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BI-OBJECTIVE OPTIMIZATION MODELS FOR NETWORK INTERDICTION

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Abstract. This paper designs models for the network interdiction problem. The interdiction problem under study has two contradicting goals: disrupting the network to minimize the profit of one set of agents, while as much as possible preserve the profit of another set of agents. Three bi-objective optimization methods are employed to form the optimal objectives. Also, we develop two formulations (MILP and multi-stage LP) used to deal with congestion cost which is a piecewise cost function. A numerical instance is also presented to better illustrate those models.

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1. Introduction

Network systems are currently playing an essential role in many practical problems, including power systems, supply chain, military and so forth. In real world, the network faces many kinds of disruptions. Imagine a military operation aims to destroy the enemy's supply system, for an example. Network interdiction studies the problem when the arcs of the network is disrupted by a *disrupter*. In the most basic form of the interdiction problem, the process can be viewed as a game, where the disrupter aims to minimize the profit of a profit-maximizing agent, by eliminate a limited number of arcs. A more complex interdiction problem, such as the one studied in this paper, includes more generalizations. In our case, the disrupter faces a trade-off between disrupting some agents and protecting other agents who use the same network system.

Wood [14] used a min-max formulation to model the basic form of interdiction problem mentioned above. The paper also studied generalizations to partial arc interdiction, and to multiple sources and sinks. The models in this paper are what we referred to when forming the simple model of interdiction in Section 3.

Contreras *et al.* [8] and Snyder *et al.* [10] included cost in the model and considered a capacitated network generalization. They described the supply chain network interdiction as follows: Given a network, the transportation costs and congestion costs constitutes the total expense in the system. And resources transported into each node satisfying the demands account for overall profit. Wood (2011) and Bertsimas *et al.* [3] viewed network interdiction as a game, in which the disrupter moves first and the disrupted profiting agents optimize accordingly. Such a viewpoint is helpful for our formulation in Section 4, where the introduction of the protected agents makes the problem more complicated.

Keywords. Bi-objective optimization, interdiction, network flows.

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Our contribution is that we modelled the interdiction problem with contradicting goals. To our best of knowledge, most papers on network interdiction have a disrupter with one single goal: to minimize the profit of the profit-maximizing agent. However, in practice the disrupter may be restrained in the destruction of network, if they want to protect the benefits of other agents who use the same network. For example, an army cannot destroy all the roads in the enemy's area, because they may need to use some of those roads later themselves. A trade-off in such circumstances is what we are interested in. Also, an arc can be partially disrupted in our model, which is a generalization to the requirement that an arc should be totally destroyed in some literature.

In Section 3, we propose a minimize-maximize linear programming to solve a simple problem, where the disrupting agent aims to minimize the profit of a single profit-maximizing agent. In Section 4, we model the main problem with three different methods. Those models show how bi-objective optimization is used to formulate contradicting goals of the disrupter. Besides, two kinds of constraints are formulated to deal with piecewise cost and fees. An instance and some analysis are attached in Section 5 to illustrate the use of the models.

2. Problem description

We consider the following optimization problem: There is a set of agents which is denoted by J. They transport resources from production centers to demand centers *via* a transportation network to generate profits. There is a constraint in the capacity of each link. A disrupting agent wants to minimize the profit of a subset of J_- agents in J, at the same time preserve as much as possible the profit of a subset of J_+ agents in J. And $J_+ \cup J_- = J$, $J_+ \cap J_- = \emptyset$. The disrupting agent attains its aim by reducing or disabling the capacity of links. However, there is a budget as to how much in total it can affect the capacity.

The transportation network is described by the graph $G(V, E)$ where V is a set of nodes indexed by n and $E \subset V \times V$ is a set of links indexed by l. The agents in J are indexed by j. Denote the source nodes by $S_j \in V$ and the sink nodes by $T_j \in V$. Demand is defined at each node n and for each agent j by d_{nj}, where $d_{nj} > 0$ if $n \in T_j$ and 0 otherwise. Demand can be fully satisfied or just in part. The price of the resource that is sold by agent j at node n is P_{nj} per unit, such that $P_{nj} > 0$ if $n \in T_j$ and 0 otherwise. The agents get their resource at the source nodes in S_j, and there is no cost or limit on quantity. The agent pays base transportation cost along each link, which is linear to the quantity transited on the link. The cost on link l is $b_l > 0$ per unit of resource. The upper bound of capacity on link l is w_l, which is the total limit of resources transported in both directions (all the links are bidirectional). There is a congestion fee on each link, which is levied when the capacity of the link is not enough to satisfy the needs of all agents. The congestion fee is linear to the quantity of resources transported by each agent and equals to the minimal number that can keep the total number of transported resources exactly at link capacity.

A Simple Model with One Profit-Maximizing Agent. (Sect. 3) is to consider the disruption of a single profit-maximizing agent. In this case, our goal is to reduce the profit of the regular agent maximally.

Disruption with Contradicting Goals. (Sect. 4) considers a disruption problem with bi-objective programming. We model the problem with three designs of objective functions and two formulations of constraints.

3. A simple model with one profit-maximizing agent

We formulate a basic model where there is one profit-maximizing agent, and one disrupter. The disrupter aims to minimize the profit of this agent by decreasing the transportation capacity of the edges in the network.

3.1. Notations

The notations used in the formulation of the single agent model is summarized as follows:
Sets:
V: the set of nodes in the network.

E: the set of edges in the network.
T: the set of nodes with positive demands.
S: the set of source node.
Parameters:
p_i: the price of unit resource in the demand node $i \in V$.
d_i: the demand at node $i \in V$.
w^*_{ij}: the capacity of $(i,j) \in E$ before disruption.
b_{ij}: the unit transportation fee in $(i,j) \in E$.
β: the percentage of the total transportation capacity that can be disrupted.
Variables:
x_{ij}: the amount of resources that are transported in $(i,j) \in E$.
w_{ij}: the amount of actual transportation capacity in $(i,j) \in E$ after disruption.
z_i: the demand that is satisfied at node $i \in E$.

3.2. Model formulation

We formulate disruption problem with single disruptive agent as follows:

$$
\begin{aligned}
\min_{w_{ij}} \max_{x_{ij}} \quad & \sum_{i \in T} p_i z_i - \sum_{(i,j)\in E} b_{ij} x_{ij} \\
s.t. \quad & \sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = 0, \forall i \in V/(S \cup T), \\
& \sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = z_i, \forall i \in T, \\
& \sum_{(i,j)\in E} w_{ij} \geq (1-\beta) \sum_{(i,j)\in E} w^*_{ij}, \\
& 0 \leq z_i \leq d_i, \forall i \in T, \\
& 0 \leq w_{ij} \leq w^*_{ij}, \forall (i,j) \in E, \\
& 0 \leq x_{ij} \leq w_{ij}, \forall (i,j) \in E.
\end{aligned}
\tag{3.1}
$$

This is the single objective formulation for solving One-Profit-Maximizing Agent. An important assumption is that the capacity in each arc is limited to one-way flow which means we require $x_{ij} \leq w_{ij}$ for each $(i,j) \in E$. The minimize-maximize objective function, which was motivated from [6, 12, 13], is built for optimal network design for the single agent. The objective is to firstly maximize the profit of the agent by operating flow x_{ij}, and then minimize the profit in respect to disrupted capacity. The overall profits are explicitly measured by income from selling resources minus transportation costs. By considering the inner maximum sub-optimization problem, we obtain the maximum profits without disruptions. Then we minimize the maximized profit by setting disruption variables.

The first two constraints are flow conservation. The third constraint ensures the total disrupted capacity is no more than β of the original capacity. Since the demand can be partially satisfied, the net inflow at node $i \in T$, z_i, should be no more than d_i. The model formulation is a *Linear Programming* directly solved by Gurobi Solver after we use *epi-graph* formulation by letting a new variable t equal to inner maximized objective and then transforming the inner maximization term into a new constraint.

4. Disruption with contradicting goals

In this section, we consider a model with two sets of agents. The goal of the disrupter is twofold: (a) to reduce the total profit of the agents in set J_-, while (b) protecting the total profit of the agents in J_+. What makes the problem difficult is that the two sets of agents share the same transportation facility. When the disrupter reduces the capacity of some edges, all the agents are suffered. Thus, the two goals of the disrupter are contradicting. Also, there is a congestion fee added to the problem, which is set at the minimum price to keep the use of the edge at its capacity.

To simplify the problem, assume there is only one agent in each of set J_+ and set J_-. The disruption problem is now considered as a bi-objective optimization problem. We formulate the objective with three different methods and formulate the constraints with two methods.

4.1. Notations

Sets:
V: the set of nodes in the network.
E: the set of edges in the network.
T_k: the set of nodes with positive demands. k=1 means for agent in J_+, k=2 for agent in J_- (same rule applied for the k's in notations below).
S_k: the set of source node.
Parameters:
p_i: the price of unit resource in the demand node $i \in V$.
d_i^k: the demand at node $i \in V$.
w_{ij}^*: the capacity of $(i,j) \in E$ before disruption.
b_{ij}: the unit transportation fee in $(i,j) \in E$.
β: the percentage of the total transportation capacity that can be disrupted.
Variables:
x_{ij}: the amount of resources that agent in set J_+ transports in $(i,j) \in E$.
y_{ij}: the amount of resources that agent in set J_- transports in $(i,j) \in E$.
f^*: the optimal profit of agent in set J_+, before disruption.
w_{ij}: the amount of transportation capacity in $(i,j) \in E$ after disruption.
z_i^k: the demand that is satisfied at node $i \in E$.
β_{ij}^k: the congestion fee levied for $(i,j) \in E$.
γ_{ij}^k: the transportation cost on $(i,j) \in E$.
t_k: the total revenue of an agent.

4.2. Objective designs

We discuss three ways to formulate the objective function of the bi-objective optimization with disruption:
(A). The two-stage optimization model with a penalty: the first step is to solve the single maximum objective to obtain the optimal profit of the agent in J_+ without any disruption (which is denoted by f^*). The objective can be written as

$$\begin{aligned} \min \quad & \lambda||f^* - t_1|| + \max t_2 \\ \text{s.t.} \quad & t_1 = \sum_{i\in T} p_i z_i^1, \\ & t_2 = \sum_{i\in T} p_i z_i^2. \end{aligned} \tag{4.1}$$

Where λ is a coefficient that shows a trade-off between two objectives. When *lambda* increases, the disrupter puts more emphasis on preserving the profit of agent in J_+. Thus, it will be less able to decrease the profit of agent in J_+.

The expression $||f^* - t_1||$ basically means the difference between the optimal profit and the actual profit (after disruption) for agent in J_+. The disrupter wants that difference as small as possible.

And f^* is the optimal value of the following problem (for now we only consider the revenue and ignore the costs and fees):

$$\begin{aligned}
\max \quad & \sum_{i \in T} p_i z_i^1 \\
\text{s.t.} \quad & \sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = 0, \forall i \in V/(S_1 \cup T_1), \\
& \sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = z_i^1, \forall i \in T_1, \\
& 0 \le z_i \le d_i, \forall i \in T_1, \\
& 0 \le x_{ij} \le w_{ij}^*, \forall (i,j) \in E.
\end{aligned} \tag{4.2}$$

The optimization of the profit for agent in J_+ is similar to the formulation of simple model in Section 3, thus is easy to understand.

(B). The second method is to seek the Pareto optimal by simply scalarizing the bi-objective optimization:

$$\begin{aligned}
\max \quad & \lambda_1 \min t_1 - \lambda_2 t_2 \\
\text{s.t.} \quad & t_1 = \sum_{i \in T} p_i z_i^1, \\
& t_2 = \sum_{i \in T} p_i z_i^2
\end{aligned} \tag{4.3}$$

where $\lambda_1 + \lambda_2 = 1, \lambda_1 \ge 0, \lambda_2 \ge 0$. The convex composition of two goals can get the Pareto optimal solution.

(C). We can also use the ε-constraints method [4] to formulate the bi-objective problem. In order to protect the profit of agent J_+, the ε-constraints is used to control the gap between the ideal profits and the actual profits. The objective function is as follows:

$$\begin{aligned}
& \min \max t_2 \\
\text{s.t.} \quad & ||f^* - t_1|| \le \varepsilon_0, \\
& t_1 = \sum_{i \in T} p_i z_i^1, \\
& t_2 = \sum_{i \in T} p_i z_i^2
\end{aligned} \tag{4.4}$$

where f^* has the same definition as that in the first objective design, and ε_0 is the ε of the ε-constraint method. The optimal solutions can be controlled by the trade-off of two objectives, by adjusting the ε_0. Note that the optimization problem may become infeasible for a sufficiently small ε_0.

4.3. Constraints and formulation

Now we introduce the transportation costs, congestion fees and the constraint about disruptions to the formulation. Since the costs and congestion fees are represented as piecewise functions, we employ two methods to deal with them: one in mixed integer linear programming (MILP), another in multi-stage linear programming.

4.3.1. Mixed integer linear programming method

The congestion fee for agent in J_+ can be written as:

$$\beta_{ij}^1 = \begin{cases} b_{ij} x_{ij} & x_{ij} + y_{ij} > w_{ij}, \\ 0 & x_{ij} + y_{ij} \le w_{ij}. \end{cases}$$

And we compute costs by:

$$\gamma_{ij}^1 = \begin{cases} b_{ij}w_{ij} & x_{ij}+y_{ij} > w_{ij}, \\ b_{ij}x_{ij} & x_{ij}+y_{ij} \le w_{ij}. \end{cases}$$

By changing the subscriptions of $beta_{ij}^1$ and γ_{ij}^1 to 2, we get the expressions for agnet in J_-.

Adding the costs and fees, we can reformulate the three objective designs (in Sect. 4.1) as:

(A). The first formulation:

$$\begin{aligned} \min \quad & \lambda \left\| f^* - \left(t_1 - \sum_{(i,j)\in E} \beta_{ij}^1 - \sum_{(i,j)\in E} \gamma_{ij}^1 \right) \right\| + \max \left(t_2 - \sum_{(i,j)\in E} \beta_{ij}^2 - \sum_{(i,j)\in E} \gamma_{ij}^2 \right) \\ \text{s.t.} \quad & t_1 = \sum_{i\in T_1} p_i z_i^1, \\ & t_2 = \sum_{i\in T_2} p_i z_i^2. \end{aligned} \tag{4.5}$$

And f^* is the optimal value of the following:

$$\begin{aligned} \max \quad & \sum_{i\in T} p_i z_i^1 - \sum_{(i,j)\in E} \beta_{ij}^1 - \sum_{(i,j)\in E} \gamma_{ij}^1 \\ \text{s.t.} \quad & \sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = 0, \forall i \in V/(S_1 \cup T_1), \\ & \sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = z_i^1, \forall i \in T_1, \\ & 0 \le z_i \le d_i, \forall i \in T_1, \\ & 0 \le x_{ij} \le w_{ij}^*, \forall (i,j) \in E. \end{aligned} \tag{4.6}$$

(B). The second formulation:

$$\begin{aligned} \max \quad & \lambda_1 \min \left(t_1 - \sum_{(i,j)\in E} \beta_{ij}^1 - \sum_{(i,j)\in E} \gamma_{ij}^1 \right) - \lambda_2 \left(t_2 - \sum_{(i,j)\in E} \beta_{ij}^2 - \sum_{(i,j)\in E} \gamma_{ij}^2 \right) \\ \text{s.t.} \quad & t_1 = \sum_{i\in T_1} p_i z_i^1, \\ & t_2 = \sum_{i\in T_2} p_i z_i^2. \end{aligned} \tag{4.7}$$

(C). The third formulation:

$$\begin{aligned} \min\max \quad & t_2 - \sum_{(i,j)\in E} \beta_{ij}^2 - \sum_{(i,j)\in E} \gamma_{ij}^2 \\ \text{s.t.} \quad & \left\| f^* - (t_1 - \sum_{(i,j)\in E} \beta_{ij}^1 - \sum_{(i,j)\in E} \gamma_{ij}^1) \right\| \le \varepsilon_0, \\ & t_1 = \sum_{i\in T_1} p_i z_i^1, \\ & t_2 = \sum_{i\in T_2} p_i z_i^2. \end{aligned} \tag{4.8}$$

Now we form the constraints. The constraints relevant to all three different objective designs are as follows,

$$\sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = 0, \forall i \in V/(S_1 \cup T_1), \tag{4.9}$$

$$\sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = z_i^1, \forall i \in T_1, \tag{4.10}$$

$$\sum_{j|(j,i)\in E} y_{ji} - \sum_{j|(i,j)\in E} y_{ij} = 0, \forall i \in V/(S_2 \cup T_2), \tag{4.11}$$

$$\sum_{j|(j,i)\in E} y_{ji} - \sum_{j|(i,j)\in E} y_{ij} = z_i^2, \forall i \in T_2, \tag{4.12}$$

$$-M(1-\alpha_{ij}^1) + w_{ij} + \varepsilon \leq x_{ij} + y_{ij}, \forall (i,j) \in E, \tag{4.13}$$

$$-M\alpha_{ij}^1 \leq x_{ij} + y_{ij} \leq w_{ij} + M\alpha_{ij}^1, \forall (i,j) \in E, \tag{4.14}$$

$$-M(1-\alpha_{ij}^2) + w_{ij} + \varepsilon \leq x_{ij}, \forall (i,j) \in E, \tag{4.15}$$

$$-M\alpha_{ij}^2 \leq x_{ij} \leq w_{ij} + M\alpha_{ij}^2, \forall (i,j) \in E, \tag{4.16}$$

$$-M(1-\alpha_{ij}^3) + w_{ij} + \varepsilon \leq y_{ij}, \forall (i,j) \in E, \tag{4.17}$$

$$-M\alpha_{ij}^3 \leq y_{ij} \leq w_{ij} + M\alpha_{ij}^3, \forall (i,j) \in E, \tag{4.18}$$

$$-M(1-\alpha_{ij}^1) + b_{ij}x_{ij} \leq \beta_{ij}^1 \leq b_{ij}x_{ij} + M(1-\alpha_{ij}^1), \forall (i,j) \in E, \tag{4.19}$$

$$-M\alpha_{ij}^1 \leq \beta_{ij}^1 \leq M\alpha_{ij}^1, \forall (i,j) \in E, \tag{4.20}$$

$$-M(1-\alpha_{ij}^1) + b_{ij}y_{ij} \leq \beta_{ij}^2 \leq b_{ij}y_{ij} + M(1-\alpha_{ij}^1), \forall (i,j) \in E, \tag{4.21}$$

$$-M\alpha_{ij}^1 \leq \beta_{ij}^2 \leq M\alpha_{ij}^1, \forall (i,j) \in E, \tag{4.22}$$

$$-M(1-\alpha_{ij}^2) + b_{ij}w_{ij} \leq \gamma_{ij}^1 \leq b_{ij}w_{ij} + M(1-\alpha_{ij}^2), \forall (i,j) \in E, \tag{4.23}$$

$$-M\alpha_{ij}^2 + b_{ij}x_{ij} \leq \gamma_{ij}^1 \leq b_{ij}x_{ij} + M\alpha_{ij}^2, \forall (i,j) \in E, \tag{4.24}$$

$$-M(1-\alpha_{ij}^3) + b_{ij}w_{ij} \leq \gamma_{ij}^2 \leq b_{ij}w_{ij} + M(1-\alpha_{ij}^3), \forall (i,j) \in E, \tag{4.25}$$

$$-M\alpha_{ij}^3 + b_{ij}y_{ij} \leq \gamma_{ij}^2 \leq b_{ij}y_{ij} + M\alpha_{ij}^3, \forall (i,j) \in E, \tag{4.26}$$

$$0 \leq z_i^1 \leq d_i^1, i \in T_1, \tag{4.27}$$

$$0 \leq z_i^2 \leq d_i^2, i \in T_2, \tag{4.28}$$

$$\sum_{(i,j)\in E} w_{ij} = (1-\beta) \sum_{(i,j)\in E} w_{ij}^*, \tag{4.29}$$

$$0 \leq w_{ij} \leq w_{ij}^*, \forall (i,j) \in E, \tag{4.30}$$

$$x_{ij} \geq 0, y_{ij} \geq 0, \alpha_{ij}^k = \{0,1\}, k = 1,2,3, \forall (i,j) \in E, \tag{4.31}$$

$$\beta_{ij}^k \geq 0, \gamma_{ij}^k \geq 0, k = 1,2, \forall (i,j) \in E. \tag{4.32}$$

4.4. Multi-stage linear programming method

Despite the piecewise function shown above, another way to compute congestion fee is:

$$b_{ij} \max\left(x_{ij} + y_{ij} - w_{ij}, 0\right)$$

Then it can be distributed proportionally:

$$\left(\frac{x_{ij}^*}{x_{ij}^* + y_{ij}^*}, \frac{y_{ij}^*}{x_{ij}^* + y_{ij}^*}\right)$$

Denote

$$\xi_{ij} = \max\left(x_{ij} + y_{ij} - w_{ij}, 0\right).$$

Which is equivalent to

$$\begin{aligned} \min \quad & \xi_{ij} \\ \text{s.t.} \quad & \xi_{ij} \geq 0, \\ & \xi_{ij} \geq x_{ij} + y_{ij} - w_{ij}. \end{aligned} \tag{4.33}$$

The transportation costs are $\gamma^1_{ij} = b_{ij}x_{ij}$ for the agent in J_+, and $\gamma^2_{ij} = b_{ij}y_{ij}$ for the agent in J_-. Again, we reformulate the constraints to all three objective designs:

(A). The first formulation:

$$\begin{aligned} \max \quad & \left\{ \min \sum_{(i,j)\in E} \xi_{ij} - \lambda \left\| f^* - \left(t_1 - \sum_{(i,j)\in E} \gamma^1_{ij} \right) \right\| - \max\left(t_2 - \sum_{(i,j)\in E} \gamma^2_{ij} \right) \right\} \\ \text{s.t.} \quad & t_1 = \sum_{i\in T_1} p_i \left(\sum_{j\in V^-_{ij}} x_{ji} - \sum_{j\in V^+_{ij}} x_{ij} \right), \\ & t_2 = \sum_{i\in T_2} p_i \left(\sum_{j\in V^-_{ij}} y_{ji} - \sum_{j\in V^+_{ij}} y_{ij} \right). \end{aligned} \tag{4.34}$$

f^* is calculated the same way as in equation (4.6).
(B). The second formulation:

$$\begin{aligned} \max \quad & \left\{ \min \sum_{(i,j)\in E} \xi_{ij} + \lambda_1 \min\left(t_1 - \sum_{(i,j)\in E} \gamma^1_{ij} \right) - \lambda_2 (t_2 - \sum_{(i,j)\in E} \gamma^2_{ij}) \right\} \\ \text{s.t.} \quad & t_1 = \sum_{i\in T_1} p_i \left(\sum_{j\in V^-_{ij}} x_{ji} - \sum_{j\in V^+_{ij}} x_{ij} \right). \\ & t_2 = \sum_{i\in T_2} p_i \left(\sum_{j\in V^-_{ij}} y_{ji} - \sum_{j\in V^+_{ij}} y_{ij} \right). \end{aligned} \tag{4.35}$$

(C). The third formulation:

$$\begin{aligned} \max \quad & \left\{ \min \sum_{(i,j)\in E} \xi_{ij} - \max\left(t_2 - \sum_{(i,j)\in E} \gamma^2_{ij} \right) \right\} \\ \text{s.t.} \quad & \left\| f^* - \left(t_1 - \sum_{(i,j)\in E} \gamma^1_{ij} \right) \right\| \leq \varepsilon_0, \\ & t_1 = \sum_{i\in T_1} p_i \left(\sum_{j\in V^-_{ij}} x_{ji} - \sum_{j\in V^+_{ij}} x_{ij} \right), \\ & t_2 = \sum_{i\in T_2} p_i \left(\sum_{j\in V^-_{ij}} y_{ji} - \sum_{j\in V^+_{ij}} y_{ij} \right). \end{aligned} \tag{4.36}$$

The corresponding constraints are as follows,

$$\sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = 0, \forall i \in V/(S_1 \cup T_1), \tag{4.37}$$

$$\sum_{j|(j,i)\in E} x_{ji} - \sum_{j|(i,j)\in E} x_{ij} = z_i^1, \forall i \in T_1, \tag{4.38}$$

$$\sum_{j|(j,i)\in E} y_{ji} - \sum_{j|(i,j)\in E} y_{ij} = 0, \forall i \in V/(S_2 \cup T_2), \tag{4.39}$$

$$\sum_{j|(j,i)\in E} y_{ji} - \sum_{j|(i,j)\in E} y_{ij} = z_i^2, \forall i \in T_2, \tag{4.40}$$

$$\xi_{ij} \geq 0, \xi_{ij} \geq x_{ij} + y_{ij} - w_{ij}, \forall (i,j) \in E, \tag{4.41}$$

$$\gamma_{ij}^1 = b_{ij}x_{ij}, \gamma_{ij}^2 = b_{ij}y_{ij}, \forall (i,j) \in E, \tag{4.42}$$

$$x_{ij} + y_{ij} \leq 2w_{ij}, \forall (i,j) \in E, \tag{4.43}$$

$$0 \leq z_i^1 \leq d_i^1, i \in T_1, \tag{4.44}$$

$$0 \leq z_i^2 \leq d_i^2, i \in T_2, \tag{4.45}$$

$$\sum_{(i,j)\in E} w_{ij} = (1-\beta) \sum_{(i,j)\in E} w_{ij}^*, \tag{4.46}$$

$$0 \leq w_{ij} \leq w_{ij}^*, \forall (i,j) \in E, \tag{4.47}$$

$$x_{ij} \geq 0, y_{ij} \geq 0, \alpha_{ij}^k = \{0,1\}, k = 1,2,3, \forall (i,j) \in E, \tag{4.48}$$

$$\beta_{ij}^k \geq 0, \gamma_{ij}^k \geq 0, k = 1,2, \forall (i,j) \in E. \tag{4.49}$$

With this method, the transportation cost is γ_{ij} and the congestion fee at ξ_{ij} is β_{ij}^k.

Compared with MILP, we need not introduce constraints for integer variables, or introduce any binary variable. Although we can not obtain the congestion cost directly, we can distribute the optimally common congestion costs ξ_{ij} into two parts based on the optimal ratio $(\frac{x_{ij}^*}{x_{ij}^*+y_{ij}^*}, \frac{y_{ij}^*}{x_{ij}^*+y_{ij}^*})$.

5. Numerical results and analysis

Here we present a numerical example to illustrate how the models work. We use an instance with 33 nodes and 72 links. With Matlab, we compute the results for all three objective designs and both formulation of constraints. We also analyze the trade-off between the two objectives and conduct sensitivity analysis regarding $1-\beta$. All the numerical results are listed in Appendix A and Appendix B where J_+, J_- denote total optimal profits described in Section 4.

The results for MILP method is in Table A.1. In this case we fix β at 80%, and uniformly increase penalty λ and weight λ_1, as well as uniformly decrease ε_0 within 20 steps. λ has uniform 20 points in (1,200), λ_1 has uniform points in (0.1, 0.99), and ε_0 uniformly decreases from 80 550 to 42 500 where 42 500 is the asymptotically best lower bound for guaranteeing the feasibility of optimization models which implies there is no feasible policy if we set the gap between protected profit and primal profit f^* is less than 42500. Also, the 80 550 is also a proper upper bound because f^* solved from (2) is 80 550. If we set the upper bound is more than 80 550, the constraints in (4) and (8) are useless for optimization. We solve our six formulations of MILP and LP by Gurobi solver in CVX with tolerance gap of primal objective and dual objective. The optimal solutions of MILP formulations are obtained within tolerance $1e-04$ and 1000 iterations by Cutting Plane Method and of LP formations are competely solved within 200 iterations of Simplex Method.

The results for multi-stage LP method is in Table A.2. The value of β is same as in MILP method, so does the variation of λ, λ_1 and ε_0.

We observe the trade-off of two objectives in both Tables A.1 and A.2 with changes in the weights of two objectives. MILP method is more sensitive in the trade-off while the computational cost for that is much less expensive.

We also conduct the sensitivity analysis for percentage of total disrupted transportation capacity by fixing λ at 1, λ_1 at 0.5 and ε_0 at 50 000 and $f^* = 80\,550$ as a proper case in all optimization models. The results are illustrated in Tables B.1 and B.2 in Appendix B.

6. Conclusion

We have modeled the network interdiction with contradicting goals with three methods. We employed both scalarization method and ε-constraint method in the bi-optimization optimization, to get the Pareto-optimal points. Also, we used both MILP and multi-stage linear programming for the piecewise cost functions. A simple computational instance is used to illustrate those methods. It showed that MILP is more sensitive in the trade-off of objectives, while multi-stage linear programming is faster in its computation.

There are several problems worth future study. Generalizations to multiple resources, undirected networks and randomized disruptions requires further consideration into modeling. As for the computation, Wood [14] showed a simple problem such as a disrupter wanting to only disrupt the profiting agents, is NP-complete. Thus, more instances need to be generated to better study the computation of the network.

Appendix A. Tables for optimal trade-off results

Table A.1. Optimal trade-off results of MILP method.

	First formulation		Second formulation		Third formulation	
Step	J_+	J_-	J_+	J_-	J_+	J_-
1	2.80E + 04	−1.75E + 05	2.80E + 04	−1.75E + 05	5.25E + 03	−1.75E + 05
2	2.80E + 04	−1.75E + 05	3.24E + 04	−1.74E + 05	2.00E + 03	−1.75E + 05
3	2.80E + 04	−1.75E + 05	3.24E + 04	−1.74E + 05	4.01E + 03	−1.75E + 05
4	3.24E + 04	−1.74E + 05	3.68E + 04	−1.73E + 05	2.22E + 04	−1.75E + 05
5	3.24E + 04	−1.74E + 05	3.70E + 04	−1.73E + 05	1.19E + 04	−1.75E + 05
6	3.70E + 04	−1.73E + 05	3.70E + 04	−1.73E + 05	1.00E + 04	−1.75E + 05
7	3.70E + 04	−1.73E + 05	3.70E + 04	−1.73E + 05	2.08E + 04	−1.75E + 05
8	3.70E + 04	−1.73E + 05	3.70E + 04	−1.73E + 05	1.40E + 04	−1.75E + 05
9	3.70E + 04	−1.73E + 05	3.70E + 04	−1.73E + 05	2.11E + 04	−1.75E + 05
10	3.70E + 04	−1.73E + 05	3.70E + 04	−1.73E + 05	2.28E + 04	−1.74E + 05
11	3.70E + 04	−1.73E + 05	3.70E + 04	−1.73E + 05	2.00E + 04	−1.74E + 05
12	3.70E + 04	−1.73E + 05	3.73E + 04	−1.72E + 05	2.20E + 04	−1.74E + 05
13	3.73E + 04	−1.72E + 05	3.73E + 04	−1.72E + 05	2.59E + 04	−1.74E + 05
14	3.73E + 04	−1.72E + 05	3.73E + 04	−1.72E + 05	2.60E + 04	−1.74E + 05
15	3.75E + 04	−1.72E + 05	3.75E + 04	−1.72E + 05	2.80E + 04	−1.74E + 05
16	3.75E + 04	−1.72E + 05	3.78E + 04	−1.70E + 05	3.00E + 04	−1.73E + 05
17	3.78E + 04	−1.70E + 05	3.78E + 04	−1.70E + 05	3.20E + 04	−1.73E + 05
18	3.78E + 04	−1.70E + 05	3.78E + 04	−1.70E + 05	3.40E + 04	−1.73E + 05
19	3.82E + 04	−1.65E + 05	3.82E + 04	−1.65E + 05	3.60E + 04	−1.73E + 05
20	3.83E + 04	−1.63E + 05	3.83E + 04	−1.63E + 05	3.81E + 04	−1.68E + 05

TABLE A.2. Optimal trade-off results of multi-stage LP method.

	First formulation		Second formulation		Third formulation	
Step	J_+	J_-	J_+	J_-	J_+	J_-
1	2.71E + 04	−2.39E + 05	2.03E + 04	−2.84E + 05	0.00E + 03	−2.86E + 05
2	2.72E + 04	−2.39E + 05	3.09E + 04	−2.83E + 05	1.91E + 03	−2.85E + 05
3	3.16E + 04	−2.38E + 05	3.53E + 04	−2.82E + 05	3.79E + 03	−2.85E + 05
4	3.60E + 04	−2.37E + 05	3.53E + 04	−2.82E + 05	5.68E + 03	−2.85E + 05
5	3.60E + 04	−2.37E + 05	3.53E + 04	−2.82E + 05	7.49E + 03	−2.85E + 05
6	3.60E + 04	−2.37E + 05	3.53E + 04	−2.82E + 05	9.29E + 03	−2.85E + 05
7	3.60E + 04	−2.37E + 05	3.53E + 04	−2.82E + 05	1.10E + 04	−2.84E + 05
8	3.60E + 04	−2.37E + 05	3.51E + 04	−2.80E + 05	1.26E + 04	−2.84E + 05
9	3.60E + 04	−2.37E + 05	3.51E + 04	−2.80E + 05	1.43E + 04	−2.84E + 05
10	3.60E + 04	−2.37E + 05	3.56E + 04	−2.80E + 05	1.62E + 04	−2.84E + 05
11	3.60E + 04	−2.37E + 05	3.56E + 04	−2.80E + 05	1.82E + 04	−2.84E + 05
12	3.60E + 04	−2.37E + 05	3.56E + 04	−2.80E + 05	2.01E + 04	−2.84E + 05
13	3.60E + 04	−2.37E + 05	3.56E + 04	−2.80E + 05	2.20E + 04	−2.83E + 05
14	3.60E + 04	−2.37E + 05	3.55E + 04	−2.77E + 05	2.39E + 04	−2.83E + 05
15	3.60E + 04	−2.37E + 05	3.55E + 04	−2.77E + 05	2.58E + 04	−2.83E + 05
16	3.60E + 04	−2.37E + 05	3.55E + 04	−2.77E + 05	2.76E + 04	−2.83E + 05
17	3.60E + 04	−2.37E + 05	3.51E + 04	−2.70E + 05	2.95E + 04	−2.83E + 05
18	3.60E + 04	−2.37E + 05	3.51E + 04	−2.70E + 05	3.14E + 04	−2.82E + 05
19	3.60E + 04	−2.37E + 05	3.51E + 04	−2.70E + 05	3.33E + 04	−2.82E + 05
20	3.60E + 04	−2.37E + 05	3.51E + 04	−2.70E + 05	3.52E + 04	−2.81E + 05

APPENDIX B. TABLES FOR SENSITIVITY ANALYSIS

TABLE B.1. Sensitivity analysis results for MILP method.

	Second formulation		Third formulation	
Step	J_+	J_-	J_+	J_-
1	3.56E + 04	6.33E + 03	3.50E + 04	−4.13E + 04
2	3.61E + 04	−1.31E + 04	3.50E + 04	−6.05E + 04
3	3.61E + 04	−3.06E + 04	3.56E + 04	−7.77E + 04
4	3.61E + 04	−4.68E + 04	3.61E + 04	−9.33E + 04
5	3.61E + 04	−6.21E + 04	3.59E + 04	−1.08E + 05
6	3.61E + 04	−7.59E + 04	3.50E + 04	−1.22E + 05
7	3.61E + 04	−8.89E + 04	3.58E + 04	−1.34E + 05
8	3.61E + 04	−1.01E + 05	3.50E + 04	−1.46E + 05
9	3.61E + 04	−1.13E + 05	3.50E + 04	−1.58E + 05
10	3.61E + 04	−1.24E + 05	3.50E + 04	−1.69E + 05
11	3.61E + 04	−1.34E + 05	3.51E + 04	−1.80E + 05
12	3.61E + 04	−1.44E + 05	3.51E + 04	−1.88E + 05
13	3.59E + 04	−1.52E + 05	3.57E + 04	−1.96E + 05
14	3.61E + 04	−1.59E + 05	3.57E + 04	−2.03E + 05
15	3.59E + 04	−1.66E + 05	3.57E + 04	−2.10E + 05
16	3.64E + 04	−1.71E + 05	3.50E + 04	−2.15E + 05
17	3.70E + 04	−1.73E + 05	3.71E + 04	−2.15E + 05
18	3.70E + 04	−1.73E + 05	3.72E + 04	−2.15E + 05
19	3.70E + 04	−1.73E + 05	3.75E + 04	−2.15E + 05
20	3.70E + 04	−1.73E + 05	3.76E + 04	−2.15E + 05

Table B.2. Sensitivity analysis results of multi-stage LP method.

	First formulation		Second formulation		Third formulation	
Step	J_+	J_-	J_+	J_-	J_+	J_-
1	3.48E + 04	−6.81E + 03	3.28E + 04	−7.81E + 03	2.78E + 04	−1.22E + 04
2	3.49E + 04	−3.09E + 04	3.33E + 04	−3.17E + 04	2.80E + 04	−3.56E + 04
3	3.50E + 04	−5.24E + 04	3.35E + 04	−5.32E + 04	2.81E + 04	−5.70E + 04
4	3.51E + 04	−7.22E + 04	3.37E + 04	−7.30E + 04	2.82E + 04	−7.65E + 04
5	3.52E + 04	−9.08E + 04	3.38E + 04	−9.16E + 04	2.83E + 04	−9.48E + 04
6	3.53E + 04	−1.08E + 05	3.40E + 04	−1.09E + 05	2.84E + 04	−1.12E + 05
7	3.55E + 04	−1.24E + 05	3.43E + 04	−1.25E + 05	2.85E + 04	−1.27E + 05
8	3.56E + 04	−1.39E + 05	3.44E + 04	−1.40E + 05	2.86E + 04	−1.42E + 05
9	3.56E + 04	−1.54E + 05	3.45E + 04	−1.55E + 05	2.86E + 04	−1.57E + 05
10	3.57E + 04	−1.68E + 05	3.48E + 04	−1.67E + 05	2.87E + 04	−1.70E + 05
11	3.58E + 04	−1.80E + 05	3.50E + 04	−1.80E + 05	2.88E + 04	−1.83E + 05
12	3.59E + 04	−1.92E + 05	3.50E + 04	−1.91E + 05	2.87E + 04	−1.94E + 05
13	3.59E + 04	−2.02E + 05	3.51E + 04	−2.01E + 05	2.84E + 04	−2.03E + 05
14	3.60E + 04	−2.11E + 05	3.50E + 04	−2.09E + 05	2.85E + 04	−2.12E + 05
15	3.60E + 04	−2.19E + 05	3.51E + 04	−2.18E + 05	2.85E + 04	−2.21E + 05
16	3.60E + 04	−2.27E + 05	3.53E + 04	−2.27E + 05	2.82E + 04	−2.28E + 05
17	3.61E + 04	−2.33E + 05	3.57E + 04	−2.33E + 05	2.82E + 04	−2.35E + 05
18	3.60E + 04	−2.39E + 05	3.55E + 04	−2.38E + 05	2.81E + 04	−2.41E + 05
19	3.58E + 04	−2.44E + 05	3.57E + 04	−2.44E + 05	2.80E + 04	−2.46E + 05
20	3.57E + 04	−2.49E + 05	3.57E + 04	−2.49E + 05	2.79E + 04	−2.50E + 05

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