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THE *ST*-BOND POLYTOPE ON SERIES-PARALLEL GRAPHS

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Abstract. The *st*-bond polytope of a graph is the convex hull of the incidence vectors of its *st*-bonds, where an *st*-bond is a minimal *st*-cut. In this paper, we provide a linear description of the *st*-bond polytope on series-parallel graphs. We also show that the *st*-bond polytope is the intersection of the *st*-cut dominant and the bond polytope.

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1. Introduction

In combinatorial optimization, *st*-bonds are well-known objects since they are precisely the *st*-cuts involved in the famous max-flow min-cut theorem [11]. However, they are not well described from a polyhedral point of view. In this paper, we make a contribution in this regard by providing a linear description of the *st*-bond polytope on series-parallel graphs. These graphs are precisely those with no K_4-minor [8].

In an undirected graph $G = (V, E)$, a *cut* is the set of edges between a subset of vertices and its complement. A cut containing only itself and the emptyset as cuts is called a *bond*. Nonempty bonds are the cuts whose removal yields exactly two connected components.

Although cuts and bonds are similar, they behave quite differently. From a complexity point of view, optimizing over bonds is harder than optimizing over cuts. The maximum cut problem – which asks for a cut of maximum weight in an edge-weighted graph – is NP-hard in general [13] but becomes polynomial for graphs with no K_5-minor [15], and hence for planar graphs [16]. In comparison, finding a bond of maximum weight in a planar graph is already NP-hard but becomes polynomial for the subclass of graphs with no K_4-minor. Indeed, bonds are planar duals of circuits, and finding a circuit of maximum weight is NP-hard for planar graphs [14] and polynomial for graphs with no K_4-minor [5].

The polyhedral aspects of these problems reflect these complexity results. The *cut polytope* and the *bond polytope* of a graph are the convex hulls of the incidence vectors of its cuts and its bonds, respectively. The cut polytope has been described for graphs with no K_5-minor in [3] whereas a description of the bond polytope is only known for graphs with no K_4-minor [5].

Given two distinct vertices s and t, an *st-cut* and an *st-bond* are respectively a cut and a bond whose removal disconnects s and t. It follows that *st*-bonds are inclusionwise minimal *st*-cuts. The *st-cut polytope* and the *st-bond polytope* are the convex hull of the incidence vectors of *st*-cuts and *st*-bonds, respectively.

Keywords and phrases: Bond, minimal *st*-cut, *st*-bond polytope, series-parallel graph.

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The behaviour of st-cuts and st-bonds is different from a complexity point of view. For instance, consider the class of planar graphs where s and t belong to a same face, hereafter called *st-planar graphs*. In these graphs, finding an st-bond of maximum weight is NP-hard whereas an st-cut of maximum weight can be found in polynomial time. Indeed, a nonempty bond is an st-bond for some pair of vertices s and t on a same face. Hence, finding an st-bond of maximum weight is at least as hard as finding a bond of maximum weight in a planar graph. Computing an st-cut of maximum weight is polynomial-time solvable because adding an edge st with a sufficiently large weight reduces the problem to a maximum cut problem in a graph with no K_5-minor.

From a polyhedral point of view, remark that when the graph contains the edge st, the st-cut (resp. st-bond) polytope is a face of the cut (resp. bond) polytope. Hence, a polyhedral description for the st-cut (resp. st-bond) polytope is known for graphs with no K_5-minor (resp. K_4-minor) but containing the edge st. In this paper, we extend this polyhedral characterization by describing the st-bond polytope for graphs with no K_4-minor for any pair of vertices s and t.

By definition, the st-bond polytope is contained in the st-cut polytope. Nevertheless, these two polyhedra have the same dominant. This so-called *st-cut dominant* has been thoroughly studied in [18].

In general, the intersection of two integer polyhedra is not integer. Even adding a simple constraint to a well-structured polyhedron may destroy its integrality – see *e.g.* [6]. Few families of polyhedra are known to remain integer after intersection. For instance, box-TDI polyhedra keep their integrality after being intersected with any integer box [17]. The intersection of two polymatroids [9] or of two lexicographical polytopes [4] is integer. Calvillo characterizes the graphs whose stable set polytope remains integer after being intersected with specific hyperplanes [6]. For st-planar graphs, the intersection of the cut polytope [3] and the st-cut dominant [18] is integer: it is nothing but the st-cut polytope. For series-parallel graphs, we prove that the st-bond polytope is the intersection of the bond polytope and the st-cut dominant.

The contributions of this paper concern st-bonds in series-parallel graphs. We prove that finding an st-bond of maximum weight in a series-parallel graph can be performed in polynomial time. Moreover, we provide a polyhedral description of the st-bond polytope for such graphs. The approach is as follows. We describe the st-bond polytope on the subclass of 2-connected outerplanar graphs. We also provide a sufficient condition for a series-parallel graph to be outerplanar. We exploit this result to extend this polyhedral description of the st-bond polytope to series-parallel graphs. A consequence of our result is that intersecting the bond polytope and the st-cut dominant of a series-parallel graph does not destroy integrality: it precisely gives the st-bond polytope.

The outline of the paper is as follows. In Section 2, we first give the main definitions and notation used in this paper. We then show that a simple series-parallel graph with two vertices of degree at most two is outerplanar. We use it to show that finding an st-bond of maximum weight in series-parallel graphs can be done in polynomial time. In Section 3, we describe the st-bond polytope for 2-connected outerplanar graphs. In Section 4, we extend this description to series-parallel graphs. As a consequence, we prove that, for these graphs, the st-bond polytope is the intersection of the bond polytope and the st-cut dominant.

2. Bonds and st-bonds in series-parallel graphs

Throughout the paper, $G=(V,E)$ will denote a loopless connected undirected graph and s and t two distinct vertices of G. Given a vertex v of G, the graph $G \setminus v$ is obtained from G by removing the vertex v and its incident edges. Let uv be an edge of G. The graph G/uv is the graph obtained from G by *contracting* uv, that is, by identifying u and v into a new vertex which becomes adjacent to all the former neighbors of u and v. We define $G \setminus uv = (V, E \setminus \{uv\})$. Let F be a subset of E. A set of vertices is *covered* by F if each of its vertices is incident to at least one edge of F. Given a subgraph H of G, the set $F|_H$ denotes the set of edges of F which belong to H. Given two sets A and B, the symmetric difference $A\Delta B$ is $A\Delta B = (A\cup B)\setminus(A\cap B)$.

A subset of edges is called a *cycle* if it induces a graph in which every vertex has even degree. A *circuit* is a cycle inducing a connected graph in which every vertex has degree two. An *st-circuit* is a circuit covering both s and t. A *uv-path* P is a set of edges inducing a connected subgraph in which u and v have degree one and the other vertices have degree two. The vertices u and v are called *end vertices* of P whereas the other vertices covered by P are called *internal vertices*. A *path* is a uv-path for some distinct vertices u and v. Let P be an uv-path and W a set of vertices covered by P. Running P from u to v induces a (total) order on W which corresponds to the order in which the vertices of W are traversed by P starting from u. This order is called the *order of W induced by P starting from u*.

A subset F of E is a *cut* if it is the set of edges having exactly one extremity in X, for some $X \subseteq V$; it is denoted by $F = \delta(X)$. If $s \in X$ and $t \in V \setminus X$, the cut $\delta(X)$ is an *st-cut* and *separates* s and t. A nonempty *bond* is a cut containing no other nonempty cut. Equivalently, a nonempty bond is a cut whose removal gives exactly two connected components. An *st-bond* is an *st*-cut which is a bond. A *bridge* is an edge whose removal disconnects the graph. Equivalently, a bridge is a bond composed of a single edge.

A vertex whose removal yields several connected components is a *cut vertex*. If s and t belong to different connected components after the removal of a cut vertex v, then v is an *st-cut vertex*. When no vertex removal disconnects a connected graph with at least two vertices, the latter is said 2*-connected*. The 2*-connected components* of a graph are the inclusionwise maximal 2-connected subgraphs of the graph. A 2-connected graph composed of a single edge is called *trivial*. Note that the trivial 2-connected components of a graph $G = (V, E)$ are the graphs induced by the bridges of E.

2.1. Series-parallel and outerplanar graphs

A nontrivial 2-connected graph is *series-parallel* if it can be built, starting from the circuit of length two, by repeatedly applying the following operations:

- *parallel addition:* add a parallel edge to an existing edge,
- *subdivision:* replace an edge uv by the path $\{uw, wv\}$ where w is a new vertex.

A graph is *series-parallel* if all its nontrivial 2-connected components are.

Series-parallel graphs are also known to be those admitting the following kind of decomposition. An *open nested ear decomposition* [10] of a graph $G = (V, E)$ is a partition $\mathcal{E}$ of E into a sequence $E_0, \dots, E_k$ such that E_0, also denoted by $C_{\mathcal{E}}$, is a circuit of G and the *ears* E_i, for $i = 1, \dots, k$, are paths with the following properties:

- the end vertices of E_i are both covered by E_j for some $j < i$,
- no internal vertex of E_i is covered by E_j for all $j < i$,
- if two ears E_i and $E_{i'}$ have both their end vertices in the same member E_j of $\mathcal{E}$, then there exit a path P_i in E_j between the end vertices of E_i and a path $P_{i'}$ in E_j between the end vertices of $E_{i'}$ such that P_i and $P_{i'}$ are either disjoint or contained one in another.

A nontrivial 2-connected graph is series-parallel if and only if it admits an open nested ear decomposition [10].

An important subclass of series-parallel graphs are *outerplanar* graphs. They are the graphs admitting a planar drawing such that all the vertices lie on the unbounded face of the drawing, called the *external* face. An outerplanar graph is always supposed to be drawn in this way and the same terminology is used for the graph and the drawing. The *chords* are the edges which do not belong to the external face. In a simple outerplanar graph, the external face and the chords are uniquely defined. The faces other than the external one are called *inner faces*.

Lemma 2.2 provides a sufficient condition for a series-parallel graph to be outerplanar. We first give the following lemma since it is used in the proof of Lemma 2.2. In both proofs, we implicitly use that deleting a parallel edge or contracting an edge incident to a vertex of degree two in a graph preserves both 2-connectivity and series-parallelness.

Lemma 2.1. *A nontrivial simple* 2*-connected series-parallel graph has at least two vertices of degree two.*

Proof. Let $G = (V, E)$ be a counter example with a minimum number of vertices, that is, G is a nontrivial simple 2-connected series-parallel graph with at most one vertex of degree two. By construction of nontrivial 2-connected series-parallel graphs, G, being simple, has exactly one vertex of degree two, say v. Let xv and vy be its incident edges. Contracting xv preserves the degrees in $V \setminus \{v\}$ and maintains 2-connectivity and series-parallelness. Hence, by the minimality of G, the graph $G' = G/xv$ is not simple, and thus $xy \in E$. Thus, $G \setminus v$ is a simple 2-connected series-parallel graph with less vertices than G. By the minimality assumption, $G \setminus v$ has at least two vertices of degree two. All the vertices except x and y have the same degree in G and $G \setminus v$, so they all are of degree greater than two. Therefore, the degrees of x and y equal two in $G \setminus v$ and hence three in G. But then $(G \setminus v)/xy$ is a counter example – a contradiction to the minimality of G. □

Lemma 2.2. *A simple series-parallel graph having exactly two vertices of degree at most two is outerplanar.*

Proof. Let κ_G denote the number of vertices of degree at most two of a graph G.

We first prove the assertion for nontrivial 2-connected graphs. Let $G = (V, E)$ be a counter example with a minimum number of vertices, that is, G is a nontrivial simple 2-connected series-parallel graph with $\kappa_G = 2$ but is not outerplanar. Then, G differs from the circuit of length two. Since G is 2-connected, it has exactly two vertices of degree two. Let v be a vertex of degree two of G and xv and vy its incident edges. The graph $G' = G/xv$ is nontrivial, 2-connected and series-parallel. As $\kappa_{G'} = 1$, Lemma 2.1 implies that G' is not simple, and hence $xy \in E$. Hence, $G'' = G \setminus v$ is a simple 2-connected series-parallel graph. By Lemma 2.1, G'' has at least two vertices of degree two. Thus, by construction, at least one among x and y has degree two in G''. If G'' was outerplanar, then it would admit a planar drawing in which all its vertices, and hence also the edge xy, lie on the external face. But then G would be outerplanar – a contradiction. Hence, G'' is not outerplanar and, by the minimality assumption, G'' has at least three vertices of degree two. Since G has exactly two vertices of degree two, G'' has three vertices of degree two including x and y. Then, G''/xy is a counter example having fewer vertices than G – a contradiction. Thus, the assertion is proved for nontrivial 2-connected graphs. Note that the assertion also holds for trivial 2-connected graphs.

We now prove the result for simple series-parallel graphs. Given $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, identifying a vertex $v_1 \in V_1$ and a vertex $v_2 \in V_2$ yields a graph G satisfying $\kappa_G \geq \kappa_{G_1} + \kappa_{G_2} - 2$. Let G be a simple series-parallel graph with $\kappa_G = 2$ and $G_1, \ldots, G_k$ be its 2-connected components. By Lemma 2.1 and since $\kappa_G \geq \sum_{i=1}^{k} \kappa_{G_i} - 2(k-1)$, we deduce $\kappa_{G_i} = 2$ for $i = 1, \ldots, k$. Therefore, as proved above, each 2-connected component of G is outerplanar, hence so is G. □

2.2. Bonds and *st*-bonds

A well-known characterization of cuts is that they are the sets of edges intersecting every circuit an even number of times. Bonds being cuts, each bond intersects every circuit an even number of times. For series-parallel graphs, this property can be refined as follows.

Observation 2.3 (Chakrabarti *et al.* [7]). *In a series-parallel graph, a bond and a circuit intersect in zero or two edges.*

If the graph is also 2-connected, this property becomes a characterization of bonds.

Lemma 2.4 (Borne *et al.* [5]). *In a 2-connected series-parallel graph, a set of edges is a bond if and only if it intersects every circuit in zero or two edges.*

Let W be the set of st-cut vertices of G together with s and t. Then, every st-path of G covers W. Moreover, the order of W induced by running any st-path from s to t is the same. This order will be denoted by $v_1, v_2, \ldots, v_{|W|}$ throughout. Note that $s = v_1$ and $t = v_{|W|}$. Let $k = |W| - 1$ and ℓ be the number of 2-connected components of G. Let $H_1 = (V_1, E_1), \ldots, H_\ell = (V_\ell, E_\ell)$ be the 2-connected components of G numbered in such a way that, for $i = 1, \ldots, k$, $H_i = (V_i, E_i)$ contains v_i and v_{i+1}. Each bond separating two vertices of W being an st-bond, we obtain the following observation.

Observation 2.5. *The set of st-bonds of G is the union of the sets of $v_i v_{i+1}$-bonds of H_i over $i = 1, \ldots, |W| - 1$.*

By Observation 2.5, in order to solve the maximum st-bond problem, it is enough to solve the problem in each 2-connected component. In series-parallel graphs, this remark, combined with Lemma 2.2, gives the following complexity result.

Corollary 2.6. *Finding an st-bond of maximum weight in a series-parallel graph can be done in polynomial time.*

Proof. Let $G = (V, E)$ be a series-parallel graph with edge weights $w \in \mathbb{R}^E$. By Observation 2.5, we may suppose that G is 2-connected. We may suppose that G is nontrivial as otherwise it contains a unique bond. The following graph operations may be applied on G without changing the weight of a maximum st-bond. If e and f are two parallel edges, they can be replaced by a single edge with weight $w_e + w_f$. If e and f are in series, that is, there exists a vertex v of degree two incident to e and f, then the edge among e and f with the minimum weight may be contracted whenever v differs from s and t. By definition of series-parallelness, applying these two types of operations as long as possible yields a simple 2-connected series-parallel graph G' in which no vertex is of degree two except possibly s and t. Hence, by Lemmas 2.1 and 2.2, G' is outerplanar. From the definition of outerplanar graphs, the number of st-bonds in G' is quadratic in the number of its vertices. Thus, finding an st-bond of maximum weight in G' can be done in polynomial time by enumeration. Moreover, by construction, one can retrieve from such an st-bond one of G having the same weight. ☐

3. The *st*-bond polytope on 2-connected outerplanar graphs

In this section, we give a linear description of the st-bond polytope in 2-connected outerplanar graphs. Let $G = (V, E)$ be such a graph. If G is trivial, then it only has s and t as vertices and its only edge is st. The st-bond polytope of G is then $\{x_{st} = 1\}$. We now suppose, for the rest of this section, that G is nontrivial. We denote by P_{st} and Q_{st} the two st-paths of the external face F_{ext}, that is, $F_{\text{ext}} = P_{st} \cup Q_{st}$. We denote by $B_{st}(G)$ the convex hull of the incident vectors of the st-bonds of G.

We denote by $P(G)$ the polytope defined by the set of $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$x_e \geq 0, \qquad \text{for all } e \in E, \tag{3.1}$$
$$x_e \leq x(F \setminus e), \qquad \text{for all chords } e \text{ and faces } F \text{ containing } e, \tag{3.2}$$
$$x(P) \geq 1, \qquad \text{for all } st\text{-paths } P, \tag{3.3}$$
$$x(P) = 1, \qquad \text{for all } st\text{-paths } P \text{ contained in a circuit.} \tag{3.4}$$

We now prove in the next three lemmas that $P(G)$ is a linear description of $B_{st}(G)$. More precisely, we show the validity of inequalities (3.1)–(3.4) for $B_{st}(G)$ in Lemma 3.1. We then show that every integer vector satisfying these inequalities is the incident vector of an st-bond in Lemma 3.2. Finally, we end the proof by showing that $P(G)$ is an integer polytope in Lemma 3.4.

Lemma 3.1. $B_{st}(G) \subseteq P(G)$.

Proof. We show that the incidence vectors of st-bonds satisfy (3.1)–(3.4). A bond being a cut and a face being a circuit, a bond cannot intersect a face in exactly one edge by Observation 2.3. This ensures the validity of (3.2). The validity of (3.3) stems from the fact that removing the edges of an st-bond separates s and t. If an st-path is contained in a circuit, then this circuit is the union of two disjoint st-paths. By (3.3), any st-bond intersects this circuit at least twice. Thus, by Observation 2.3, inequalities (3.4) are valid. ☐

Lemma 3.2. $P(G) \cap \mathbb{Z}^E \subseteq B_{st}(G)$.

Proof. Let $\bar{x}$ be a point of $P(G) \cap \mathbb{Z}^E$. We need to show that it is the incident vector of an st-bond.

If the intersection of a circuit C and an inner face F is a chord of G, then, by (3.2), we have that $\bar{x}(C) = \bar{x}(C \setminus e) + \bar{x}_e \leq \bar{x}(C \setminus e) + \bar{x}(F \setminus e) = \bar{x}(C')$ with $C' = C \Delta F$. By repeating this argument, and since $\bar{x}(F_{\text{ext}}) = 2$ due to (3.4) associated with P_{st} and Q_{st}, we obtain that

$$\bar{x}(C) \leq 2, \quad \text{for each circuit } C. \tag{3.5}$$

Inequalities (3.2) can be generalized to any circuit and any edge of this circuit as shown in the following claim.

Claim 3.3. *$\bar{x}_e \leq \bar{x}(C \setminus e)$ for all circuits C and for all edges $e \in C$.*

Proof. Let F be an inner face of G and e an edge of $F \cap F_{\text{ext}}$. Without loss of generality, we suppose that $e \in P_{st}$. Let $P = P_{st} \cup F \setminus e$. Then, P contains an st-path and by (3.1), (3.3) and (3.4), we have $0 \leq \bar{x}(P) - \bar{x}(P_{st}) = \bar{x}(F \setminus P_{st}) - \bar{x}_e \leq \bar{x}(F \setminus e) - \bar{x}_e$. This, together with (3.2), gives

$$\bar{x}_e \leq \bar{x}(F \setminus e), \quad \text{for all inner faces } F \text{ and for all edges } e \in F. \tag{3.6}$$

In a planar graph, a circuit is the symmetric difference of inner faces. Suppose that a circuit C is the symmetric difference of two inner faces F_1 and F_2. Let f be the edge of $F_1 \cap F_2$ and let e be an edge of $F_1 \setminus F_2$. Adding inequalities (3.6) associated with e and F_1, and with f and F_2 gives $\bar{x}_e \leq \bar{x}(C \setminus e)$. The face F_1 and the edge e have been chosen arbitrarily so the inequality holds for every circuit consisting in the symmetric difference of two inner faces and for every edge it contains. By induction on the number of inner faces contained in a circuit, we obtain the desired result. ☐

For any edge $e \in E$, let F be an inner face containing e. Combining inequality (3.5) associated with F and inequality (3.6) associated with F and e gives $\bar{x}_e \leq 1$. Thus, by (3.1), $\bar{x}$ is the incidence vector of a subset of E, say $\bar{E}$.

Since $\bar{x}$ satisfies (3.3), removing the set of edges $\bar{E}$ disconnects s and t. To end the proof, it remains to show that $\bar{x}$ is the incident vector of a bond. Since outerplanar graphs are series-parallel, by Lemma 2.4, one just needs to show that $\bar{x}(C)$ equals 0 or 2 for each circuit C as $\bar{x}$ is a 0–1 vector. This is immediate by inequalities (3.5) and Claim 3.3. ☐

Lemma 3.4. *$P(G) = B_{st}(G)$.*

Proof. By Lemmas 3.1 and 3.2, it remains to show that $P(G)$ is an integer polytope. Suppose this is not true and let G be a counter example with a minimal number of edges. By hypothesis, there exists a fractional extreme point of $P(G)$, say $\bar{x}$. The minimality assumption implies that G has no parallel edges because two parallel edges have the same value in $\bar{x}$ by (3.2). In the following, an st-path P satisfying $\bar{x}(P) = 1$ will be called a *tight path*.

Claim 3.5. *$\bar{x}_e > 0$ for all the edges e of the external face.*

Proof. If $\bar{x}_e = 0$ for some $e \in F_{\text{ext}}$, then contract e. The resulting graph G/e is still outerplanar. By minimality, its st-bond polytope is described by (3.1)–(3.4), yet these are precisely the inequalities obtained by setting x_e to zero in $P(G)$, thus $\bar{x}$ yields a fractional extreme point in $P(G/e)$ – a contradiction to the minimality assumption. ☐

Claim 3.6. *No vertex of $V \setminus \{s, t\}$ has degree two.*

Proof. Suppose that $u \in V \setminus \{s, t\}$ is such that $\delta(u) = \{e, f\}$. Note that $e, f \in F_{\text{ext}}$, hence we may assume $e, f \in P_{st}$. By Claim 3.5 and equation (3.4) associated with P_{st}, we have $0 < \bar{x}_e < 1$ and $0 < \bar{x}_f < 1$. Let $\tilde{x}$ be the vector obtained from $\bar{x}$ by adding ϵ to $\bar{x}_e$ and $-\epsilon$ to $\bar{x}_f$ for some positive scalar ϵ. If ϵ is small enough, then $\tilde{x}$ belongs to P_G. As $\tilde{x}$ satisfies with equality all the inequalities (3.1)–(3.4) which are tight for $\bar{x}$, it contradicts the extremality of $\bar{x}$. ☐

Since G is a simple nontrivial 2-connected outerplanar graph, Lemma 2.1 and Claim 3.6 imply that s and t have degree two. Moreover, $st \notin E$ as otherwise G/st would have only one vertex of degree 2 – a contradiction to Lemma 2.1. Let $p = |E \setminus F_{\text{ext}}|$. Then, F_{ext} has p chords and G has $p+1$ inner faces. Number the faces in such a way that F_1 is the face containing s and $|F_i \cap F_{i+1}| = 1$ for $i = 1, \ldots, p$. Then, F_{p+1} contains t. Moreover, F_1 and F_{p+1} are triangles and every inner face has at least one edge on the external face. Denote the chords of F_{ext} by $c_1, \ldots, c_p$ such that $\{c_i\} = F_i \cap F_{i+1}$, for $i = 1, \ldots, p$ – see Figure 1 for an illustration.

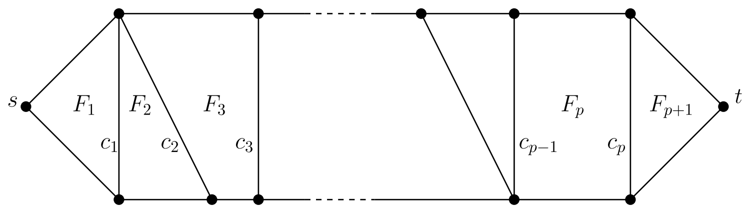

FIGURE 1. Numbered inner faces and chords.

Now, inequalities (3.2) can be rewritten as:

$$x_{c_j} \leq x(F_j \setminus c_j), \qquad \text{for } j = 1\ldots,p, \tag{3.7}$$

$$x_{c_{j-1}} \leq x(F_j \setminus c_{j-1}), \qquad \text{for } j = 2,\ldots,p+1. \tag{3.8}$$

We have the following property.

Claim 3.7. *For each $j = 2,\ldots,p$, at most one inequality among (3.7) and (3.8) associated with j is tight for $\bar{x}$.*

Proof. By contradiction, suppose there exists $j \in \{2,\ldots,p\}$ such that $\bar{x}_{c_j} = \bar{x}(F_j \setminus c_j)$ and $\bar{x}_{c_{j-1}} = \bar{x}(F_j \setminus c_{j-1})$. Then $\bar{x}_e = 0$ for all $e \in F_j \setminus \{c_{j-1}, c_j\}$. Since F_j contains at least an edge of the external face, this contradicts Claim 3.5. □

Let k be the maximum integer such that inequality (3.7) is satisfied with equality for every $j = 1,\ldots,k$. Set $k = 0$ if inequality (3.7) associated with $j = 1$ is not tight. Similarly, let ℓ be the minimum number such that inequality (3.8) is satisfied with equality for every $j = \ell,\ldots,p+1$. By Claim 3.7, we have $k < \ell$.

Claim 3.8. *For $j \in \{1,\ldots,k\} \cup \{\ell-1,\ldots,p\}$, we have $\bar{x}_{c_j} > 0$ and c_j belongs to no tight path.*

Proof. Since every inner face intersects the external face, the first part stems from Claim 3.5 and the tightness of inequalities (3.7) for $j = 1,\ldots,k$ and (3.8) for $j = \ell,\ldots,p+1$. We now prove the second part of the assertion. Suppose it is not true for some $j \in \{1,\ldots,k\}$. Let $1 \leq j' \leq k$ be the minimum index such that $c_{j'}$ is in a tight path P. If $j' = 1$, then $1 = \bar{x}(P) = \bar{x}(P \setminus c_1) + \bar{x}(F_1 \setminus c_1)$ because $\bar{x}_{c_1} = \bar{x}(F_1 \setminus c_1)$. Since $(P \cup F_1) \setminus c_1$ is the disjoint union of an st-path and an edge of the external face, this contradicts (3.3) by Claim 3.5. Thus, $j' \geq 2$. But now, $1 = \bar{x}(P) = \bar{x}(P \setminus c_{j'}) + \bar{x}(F_{j'} \setminus c_{j'}) \geq \bar{x}(P') \geq 1$, where $P' \subseteq P \cup F_{j'} \setminus c_{j'}$ is an st-path containing $c_{j'-1}$. This implies that $c_{j'-1}$ is in a tight path – a contradiction to the minimality of j'. The case $j \in \{\ell-1,\ldots,p\}$ can be proved similarly. □

Let f_s be an edge of $F_{k+1} \cap F_{\text{ext}}$. Without loss of generality, we assume $f_s \in P_{st}$. Let X_s be the set of vertices of the component of P_{st} containing s after the removal of f_s. Note that $\delta(X_s) = \{c_1,\ldots,c_k,f_s,g_s\}$, where g_s is the edge incident to s which does not belong to P_{st}. By construction of X_s and by Claim 3.8, every tight path of G intersects $\delta(X_s)$ exactly once.

We define a vertex set X_t with respect to t in a similar way. Then, $\delta(X_t) = \{c_{\ell-1},\ldots,c_p,f_t,g_t\}$, where $f_t \in F_{\ell-1} \cap F_{\text{ext}}$ and $g_t \in \delta(t)$. Moreover, every tight path of G intersects $\delta(X_t)$ exactly once. We refer the reader to Figure 2 for an example of X_s and X_t.

Given an edge set $F \subseteq E$, we denote by χ^F its incidence vector. Let $z = \chi^{\delta(X_s)} - \chi^{\delta(X_t)}$. By construction, $z(P) = 0$ for all tight paths. Furthermore, all the inequalities (3.7) except the one associated with $j = k+1$ and all the inequalities (3.8) are satisfied with equality by $\chi^{\delta(X_s)}$. Moreover, all the inequalities (3.8) except the one associated with $j = \ell-1$ and all the inequalities (3.7) are tight for $\chi^{\delta(X_t)}$. Hence, z satisfies with equality all the inequalities (3.7) and (3.8) except two: the inequality (3.7) associated with $k+1$ and the inequality (3.8) associated with $\ell-1$. Since none of these two inequalities is tight for $\bar{x}$, and since $\bar{x}_e > 0$ for all

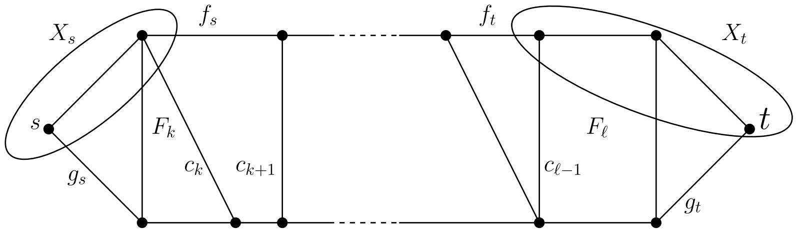

FIGURE 2. Example of X_s and X_t.

$e \in \delta(X_s) \cup \delta(X_t)$ by Claims 3.5 and 3.8, the two points $\bar{x} \pm \epsilon z$ belong to $P(G)$ for some $\epsilon > 0$ small enough. As $z \neq \mathbf{0}$, this contradicts the extremality of $\bar{x}$. □

4. THE st-BOND POLYTOPE ON SERIES-PARALLEL GRAPHS

We start this section by describing the st-bond polytope on 2-connected series-parallel graphs, see Section 4.1. We then extend this description to general series-parallel graphs, see Section 4.2. We conclude by showing that for these graphs, the st-bond polytope is actually the intersection between the bond polytope and the st-cut dominant, see Section 4.3. Recall that $B_{st}(G)$ denotes the convex hull of the incidence vectors of the st-bonds of G. We define $B(G)$ as the convex hull of the incidence vectors of the bonds of G.

4.1. 2-Connected series-parallel graphs

An ear of an open nested ear decomposition $\mathcal{E}$ whose circuit $C_{\mathcal{E}}$ is an st-circuit is called an *st-ear*.

Lemma 4.1. *Given a 2-connected series-parallel graph $G = (V, E)$ and distinct vertices s and t, the st-bond polytope $B_{st}(G)$ is the set of $x \in \mathbb{R}^E$ satisfying the following inequalities.*

$$x_e \geq 0, \quad \text{for all } e \in E, \tag{4.1}$$
$$x(Q) \leq x(C \setminus Q), \quad \text{for all } st\text{-ears } Q \text{ and circuits } C \supseteq Q, \tag{4.2}$$
$$x(P) \geq 1, \quad \text{for all } st\text{-paths } P, \tag{4.3}$$
$$x(P) = 1, \quad \text{for all } st\text{-paths } P \text{ contained in a circuit}, \tag{4.4}$$
$$x_{st} = 1, \quad \text{whenever } st \in E. \tag{4.5}$$

Proof. The result holds if G is trivial because $B_{st}(G) = \{x_{st} = 1\}$. Suppose now that G is nontrivial. Equation (4.5) can then be removed because if st is an edge of G, then it is an st-path of G contained in a circuit and the equation is of type (4.4). The result also holds when G is the circuit of length two, say $\{e, f\}$, since $B_{st}(G) = \{x_e = x_f = 1\}$.

Let us prove the result by induction on the number of edges of G. Assume that the bond polytope for every 2-connected series-parallel graph with less edges than G is given by (4.1)–(4.4).

First, suppose that G has two parallel edges g and h. By induction, $B_{st}(G \setminus h)$ is given by (4.1)–(4.4). Remark that $x_g = x_h$ for all $x \in B_{st}(G)$ as every bond contains either none of g and h or both of them. This equation is given in (4.1)–(4.4) for G by the two inequalities (4.2) associated with the circuit $C = \{g, h\}$. Moreover, for any other inequality $ax \leq b$, at most one among a_g and a_h is nonzero. Finally, the system (4.1)–(4.4) is such that for every inequality $ax \leq b$, there exists another $a'x \leq b$ where a' is obtained from a by exchanging a_g and a_h. These remarks imply that $B_{st}(G)$ is described by (4.1)–(4.4).

Assume now that G has no parallel edge. By Lemma 2.1, G has at least two vertices of degree two. Suppose that there are exactly two such vertices. Then G is outerplanar by Lemma 2.2. In this case, in the open nested ear decomposition $\mathcal{E}$ with $C_{\mathcal{E}} = F_{\text{ext}}$, the st-ears are precisely the chords of the graph. Thus, inequalities (4.2) contain inequalities (3.2). We now prove the validity of (4.2) for $B_{st}(G)$. Let Q be any st-ear and let $\mathcal{E}$ be an open nested ear decomposition containing Q such that $C_{\mathcal{E}}$ is an st-circuit. For each $e \in C_{\mathcal{E}}$, since G is a 2-connected series-parallel graph, there exists a circuit containing both e and Q. By Observation 2.3, a bond intersecting twice Q does not intersect $C_{\mathcal{E}}$, which implies that it is not an st-bond. Observation 2.3 gives the validity of (4.2). Therefore, the result follows from Lemma 3.4.

The last case to consider is when G has at least three vertices of degree two. Then, one of them, say v, is different from s and t. Let f and g be the two edges incident to v and H be the 2-connected series-parallel graph obtained from G by replacing the path $\{f, g\}$ by a single edge e. By the induction hypothesis, $B_{st}(H)$ is described by (4.1)–(4.4). Let Q be the polytope obtained by replacing x_e by $x_f + x_g$ in $B_{st}(H)$, and adding $x_f \geq 0$ and $x_g \geq 0$. This operation preserves integrality. Indeed, let $\bar{z}$ be a vertex of Q. At least one of $\bar{z}_f$ and $\bar{z}_g$ equals zero, as otherwise the points $\bar{z} \pm \epsilon(\chi^f - \chi^g)$ both belong to Q for some $\epsilon > 0$. Without loss of generality, assume that $\bar{z}_g = 0$. Then, the point $\bar{x}$ defined by $\bar{x}_h = \bar{z}_h$ for $h \neq e$ and $\bar{x}_e = \bar{z}_f$ belongs to $B_{st}(H)$. By definition of Q and since $x_e \geq 0$ is an inequality of $B_{st}(H)$, $\bar{x}$ is a vertex of $B_{st}(H)$. As $B_{st}(H)$ is an integer polytope, $\bar{x}$ is integer and so is $\bar{z}$. This implies that Q is an integer polytope.

Furthermore, the st-bonds of G are obtained from those of H as follows: keep the st-bonds of H not containing e, and for every st-bond B of H containing e, take $B \setminus e \cup f$ and $B \setminus e \cup g$. This implies that $Q = B_{st}(G)$. Given the form of inequalities (4.1)–(4.4), this proves the result. □

4.2. General case

For the rest of the paper, $G = (V, E)$ will denote a series-parallel graph and W the set of st-cut vertices of G together with s and t. Recall that the vertices of W may be ordered according to the order induced by running any st-path from s to t. This order is denoted by $s = v_1, v_2, \dots, v_{|W|} = t$. Let $k = |W| - 1$ and ℓ be the number of 2-connected components of G. Let $H_1 = (V_1, E_1), \dots, H_\ell = (V_\ell, E_\ell)$ be the 2-connected components of G numbered in such a way that, for $i = 1, \dots, k$, $H_i = (V_i, E_i)$ contains v_i and v_{i+1}. Observation 2.5 may then be translated in terms of polytopes as follows.

Observation 4.2. $B_{st}(G) = conv\left(\bigcup_{i=1}^{k} B_{v_i v_{i+1}}(H_i)\right)$.

The following observation stems from the definition of the 2-connected components $H_1, \dots, H_k$.

Observation 4.3. *Every st-path of G is the union of $v_i v_{i+1}$-paths of H_i over $i = 1, \dots, k$, and conversely.*

We now provide a polyhedral description of the st-bond polytope on general series-parallel graphs, see Theorem 4.4. The result is a consequence of Lemma 4.1, Observations 4.2 and 4.3, and the theorem of Balas [1, 2] on the union of polyhedra.

Theorem 4.4. *Let $G = (V, E)$ be a series-parallel graph and s and t be distinct vertices of V. The st-bond polytope $B_{st}(G)$ is the set of $x \in \mathbb{R}^E$ satisfying the following inequalities.*

$$x_e = 0, \quad \text{for all } e \text{ which belongs to no } st\text{-path}, \tag{4.6}$$

$$x_e \geq 0, \quad \text{for all } e \in E, \tag{4.7}$$

$$x(Q) \leq x(C \setminus Q), \quad \text{for all } v_i v_{i+1}\text{-ears } Q \text{ and circuits } C \supseteq Q \text{ of } H_i,\ i = 1, \dots, k, \tag{4.8}$$

$$x(P) \geq 1, \quad \text{for all } st\text{-paths } P \text{ of } G, \tag{4.9}$$

$$x(P) \leq 1, \quad \text{for all } st\text{-paths } P \text{ of } G \text{ such that } P|_{H_i} \text{ is contained in a circuit for all } i = 1, \dots, k \text{ with } H_i \text{ nontrivial.} \tag{4.10}$$

Proof. First, note that every edge of an st-bond belongs to an st-path, which implies the validity of inequalities (4.6). Moreover, the edges involved in (4.6) are precisely those which belong to $H_{k+1}, \dots, H_\ell$. From now on, we may suppose that the only 2-connected components of G are $H_1, \dots, H_k$.

Remark that in Lemma 4.1, equalities (4.4) and (4.5) can be replaced by inequalities of type $\leq$ because of inequalities (4.3). Moreover, (4.5) is taken into account only for trivial 2-connected components as noticed in the proof of Lemma 4.1. Since the polytopes of the right-hand side of Observation 4.2 live in different spaces, by the theorem of Balas [1, 2], $B_{st}(G)$ is the projection into the x-space of the set of $x = (x^1, \dots, x^k) \in \mathbb{R}^E$ (where $x^i \in \mathbb{R}^{E_i}$ for $i = 1, \dots, k$) and $\lambda \in \mathbb{R}^k$ satisfying:

$$x^i_e \geq 0, \qquad \text{for all } i = 1, \dots, k, \text{ for all } e \in E_i, \tag{4.11}$$

$$x^i(Q) - x^i(C \setminus Q), \leq 0 \qquad \text{for all } i = 1, \dots, k, \text{ for all } v_i v_{i+1}\text{-ears } Q \text{ and circuits } C \supseteq Q \text{ of } H_i, \tag{4.12}$$

$$-x^i(P) \leq -\lambda^i, \qquad \text{for all } i = 1, \dots, k, \text{ for all } v_i v_{i+1}\text{-paths } P \text{ of } H_i, \tag{4.13}$$

$$x^i(P) \leq \lambda^i, \qquad \text{for all } i = 1, \dots, k, \text{ for all } v_i v_{i+1}\text{-paths } P \text{ of } H_i \text{ contained in a circuit}, \tag{4.14}$$

$$x^i_{v_i v_{i+1}} \leq \lambda^i, \qquad \text{for all trivial } H_i, \tag{4.15}$$

$$0 \leq \lambda^i, \qquad \text{for all } i = 1, \dots, k, \tag{4.16}$$

$$1 = \sum_{i=1}^{k} \lambda^i. \tag{4.17}$$

The description of $B_{st}(G)$ in the natural space is obtained by projecting out λ from (4.11)–(4.17). Since (4.7)–(4.10) are valid for $B_{st}(G)$, we prove that the projected inequalities are either contained in or implied by (4.6)–(4.10), which implies our theorem.

We project out λ applying Fourier-Motzkin's elimination method [12]. Recall that, to get rid of λ^i, one has to combine every inequality where λ^i's coefficient is negative with every inequality where it is positive. Due to the structure of (4.11)–(4.17), it is enough to consider the following combinations.

Inequalities (4.11) and (4.12) have to be kept in the projection and are nothing but (4.7) and (4.8) respectively. The inequalities obtained by combination with (4.17) are of two forms. First, due to Observation 4.3, combining (4.17) and (4.13) for $i = 1, \dots, k$ yields precisely inequalites (4.9). Second, by Observation 4.3, the combination of (4.17) with, for every $i = 1, \dots, k$, an inequality of (4.14) if H_i is nontrivial and (4.15) otherwise, gives precisely (4.10). Note that, in the previous combination, replacing some inequalities of (4.14) or (4.15) by (4.16) provides redundant inequalities.

The last possible combinations are those involving inequalities containing only λ^i, for some $i \in \{1, \dots, k\}$. It gives $x^i \geq 0$, $x^i(P) \geq 0$ and $x^i(P) - x^i(P') \leq 0$ for all $v_i v_{i+1}$-paths P and P' of H_i such that P is contained in a circuit. These are redundant with respect to (4.7) and (4.9)–(4.10). □

4.3. Intersection property

The *dominant* of a polyhedron P of $\mathbb{R}^n$ is $P^+ = \{x \in \mathbb{R}^n : x \geq y \text{ for some } y \in P\}$. In this section, we show that intersecting the bond polytope and the *st*-cut dominant of a series parallel graph preserves integrality. Indeed, using Theorem 4.4 and the descriptions of the bond polytope [5] and the *st*-cut dominant [18], we prove that this intersection is nothing but the *st*-bond polytope.

We first state the result of [5] describing the bond polytope of a series-parallel graph, for which we need a few definitions. Two collections of edge subsets $\mathcal{F} = \{F_0, F_1, \dots, F_q\}$ and $\mathcal{P} = \{P_1, \dots, P_q\}$ form an *ear-cycle collection* if $\mathcal{F}$ is contained in an open nested ear decomposition $\mathcal{E}$ of G, $F_0 = C_{\mathcal{E}}$, and $F_i \Delta P_i$ is a cycle for $i = 1, \dots, q$.

Theorem 4.5 (Borne *et al.* [5])**.** *Let G be a series-parallel graph, $G_1, \dots, G_p$ its nontrivial 2-connected components and $\mathfrak{B}$ its set of bridges. The bond polytope of G is the set of $x \in \mathbb{R}^E$ satisfying the following* *inequalities.*

$$\begin{aligned} & x_e \geq 0, && \textit{for all } e \in E, && (4.18)\\ & x_e \leq x(C \setminus e), && \textit{for all circuits } C, \textit{ for all } e \in C, && (4.19)\\ & \sum_{i=1}^{p}(x(\mathcal{F}_i) - x(\mathcal{P}_i)) + 2x(\mathfrak{B}) \leq 2, && \textit{for all } i = 1,\dots,p, \textit{ for all ear-cycle collections } \mathcal{F}_i, \mathcal{P}_i \textit{ of } G_i. && (4.20)\end{aligned}$$

The st-cut dominant $B_{st}(G)^+$ of G is described as follows, see *e.g.* [18]:

$$B_{st}(G)^+ = \{x \in \mathbb{R}_+^E : x(P) \geq 1 \text{ for all } st\text{-paths } P\}.$$

An st-bond being both an st-cut and a bond, we get the inclusion $B_{st}(G) \subseteq B_{st}(G)^+ \cap B(G)$. It turns out that the converse holds for series-parallel graphs, see below.

Corollary 4.6. *If G is series-parallel, then $B_{st}(G) = B_{st}(G)^+ \cap B(G)$.*

Proof. To prove $B_{st}(G)^+ \cap B(G) \subseteq B_{st}(G)$, by Theorem 4.4, we show that (4.6)–(4.10) are valid for $B_{st}(G)^+ \cap B(G)$. Note that $B_{st}(G)^+$ is the set of $x \in \mathbb{R}^E$ satisfying (4.7) and (4.9), as mentioned above. Recall that the 2-connected components $H_1, \dots, H_\ell$ of G are numbered in such a way that, for $i = 1, \dots, k$, H_i contains both v_i and v_{i+1}. Note that $G_1, \dots, G_p$ of Theorem 4.5 correspond to the nontrivial 2-connected components among $H_1, \dots, H_\ell$.

Let P be any st-path associated with (4.10). By definition of P and Observation 4.3, there exists an st-path P' such that the restriction C_i of $P \cup P'$ to H_i is a v_iv_{i+1}-circuit, for $i = 1, \dots, k$ with H_i nontrivial. For $i = k+1, \dots, \ell$ such that H_i is nontrivial, pick a circuit C_i of H_i. Since every circuit is contained in some open nested ear decomposition, $\{C_i\}, \emptyset$ is an ear-cycle collection of H_i for $i = 1, \dots, \ell$ such that H_i is nontrivial. Let $\mathfrak{B}_H$ denote the edges of the trivial H_i's. Now, applying (4.9) for P and P', the definition of C_i and $\mathfrak{B}_H$, (4.7), and finally (4.20), yields

$$1 + 1 \leq x(P) + x(P') = \sum_{\substack{i=1 \\ \text{nontrivial } H_i}}^{k} x(C_i) + 2x(\mathfrak{B}_H) \leq \sum_{i=1}^{p} x(C_i) + 2x(\mathfrak{B}) \leq 2. \tag{4.21}$$

Hence, there is equality everywhere and the validity of (4.10) follows. Moreover, since an edge e which belongs to no st-path is either in $\mathfrak{B} \setminus \mathfrak{B}_H$ or in some circuit of a nontrivial H_j with $j > k$, we also get (4.6).

Finally, we prove the validity of (4.8). Let $j \in \{1, \dots, k\}$, Q be a v_jv_{j+1}-ear of H_j, and C be a circuit of H_j containing Q. By definition, there exists an open nested ear decomposition $\mathcal{E}$ of H_j containing Q such that $C_\mathcal{E}$ is a v_jv_{j+1}-circuit. By (4.21), we have

$$\sum_{i=1}^{p} x(C_i) + 2x(\mathfrak{B}) = 2 \tag{4.22}$$

for $C_j = C_\mathcal{E}$ and C_i, $i \neq j$, is a circuit defined as in (4.21).

Note that $\{C_\mathcal{E}, Q\}, \{C \setminus Q\}$ is an ear-cycle collection of H_j. Inequality (4.20) associated with $\mathcal{F}_j$, $\mathcal{P}_j = \{C_\mathcal{E}, Q\}, \{C \setminus Q\}$ and $\mathcal{F}_i, \mathcal{P}_i = \{C_i\}, \emptyset$ for $i \neq j$ is

$$\sum_{i=1}^{p} x(C_i) + x(Q) - x(C \setminus Q) + 2x(\mathfrak{B}) \leq 2. \tag{4.23}$$

Subtracting equation (4.22) to inequality (4.23) gives (4.8). □

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The problem of finding an *st*-bond arises in the context of partitioning a smart grid into areas having autonomy requirements. One of these requirements is to ensure the connectivity of each area for local energy transportation. Hence, partitioning a smart grid into two such areas reduces to finding an *st*-bond with additional properties.

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