b). If this point is the image of a feasible solution, then such a feasible solution optimizes all objective functions simultaneously, however, in real-world problems the decision maker is not faced with such a situation because the objective functions are conflicting (say minimizing risk against maximizing return in portfolio selection; or minimizing cost against maximizing quality in production planning). The utopia point which plays a vital role in compromise programming is obtained by some small improvements in the ideal point (see Fig. 1b).

2. Compromise programming

The main aim of compromise programming is getting feasible solutions (in the image space) as close as possible to the ideal (utopia) point [1,2].

Hereafter, it is assumed that the ideal point $\mathbf{y}^I$ exists. Let $\boldsymbol{\lambda} \in \mathbb{R}^m$ be a given weight vector. One of the popular measure functions, which has been widely used in the literature, is $d_{\boldsymbol{\lambda}}(., \mathbf{y}^U) : \mathbb{R}^m \longrightarrow \mathbb{R}$ defined by

$$d_{\boldsymbol{\lambda}}(\mathbf{y}, \mathbf{y}^U) := \|\boldsymbol{\lambda} \odot (\mathbf{y} - \mathbf{y}^U)\|_p,$$

in which p is a positive integer. Also, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$,

$$\mathbf{x} \odot \mathbf{y} := (x_1 y_1, x_2 y_2, \ldots, x_m y_m)$$

and

$$\|\mathbf{x} \odot \mathbf{y}\|_p := \left(\sum_{j=1}^m |x_j y_j|^p\right)^{\frac{1}{p}}.$$

Considering a $\boldsymbol{\lambda} \in \mathbb{R}^m_>$, the set of best approximations of the utopia point measured by $\|.\|_p$ is defined by

$$A(\boldsymbol{\lambda}, p, Y) := \left\{\bar{\mathbf{y}} \in Y : \|\boldsymbol{\lambda} \odot (\bar{\mathbf{y}} - \mathbf{y}^U)\|_p = \min_{\mathbf{y}\in Y} \|\boldsymbol{\lambda} \odot (\mathbf{y} - \mathbf{y}^U)\|_p\right\},$$

in which $Y = f(X)$. Now, the set of best approximations of $\mathbf{y}^U$ considering all positive weights is defined by

$$A(Y) := \bigcup_{\boldsymbol{\lambda}\in\Lambda^0} \bigcup_{1\le p<\infty} A(\boldsymbol{\lambda}, p, Y)$$

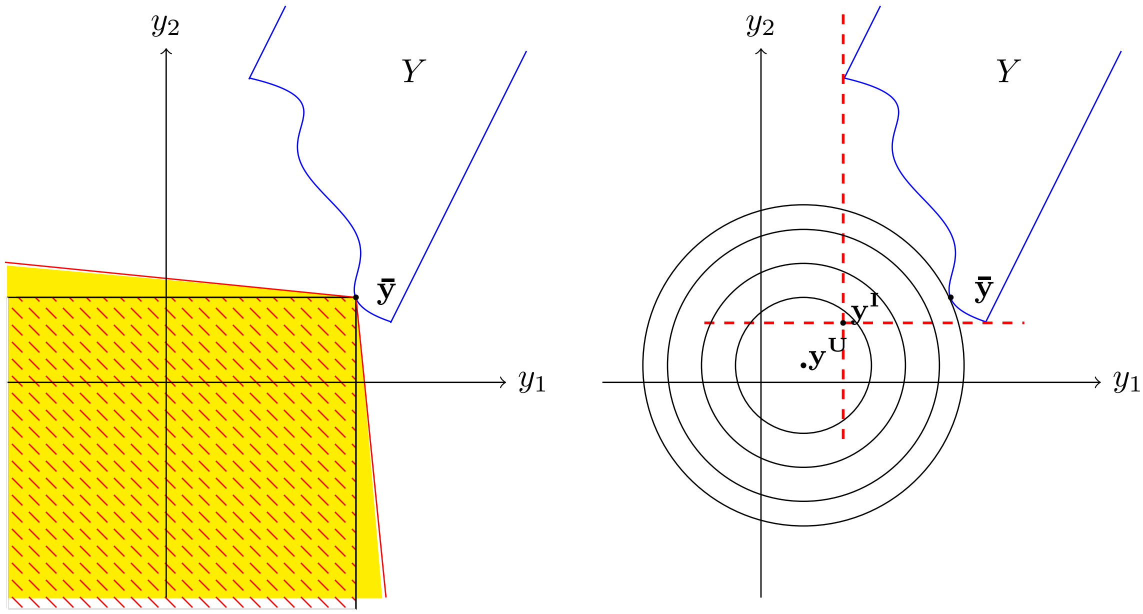
where

$$\Lambda^0 := \left\{\boldsymbol{\lambda} \in \mathbb{R}^m_> \ :\ \sum_{j=1}^m \lambda_j = 1\right\}.$$

Each member of $A(Y)$ is called a compromise solution/point of (MOP). In fact, the compromise solutions are the members of the image set $f(X)$ which are as close as possible to a utopia point with respect to some weighted p-norm with $p \in [1, \infty)$ and positive normal weights.

The following important result, which demonstrates the relationship between compromise solutions and properly efficient solutions, can be found in the literature; See, (*e.g.*, [1], p. 116 and [7], p. 79).

Theorem A. [1,2,7] *If Y is $\mathbb{R}^m_{\geqq}$−closed, then $A(Y) \subseteq Y_{PN}$.*



a: Properly efficient solutions. b: Compromise solutions.

FIGURE 1. Illustration of some solution concepts.

To have a better insight about the aforementioned notions and notations, we illustrate them invoking a figure. Consider the image set $Y \subseteq \mathbb{R}^2$ depicted in Figure 1b. The ideal point $\mathbf{y^I}$ has been shown in the figure. Utopia point can be any point in the area dominated by the ideal. One of such points, denoted by $\mathbf{y^U}$, is seen in the figure. Considering $\boldsymbol{\lambda} := (\frac{1}{2}, \frac{1}{2})$ and $p = 2$, the set $A(\boldsymbol{\lambda}, p, Y)$ is consisting of the optimal solutions of the optimization problem $\min_{\mathbf{y}\in Y} \dfrac{1}{2}\|\mathbf{y}-\mathbf{y^U}\|_2$. The contours of the function $\|\cdot - \mathbf{y^U}\|_2$, depicted in Figure 1b, show that the optimal solution of this problem in the considered example is the unique vector $\bar{\mathbf{y}}$. This vector is called a compromise point. By applying the Henig proper efficiency notion, it is not difficult to see that this compromise point is a properly nondominated point (see Fig. 1a). More compromise points can be generated by considering other values for the weight vector $\boldsymbol{\lambda}$ and the scalar p.

3. Main result

Theorem 3.1, the main result of the paper, proves the inclusion provided in Theorem **A** in a general setting without any assumption.

Theorem 3.1. *$A(Y) \subseteq Y_{PN}$ always holds.*

Proof. Without loss of generality, assume $\mathbf{y}^U = \mathbf{0}$. For every strictly positive vector $\boldsymbol{\lambda}$ and scalar $p \geq 1$, define the norm $\|\mathbf{y}\|_{\boldsymbol{\lambda},p} := (\sum_{i=1}^m |\lambda_i y_i|^p)^{\frac{1}{p}}$. The corresponding unit ball is denoted by $B_{\boldsymbol{\lambda},p}$.

Considering $\bar{\mathbf{y}} \in A(Y)$, we have $\bar{\mathbf{y}} \in A(\bar{\boldsymbol{\lambda}}, \bar{p}, Y)$ for some $\bar{\boldsymbol{\lambda}} \in \Lambda^0$ and $\bar{p} \in [1, \infty)$. Since $\mathbf{y}^U = 0$, we infer $\bar{\mathbf{y}} > 0$. Without loss of generality we may assume $\|\bar{\mathbf{y}}\|_{\bar{\boldsymbol{\lambda}},\bar{p}} = 1$, and so

$$\|\mathbf{y}\|_{\bar{\boldsymbol{\lambda}},\bar{p}} \geq 1, \quad \forall \mathbf{y} \in Y. \tag{3.1}$$

Let $\mathbf{e}^i$ $(i = 1, 2, \ldots, m)$ denote the ith unit vector. As $\bar{\boldsymbol{\lambda}} \in \Lambda^0$,

$$\left(\bar{\mathbf{y}} - \text{Pos}\left\{\mathbf{e}^i : i = 1, 2, \ldots, m\right\} \backslash \{\mathbf{0}\}\right) \cap (\mathbb{R}^m_{\geqq}) \subseteq \text{int}\left(B_{\bar{\boldsymbol{\lambda}},\bar{p}}\right). \tag{3.2}$$

We claim

$$\left(\bar{\mathbf{y}} - \text{Pos}\left\{\mathbf{e}^i + \epsilon B_{\bar{\boldsymbol{\lambda}},\bar{p}} : i = 1, 2, \ldots, m\right\} \backslash \{\mathbf{0}\}\right) \cap (\mathbb{R}^m_{\geqq}) \subseteq \text{int}\left(B_{\bar{\boldsymbol{\lambda}},\bar{p}}\right), \tag{3.3}$$

for sufficiently small $\epsilon > 0$; See Figure 2 and Remark 3.2 for better understanding (3.3).

To prove the claim, by indirect proof assume that there are sequences $\epsilon_\nu \downarrow 0$, $\{\boldsymbol{\beta}^\nu\} \subseteq \mathbb{R}^m_{\geqq}$ and $\{(\mathbf{b}^{1,\nu}, \ldots, \mathbf{b}^{m,\nu})\} \subseteq B_{\bar{\boldsymbol{\lambda}},\bar{p}} \times \ldots \times B_{\bar{\boldsymbol{\lambda}},\bar{p}}$ such that

$$\mathbf{y}^\nu := \bar{\mathbf{y}} - \sum_{i=1}^m \beta_i^\nu (\mathbf{e}^i + \epsilon_\nu \mathbf{b}^{i,\nu}) \in \left(\mathbb{R}^m_{\geqq} \backslash \{\bar{\mathbf{y}}\}\right) \setminus \text{int}\left(B_{\bar{\boldsymbol{\lambda}},\bar{p}}\right). \tag{3.4}$$

Set

$$\mathcal{D}_\nu := \left(\bar{\mathbf{y}} - \text{Pos}\left\{\mathbf{e}^i + \epsilon_\nu B_{\bar{\boldsymbol{\lambda}},\bar{p}} : i = 1, 2, \ldots, m\right\}\right).$$

If $\|\mathbf{y}^\nu\|_{\bar{\boldsymbol{\lambda}},\bar{p}} > 1$ for some ν, then taking $\mathbf{y}^\nu \in \mathcal{D}_\nu$, $\mathbf{0} \in \text{int}(\mathcal{D}_\nu)$, and the convexity of $\mathcal{D}_\nu$ into account, one can find a $\mathbf{z}^\nu \in \left(\mathcal{D}_\nu \backslash \{\bar{\mathbf{y}}\}\right) \cap \mathbb{R}^m_{\geqq}$ with $\|\mathbf{z}^\nu\|_{\bar{\boldsymbol{\lambda}},\bar{p}} = 1$. So, without loss of generality, we assume

$$\left\|\bar{\mathbf{y}} - \sum_{i=1}^m \beta_i^\nu (\mathbf{e}^i + \epsilon_\nu \mathbf{b}^{i,\nu})\right\|_{\bar{\boldsymbol{\lambda}},\bar{p}} = 1, \quad \forall \nu \in \mathbb{N}. \tag{3.5}$$

Due to (3.5) and the boundedness of $\{\epsilon_\nu \mathbf{b}^{i,\nu}\}$, by choosing an appropriate subsequence without relabeling, one may assume that for any i, the sequence β_i^ν converges to some $\beta_i \geq 0$ (see Rem. 3.3). From (3.5), we get

$$\left\|\bar{\mathbf{y}} - \sum_{i=1}^m \beta_i \mathbf{e}^i\right\|_{\bar{\boldsymbol{\lambda}},\bar{p}} = 1. \tag{3.6}$$

Furthermore, by (3.4), $\bar{\mathbf{y}} - \sum_{i=1}^m \beta_i^\nu \mathbf{e}^i - \epsilon_\nu \sum_{i=1}^m \beta_i^\nu \mathbf{b}^{i,\nu} \in \mathbb{R}^m_{\geqq}$. So, by $\nu \to \infty$, we get

$$\bar{\mathbf{y}} - \sum_{i=1}^m \beta_i \mathbf{e}^i \in \mathbb{R}^m_{\geqq}. \tag{3.7}$$

If $\sum_{i=1}^m \beta_i \mathbf{e}^i \neq 0$, then (3.6) and (3.7) contradict (3.2). If $\sum_{i=1}^m \beta_i \mathbf{e}^i = 0$, then $\mathbf{y}^\nu \to \bar{\mathbf{y}}$. Furthermore, the sequence $\frac{\mathbf{y}^\nu - \bar{\mathbf{y}}}{\|\mathbf{y}^\nu - \bar{\mathbf{y}}\|}$ tends to some nonzero vector $\mathbf{d} \in -\mathbb{R}^m_{\geqq}$ (by working with subsequences if necessary). So, $\mathbf{d} \in T[\partial B_{\bar{\boldsymbol{\lambda}},\bar{p}}; \bar{\mathbf{y}}]$. This implies $\nabla h(\bar{\mathbf{y}})^T \mathbf{d} = 0$, where $h(\mathbf{y}) = \|\mathbf{y}\|_{\boldsymbol{\lambda},\bar{p}}$. This makes a contradiction because $\nabla h(\bar{\mathbf{y}}) \in \mathbb{R}^m_{>}$ and $\mathbf{0} \neq \mathbf{d} \in -\mathbb{R}^m_{\geqq}$ (Here, $\frac{\partial h(\bar{y})}{\partial y_j} = \lambda_j^{\bar{p}} \bar{y}_j^{\bar{p}-1} (h(\bar{y}))^{1-\bar{p}} > 0$, for any j). In both cases, we got contradictions. So, (3.3) holds for sufficiently small $\epsilon > 0$.

It is not difficult to see that the cone

$$\mathcal{C} := \text{Pos}\left\{\mathbf{e}^i + \epsilon B_{\bar{\boldsymbol{\lambda}},\bar{p}} : i = 1, 2, \ldots, m\right\}$$

is closed, convex and pointed for sufficiently small $\epsilon > 0$; and it contains $\mathbb{R}^m_{\geqq} \backslash \{0\}$ in its interior. Furthermore, if there exists some $\mathbf{y} \in Y \backslash \{\bar{\mathbf{y}}\}$ belonging to $\bar{\mathbf{y}} - \mathcal{C}$, then by (3.3), we have $\|\mathbf{y}\|_{\bar{\boldsymbol{\lambda}},\bar{p}} < 1$. This contradicts (3.1). Therefore, we get

$$\left(\bar{\mathbf{y}} - \mathcal{C}\right) \cap Y = \{\bar{\mathbf{y}}\},$$

which implies $\bar{\mathbf{y}} \in Y_{PN}$ according to Definition 1.2. $\square$

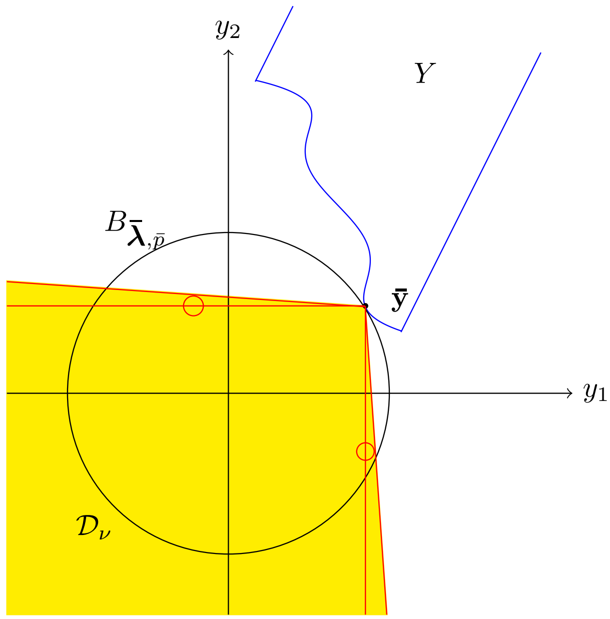


FIGURE 2. Illustration of (3.3) and its proof ($\mathcal{D}_\nu = \bar{\mathbf{y}} - Pos\{\mathbf{e}^i + \epsilon_\nu B_{\bar{\boldsymbol{\lambda}},\bar{p}} : i = 1, 2, \ldots, m\}$); see Remark 3.2.

Remark 3.2. In Figure 2, the small circles illustrate $\epsilon_\nu B_{\bar{\boldsymbol{\lambda}},\bar{p}}$. The shaded area represents the set $\mathcal{D}_\nu$. Notice that, $\mathcal{D}_\nu$ is a translated convex cone. Furthermore, the lines in the small circles denote the rays emanated from $\bar{\mathbf{y}}$ in the directions $-\mathbf{e}^i$s. Moreover, the lines on the small circles stand for the rays emanated from $\bar{\mathbf{y}}$ in two directions obtained by perturbing $-\mathbf{e}^i$s. This perturbation has been made by $\epsilon_\nu B_{\bar{\boldsymbol{\lambda}},\bar{p}}$.

Remark 3.3. In the proof of Theorem 3.1, according to equation (3.5), as $\bar{\mathbf{y}}$ is fixed, the sequence

$$\left\{ \left(\beta_1^\nu + \sum_{i=1}^m \beta_i^\nu \epsilon_\nu b_1^{i,\nu}, \ldots, \beta_m^\nu + \sum_{i=1}^m \beta_i^\nu \epsilon_\nu b_m^{i,\nu} \right) \right\}$$

is bounded. Hence, the sequence generated by summing the components of the members of the above sequence, *i.e.*

$$\boldsymbol{\Psi}^\nu := \left\{ \sum_{k=1}^m \beta_k^\nu + \sum_{k=1}^m \sum_{i=1}^m \beta_i^\nu \epsilon_\nu b_k^{i,\nu} \right\},$$

is bounded as well. It can be seen that,

$$\boldsymbol{\Psi}^\nu = \beta_1^\nu \left(1 + \epsilon_\nu \sum_{k=1}^m b_k^{1,\nu} \right) + \beta_2^\nu \left(1 + \epsilon_\nu \sum_{k=1}^m b_k^{2,\nu} \right) + \ldots + \beta_m^\nu \left(1 + \epsilon_\nu \sum_{k=1}^m b_k^{m,\nu} \right), \quad \nu \in \mathbb{N}. \tag{3.8}$$

On the other hand, $\epsilon_\nu \downarrow 0$ and $\{\sum_{k=1}^m b_k^{i,\nu}\}$ is a bounded sequence for any i. So, $\epsilon_\nu \sum_{k=1}^m b_k^{i,\nu}$, for any i, tends to zero as $\nu \to \infty$. If $\{\beta_{i_0}^\nu\}$ is unbounded for some i_0, then according to (3.8) and the nonnegativity of β_i^ν's we have $\boldsymbol{\Psi}^\nu \to \infty$ as $\nu \to \infty$. This contradicts the boundedness of $\{\boldsymbol{\Psi}^\nu\}$. Hence, $\{\beta_i^\nu\}$ is bounded for any i. Therefore, for any i, the sequence $\{\beta_i^\nu\}$ admits a convergent subsequence.

In summary, the present paper proves theorem A in general case (without any assumption).

Since in some reference publications, including [1], the authors work with only Geoffrion proper efficiency (due to its economical interpretations with respect to the trade-off notion), we close the paper by another proof for Theorem 3.1 in terms of Geoffrion proper efficiency.

Another proof. Without loss of generality, assume $\mathbf{y}^U = \mathbf{0}$. This leads to $\mathbf{y} \geqq \mathbf{y}^I > \mathbf{y}^U = \mathbf{0}$ for each $y \in Y$. Considering $\bar{\mathbf{y}} \in A(Y)$, there exists some $\boldsymbol{\lambda} \in \Lambda^0$ and some $p \in [1, \infty)$ such that $\bar{\mathbf{y}}$ solves

$$\min_{\mathbf{y} \in Y} \sum_{j=1}^{m} \lambda_j^p y_j^p. \tag{3.9}$$

Set

$$I := \{1, 2, \dots, m\}, \quad \alpha_1 := (m-1) \max_{i,j \in I:\, j \neq i} \frac{\lambda_j^p}{\lambda_i^p}, \quad \alpha_2 := \min_{i \in I} (y_i^I)^{p-1}, \alpha_3 := \max_{i \in I} \bar{y}_i, \quad M_0 := \frac{\alpha_1 + \alpha_2}{\alpha_2} (\alpha_3 + 1)^{p-1}.$$

To prove $\bar{\mathbf{y}} \in Y_{PN}$, it is sufficient to show that for any $i \in I$ and $\mathbf{y} \in Y$ satisfying $y_i < \bar{y}_i$ there exists an index $j \in I$ such that $y_j > \bar{y}_j$ and $\frac{\bar{y}_i - y_i}{y_j - \bar{y}_j} \leq M_0$. By indirect proof, assume that there exists some $i \in I$ and $\hat{y} \in Y$ such that

$$\hat{y}_i < \bar{y}_i \;\;\&\;\; \bar{y}_i - \hat{y}_i > M_0(\hat{y}_j - \bar{y}_j), \;\forall j \in I \backslash \{i\} \text{ with } \hat{y}_j > \bar{y}_j. \tag{3.10}$$

Due to (3.10), for any $j \in I \backslash \{i\}$ with $\hat{y}_j > \bar{y}_j$, we have

$$\begin{aligned} \hat{y}_j < \frac{\bar{y}_i - \hat{y}_i}{M_0} + \bar{y}_j &\leq \frac{\alpha_3 - \hat{y}_i}{M_0} + \alpha_3 \\ &\leq \frac{\alpha_3}{M_0} + \alpha_3 \\ &\leq \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{\alpha_3}{(\alpha_3 + 1)^{p-1}} + \alpha_3 \\ &\leq 1 + \alpha_3. \end{aligned}$$

So,

$$\hat{y}_j \leq \alpha_3 + 1, \;\forall j \in I \backslash \{i\} \text{ with } \hat{y}_j > \bar{y}_j. \tag{3.11}$$

Furthermore according to the convexity of the p-power function on positive real numbers, we get

$$\bar{y}_k^p - \hat{y}_k^p \geq p \hat{y}_k^{p-1} (\bar{y}_k - \hat{y}_k), \;\forall k \in I. \tag{3.12}$$

Therefore, for any $j \in I \backslash \{i\}$ with $\hat{y}_j > \bar{y}_j$,

$$\frac{\bar{y}_i^p - \hat{y}_i^p}{p \hat{y}_i^{p-1}} \underbrace{\geq}_{by\ (3.12)} \bar{y}_i - \hat{y}_i \underbrace{>}_{by\ (3.10)} M_0(\hat{y}_j - \bar{y}_j) \underbrace{\geq}_{by\ (3.12)} \frac{M_0}{p \hat{y}_j^{p-1}} (\hat{y}_j^p - \bar{y}_j^p).$$

These imply

$$\begin{aligned} \forall j \in I \backslash \{i\} \text{ with } \hat{y}_j > \bar{y}_j : (\alpha_3 + 1)^{p-1} (\bar{y}_i^p - \hat{y}_i^p) &\underbrace{\geq}_{by\ (3.11)} \hat{y}_j^{p-1} (\bar{y}_i^p - \hat{y}_i^p) \\ &> M_0 \hat{y}_i^{p-1} (\hat{y}_j^p - \bar{y}_j^p) \\ &\geq M_0 \alpha_2 (\hat{y}_j^p - \bar{y}_j^p) \\ &> \alpha_1 (\alpha_3 + 1)^{p-1} (\hat{y}_j^p - \bar{y}_j^p) \\ &\geq (m-1) \frac{\lambda_j^p}{\lambda_i^p} (\alpha_3 + 1)^{p-1} (\hat{y}_j^p - \bar{y}_j^p). \end{aligned}$$

Hence,

$$\lambda_i^p(\bar{y}_i^p - \hat{y}_i^p) > (m-1)\lambda_j^p(\hat{y}_j^p - \bar{y}_j^p),\ \forall j \in I\backslash\{i\} \text{ with } \hat{y}_j > \bar{y}_j. \tag{3.13}$$

On the other hand,

$$\lambda_i^p(\bar{y}_i^p - \hat{y}_i^p) \geq (m-1)\lambda_j^p(\hat{y}_j^p - \bar{y}_j^p),\ \forall j \in I\backslash\{i\} \text{ with } \hat{y}_j \leq \bar{y}_j. \tag{3.14}$$

So, we have

$$\lambda_i^p(\bar{y}_i^p - \hat{y}_i^p) \geq (m-1)\lambda_j^p(\hat{y}_j^p - \bar{y}_j^p) \tag{3.15}$$

for any $j \in I\backslash\{i\}$, and it is strict for some $j \in I\backslash\{i\}$. Summing (3.15) on $j \in I\backslash\{i\}$, we get

$$\sum_{j\in I\backslash\{i\}} \lambda_i^p(\bar{y}_i^p - \hat{y}_i^p) > \sum_{j\in I\backslash\{i\}} (m-1)\lambda_j^p(\hat{y}_j^p - \bar{y}_j^p)$$

$$\Longrightarrow (m-1)\lambda_i^p(\bar{y}_i^p - \hat{y}_i^p) > (m-1)\sum_{j\in I\backslash\{i\}} \lambda_j^p\hat{y}_j^p - (m-1)\sum_{j\in I\backslash\{i\}} \lambda_j^p\bar{y}_j^p$$

$$\Longrightarrow (m-1)\left[\lambda_i^p\bar{y}_i^p + \sum_{j\in I\backslash\{i\}} \lambda_j^p\bar{y}_j^p\right] > (m-1)\left[\lambda_i^p\hat{y}_i^p + \sum_{j\in I\backslash\{i\}} \lambda_j^p\hat{y}_j^p\right]$$

So, we get $\sum_{j=1}^{m} \lambda_j^p\bar{y}_j^p > \sum_{j=1}^{m} \lambda_j^p\hat{y}_j^p$, which makes a contradiction because $\bar{\mathbf{y}}$ solves (3.9). This contradiction completes the proof.

4. Conclusion

The main result of this paper proves that each compromise solution in multiple objective programming is a properly efficient solution without any closedness assumption. The established result can be useful in sketching numerical algorithms which approximate properly efficient solution set.

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