RAIRO-Oper. Res. 53 (2019) 473–486
https://doi.org/10.1051/ro/2017047

RAIRO Operations Research
www.rairo-ro.org

USE OF "E" AND "G" OPERATORS TO A FUZZY PRODUCTION INVENTORY CONTROL MODEL FOR SUBSTITUTE ITEMS

Debnarayan Khatua[1,*], Anupam De[2], Kalipada Maity[3] and Samarjit Kar[4]

Abstract. In this paper, a fuzzy optimal control model for substitute items with stock and selling price dependent demand has been developed. Here the state variables (stocks) are assumed to be fuzzy variables. So the proposed dynamic control system can be represented as a fuzzy differential system which optimize the profit of the production inventory control model through Pontryagin's maximum principle. The proposed fuzzy control problem has been transformed into an equivalent crisp differential system using "e" and "g" operators. The deterministic system is then solved by using Newton's forward-backward method through MATLAB. Finally some numerical results are presented both in tabular and graphical form.

Mathematics Subject Classification. 49J15, 90C70.

Received January 12, 2017. Accepted June 11, 2017.

1. Introduction

Many dynamical problems have been formulated as a mathematical model in real life. These problems can be formulated either as a system of ordinary or partial differential equations. But Fuzziness is a kind of uncertainty in real life problems. So, some dynamic control systems may be described by fuzzy differential equations and fuzzy control. When the information about the behavior of a dynamical system is insufficient then fuzzy differential equations are a useful tool to formulate a model. Fuzzy initial value problems arise in several areas of mathematics and science including population models [17], mathematical physics [5] and other applications [25].

Till now, many authors have been developing several techniques for controlling the fuzzy dynamical systems. Filev and Angelove [23] have formulated the problem of fuzzy optimal control of nonlinear system and solved this problem on the basis of fuzzy mathematical programming. Zhu [28] has introduced a method to solve fuzzy optimal control problem by using dynamic programming. In [21] a model of an optimal control problem with chance constraints is introduced. Also, Najariyan and Farahi [16] have formulated a technique to solve

Keywords. Fuzzy dynamical system, "e" and "g" operators, Production-inventory control, Substitute items, Stock and selling price dependent demand.

[1] Department of Basic Science and Humanities, Global Institute of Science and Technology, Haldia, Purba Medinipur – 721657, West Bengal, India.

[2] Department of Applied Sciences, Haldia Institute of Technology, Haldia, Purba Medinipur – 721657, West Bengal, India.

[3] Department of Mathematics, Mugberia Gangadhar Mahavidyalaya, Bhupatinagar, Purba Medinipur – 721425, West Bengal, India.

[4] Department of Mathematics, National Institute of Technology, Durgapur-713209, West Bengal, India.

*Corresponding author: devnarayan87@gmail.com

fuzzy control system governed by fuzzy differential equation. Recently, Ahmad *et al.* [1], Jia *et al.* [7], Jameel *et al.* [6],Wang and Wu [27] and others have developed different techniques for solving said type of problems.

Now-a-days, the product assortment has made an arena of research across several fields, including economics, analytical and empirical modelling. Common assortment decisions contain matters such as assortment size, number of categories, number of items within a group of the existing product lines, overall attraction of the goods, the relational properties of the goods, pricing strategies and the variety of goods over time. Maity and Maiti [13, 14] have developed the optimal production control problem for complementary and substitute items on the finite time horizon. Next, Chernev [4] has done an interdisciplinary review on product assortment and consumer choice. Recently, Katsifou *et al.* [9] have developed the joint product assortment inventory problem to attract the loyal and non-loyal customers. The above said papers briefly stated that demand is negative due to the effect of substitute items *i.e.* when the demand of an element increases, the demand of another element decreases.

Inventory management acts as a significant character in businesses because of it can help companies achieve the goal of ensuring without delay delivery, avoiding shortages, helping sales at competitive prices and so forth. To control an inventory system, one cannot ignore demand of the product since inventory is partially determined by demand. A manager of a company has to investigate the factors that influence demand pattern, because customers' purchasing behavior may be affected by factors such as the selling price, inventory level, seasonality, and so on. Besides this, many business practices give you an idea about that the existence of a larger quantity of goods displayed may attract more than consumers that with a smaller quantity of goods. This occurrence implies that the demand may have a optimistic correlation with stock level. Under such a situation, a firm should sincerely consider its price and order policy since the demand for their products may be affected by their selling prices and inventory level [8]. Generally, demand rate decreases with sales price yet increases with the stock quantity on display.

One of the weaknesses of present production-inventory models is the unreasonable supposition that all produced items are of good quality. But production of imperfective items is a normal phenomenon due to different difficulties in a long-run production process. The defective items as a result of imperfect quality production process were initially considered by Porteus [18] and later by several researchers such as Panda *et al.* [20], Sana [26], Khan and Jaber [10], Chen *et al.* [3], Yadav *et al.* [24], Krishnamoorthi and Panayappan [12], Sivashankari and Panayappan [22], Bartoszewicz and Lesniewski [2] and others.

In this paper, a production control problem with stock and price dependent demand has been developed for substitute items in fuzzy environment. Here the selling prices are the control variables. It is also assumed that the state variables (stock variable) are fuzzy in nature. So the dynamic control system can be described by fuzzy differential equation(FDE). Next, a optimal pricing inventory control model is considered in optimizing form and then it(FDE) is converted into crisp differential equation by using "e" and "g" operators method. Finally, total profit, which consists of the sales proceeds, inventory holding cost and production cost is formulated as an optimal control problem and solved by using Pontryagin's Maximum principle [19]. Then to obtain the numerical result by using MATLAB, an algorithm is also developed. Subsequently, the numerical results are presented both in tabular form and graphically. So first time we have developed the following methods for the multi item production inventory model:

(1) The "e" and "g" operators method to solve a optimal price control problem for substitute items in fuzzy environment.
(2) A new algorithm by using Newton's forward-backward method in MATLAB software to obtain the numerical results of optimal price, stock level, production rate, demand rate and profit function.

2. Interval Arithmetic

An interval number A is represented by closed interval $[\underline{a}, \overline{a}]$ and defined by $A = [\underline{a}, \overline{a}] = \{x : \underline{a} \leq x \leq \overline{a}, x \in R\}$. Where R is the set of all real numbers and $\underline{a}, \overline{a}$ are the left and right limits of the interval number respectively. Also every real number a can be represented by the interval number $[a, a]$, for all $a \in R$. Here we present some arithmetic operations on interval valued functions as follows:

Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two interval numbers.

Then De *et al.* [11], the following operations can be defined as

Addition: $A + B = [\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$.

Subtraction: $A - B = [\underline{a}, \overline{a}] - [\underline{b}, \overline{b}] = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$.

Scalar Multiplication:

$$\alpha A = \alpha[\underline{a}, \overline{a}] = \begin{cases} [\alpha\underline{a}, \alpha\overline{a}], & \text{if} \quad \alpha \geq 0 \\ [\alpha\overline{a}, \alpha\underline{a}], & \text{if} \quad \alpha < 0 \end{cases}$$

Multiplication:

$$A * B = \begin{cases} [\underline{a}\ \underline{b}, \overline{a}\ \overline{b}], & \text{if} \quad \underline{a} \geq 0 \text{ and } \underline{b} \geq 0 \\ [\overline{a}\ \underline{b}, \overline{a}\ \overline{b}], & \text{if} \quad \underline{a} \geq 0 \text{ and } \underline{b} < 0 < \overline{b} \\ [\overline{a}\ \underline{b}, \underline{a}\ \overline{b}], & \text{if} \quad \underline{a} \geq 0 \text{ and } \overline{b} \leq 0 \\ [\underline{a}\ \overline{b}, \overline{a}\ \underline{b}], & \text{if} \quad \underline{a} < 0 < \overline{a} \text{ and } \underline{b} \geq 0 \\ [\overline{a}\ \overline{b}, \underline{a}\ \overline{b}], & \text{if} \quad \underline{a} < 0 < \overline{a} \text{ and } \overline{b} \leq 0 \\ [\underline{a}\ \overline{b}, \overline{a}\ \underline{b}], & \text{if} \quad \overline{a} \leq 0 \text{ and } \underline{b} \geq 0 \\ [\underline{a}\ \overline{b}, \underline{a}\ \underline{b}], & \text{if} \quad \overline{a} \leq 0 \text{ and } \underline{b} < 0 < \overline{b} \\ [\overline{a}\ \overline{b}, \underline{a}\ \underline{b}], & \text{if} \quad \overline{a} \leq 0 \text{ and } \overline{b} \leq 0 \\ [\min(\overline{a}\ \underline{b}, \underline{a}\ \overline{b}), \max(\underline{a}\ \underline{b}, \overline{a}\ \overline{b})], & \text{if} \quad \underline{a} < 0 < \overline{a} \text{ and } \underline{b} < 0 < \overline{b} \end{cases}$$

Division:

$$\frac{1}{B} = \left[\frac{1}{\overline{b}}, \frac{1}{\underline{b}}\right] \qquad \text{if} \quad (0 \notin B)$$

$$\text{And, } \frac{A}{B} = A * \frac{1}{B} \qquad \text{if} \quad (0 \notin B)$$

3. "E" AND "G" OPERATORS

In this section, we have used two operators "e" and "g" [15] for quasi-level system for FDE's. Let $\mathbf{C}$ be a complex set $i.e$ $\mathbf{C} = \{a + ib : a, b \in \Re\}$. Then "e" is a identity operator and "g" corresponds to a flip about the diagonal in the complex plane, $i.e.$, $\forall a + ib \in \mathbf{C}$,

$$\begin{cases} e : a + ib \to a + ib, \\ g : a + ib \to b + ia. \end{cases} \tag{3.1}$$

3.1. Use of "e" and "g" operators in Fuzzy dynamical system

Let us consider the following non-homogeneous fuzzy dynamical system

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{f}(t), \quad \tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0, \quad t \in [0, \infty) \tag{3.2}$$

where $\mathbf{A} = [a_{ij}]_{n\times n}$, $\dot{\mathbf{x}}(t) = [\dot{\tilde{x}}_1(t)\dots\dot{\tilde{x}}_n(t)]^T$ and $\mathbf{f}(t) = [f_1(t)\dots f_n(t)]^T$.
Let $\overline{\mathbf{Y}}^{\alpha}(t) = [\bar{y}_1^{\alpha}(t)\dots\bar{y}_n^{\alpha}(t)]^T$, $\underline{\mathbf{Y}}^{\alpha}(t) = [\underline{y}_1^{\alpha}(t)\dots\underline{y}_n^{\alpha}(t)]^T$ be the solutions of quasi-level-wise system

$$\begin{cases}\dot{\underline{\mathbf{Y}}}^{\alpha}(t) + i\dot{\overline{\mathbf{Y}}}^{\alpha}(t) = B[\underline{\mathbf{Y}}^{\alpha}(t) + i\overline{\mathbf{Y}}^{\alpha}(t)] + (\underline{\mathbf{f}}(t) + i\bar{\mathbf{f}}(t)),\\ \underline{Y}^{\alpha}(0) = \underline{x}_0^{\alpha}, \quad \overline{Y}^{\alpha}(0) = \overline{x}_0^{\alpha}\end{cases} \tag{3.3}$$

where $\underline{\mathbf{f}}(t) = \bar{\mathbf{f}}(t) = \mathbf{f}(t)$ and $\mathbf{B} = [b_{ij}]_{n\times n}$, $b_{ij} = \begin{cases} a_{ij}e & a_{ij} \geq 0\\ a_{ij}g & a_{ij} < 0\end{cases}$. Then $\bar{x}_i^{\alpha}(t) = \max\limits_{t\in(0,\infty)}\{\bar{y}_i^{\alpha}(t), \underline{y}_i^{\alpha}(t)\}$ $\underline{x}_i^{\alpha}(t) = \min\limits_{t\in(0,\infty)}\{\bar{y}_i^{\alpha}(t), \underline{y}_i^{\alpha}(t)\}, i = 1, 2, \dots, n$ are also the solutions of the fuzzy dynamical system (3.2). Now if the dynamical system (3.3) is unstable, then moving the instability property to the level-wise system, we get to the same system as (3.3) *i.e.*,

$$\begin{cases}e[\dot{\underline{\mathbf{Y}}}^{\alpha}(t) + i\dot{\overline{\mathbf{Y}}}^{\alpha}(t)] = \bar{\mathbf{B}}[\underline{\mathbf{Y}}^{\alpha}(t) + i\overline{\mathbf{Y}}^{\alpha}(t)] + (\underline{\mathbf{f}}(t) + i\bar{\mathbf{f}}(t)),\\ \underline{Y}^{\alpha}(0) = \underline{x}_0^{\alpha}, \quad \overline{Y}^{\alpha}(0) = \overline{x}_0^{\alpha}\end{cases} \tag{3.4}$$

or

$$\begin{cases}g[\dot{\underline{\mathbf{Y}}}^{\alpha}(t) + i\dot{\overline{\mathbf{Y}}}^{\alpha}(t)] = \bar{\mathbf{B}}[\underline{\mathbf{Y}}^{\alpha}(t) + i\overline{\mathbf{Y}}^{\alpha}(t)] + (\underline{\mathbf{f}}(t) + i\bar{\mathbf{f}}(t)),\\ \underline{Y}^{\alpha}(0) = \underline{x}_0^{\alpha}, \quad \overline{Y}^{\alpha}(0) = \overline{x}_0^{\alpha}\end{cases} \tag{3.5}$$

where $\bar{\mathbf{B}} = [\bar{b}_{ij}]_{n\times n}$, $\bar{b}_{ij} = a_{ij}e$ or $a_{ij}g$

4. Proposed models

In this section, following assumptions and notations are used in the mathematical model for production control system in finite time horizon.

4.1. Assumptions

- (i) A single period production inventory model with finite time horizon is considered;
- (ii) Defective rate is constant;
- (iii) Shortages are not allowed;
- (iv) There is no repair or replacement of defective units over whole time period;
- (v) Demand rate depends on stock and selling price of the product simultaneously; For substitute items, demand function is defined as a linear form of the two products retail prices-downward slopping in its own price and increasing with respect to its substitute item's selling price.

4.2. Notations

For the ith ($i = 1; 2$) item,
$\tilde{X}_i(t)$: the stock level at time t which is fuzzy in nature (state variable);
$\tilde{X}_{3-i}(t)$: the stock level at time t of another item which is fuzzy in nature (state variable);
$\tilde{D}_i(t)$: demand rate at time t which is fuzzy in nature;
$\tilde{U}_i(t) = (1-\delta_i)u_{i0}(1-\frac{\tilde{X}_i(t)}{X_{i\max}})$: stock dependent production which is also fuzzy in nature with $\frac{\tilde{X}_i(t)}{X_{i\max}} < 1$ and u_{i0} is production parameter;
$\tilde{h}_i$: holding cost per unit which is fuzzy in nature;
$\tilde{d}_{i0}$: constant part of demand function which is fuzzy in nature;
$\tilde{c}_{ui}$: production cost which is also fuzzy in nature;

$\tilde{S}_i$: selling price per unit for ith item which is fuzzy in nature (control variable);
$\tilde{S}_{3-i}$: selling price per unit another ith item which is fuzzy in nature (control variable);
β_{ii}: measure of responsiveness of ith item's consumer demand to its own price;
β_{i3-i}: measure of responsiveness of another ith item's consumer demand to its own price;
η_{ii}: measure of responsiveness of ith item's consumer demand to its own stock;
η_{i3-i}: measure of responsiveness of another ith item's consumer demand to its own stock;
γ_i: marginal selling price per unit for ith item;
$\tilde{\lambda}_i(t)$: adjoint function treated as shadow price which is also fuzzy in nature;

4.3. Formulation of the defective production optimal control model for substitute items

Here, the items are produced at a rate $\tilde{U}_i(t)$ with

$$\tilde{U}_i(t) = (1-\delta_i)u_{i0}\left(1 - \frac{\tilde{X}_i(t)}{X_{\text{imax}}}\right),\ i = 1,2,\ \frac{\tilde{X}_i(t)}{X_{\text{imax}}} < 1 \tag{4.1}$$

of which δ_i is constant defective rate and u_{i0} is a production parameter. Here a two-items production-inventory problem of substitute type is considered.

In this model, the demand rate depends on the stock rate. Also demand of one item is dependent on the retail prices of its own and substitute item positively.

$$\tilde{D}_i(t) = \tilde{d}_{i0} - \beta_{ii}\tilde{S}_i(t) + \beta_{i3-i}\tilde{S}_{3-i}(t) + \eta_{ii}\tilde{X}_i(t) - \eta_{i3-i}\tilde{X}_{3-i}(t),\ i = 1,2 \tag{4.2}$$

where $\tilde{d}_{i0}$ is the constant part of demand, β_{ii} are the measure of responsiveness of ith product's consumer demand to its own price and which are positive for substitute item. η_{ii} are the measure of responsiveness of ith product's consumer demand to its own stock and which are negative for substitute item.

The differential equations of rate of change of stock for ith item representing above system during a finite time-horizon is

$$\dot{\tilde{X}}_i(t) = (1-\delta_i)u_{i0}\left(1 - \frac{\tilde{X}_i(t)}{X_{1\max}}\right) - \tilde{D}_i(t),\ i = 1,2$$

which can be written as

$$\dot{\tilde{X}}_i(t) = (1-\delta_i)u_{i0}\left(1 - \frac{\tilde{X}_i(t)}{X_{1\max}}\right) - \left(\tilde{d}_{i0} - \beta_{ii}\tilde{S}_i(t) + \beta_{i3-i}\tilde{S}_{3-i}(t) + \eta_{ii}\tilde{X}_i(t) - \eta_{i3-i}\tilde{X}_{3-i}(t)\right),\ i = 1,2 \tag{4.3}$$

Then the total profit consisting of selling prices, holding costs, and production costs leads to

$$J = \int_0^T \left(\sum_{i=1,2} \tilde{S}_i(t)\tilde{D}_i(t) - \tilde{h}_i\tilde{X}_i(t) - \tilde{c}_{ui}\tilde{U}_i(t)\right) + \sum_{i=1,2}\gamma_i\tilde{S}_i(T)\tilde{X}_i(T) \tag{4.4}$$

where the final stock $\tilde{X}_i(T)$ is selling with salvage value price $\gamma_i\tilde{S}_i(T)$ and the corresponding marginal revenue is $\gamma_i\tilde{S}_i(T)\tilde{X}_i(T)$.

Therefore the optimal price control policy for substitute items by maximizing the finite time horizon profit function is

$$\text{Maximize } J = \int_0^T \left(\sum_{i=1,2} \tilde{S}_i(t)\tilde{D}_i(t) - \tilde{h}_i\tilde{X}_i(t) - \tilde{c}_{ui}\tilde{U}_i(t)\right) + \sum_{i=1,2}\gamma_i\tilde{S}_i(T)\tilde{X}_i(T) \tag{4.5}$$

subject to (4.3).

5. Optimal control policy for substitute items

Thus the problem reduces to maximize the profit function J subject to the state constraint satisfying the dynamic price-stock relation.

$$\text{Maximize } J = \int_0^T \left(\sum_{i=1,2} \tilde{S}_i(t)\tilde{D}_i(t) - \tilde{h}_i\tilde{X}_i(t) - \tilde{c}_{ui}\tilde{U}_i(t) \right) + \sum_{i=1,2} \gamma_i \tilde{S}_i(T)\tilde{X}_i(T) \tag{5.1}$$

Subject to,

$$\dot{\tilde{X}}_i(t) = (1-\delta_i)u_{i0}\left(1 - \frac{\tilde{X}_i(t)}{X_{1\max}}\right) - \left(\tilde{d}_{i0} - \beta_{ii}\tilde{S}_i(t) + \beta_{i3-i}\tilde{S}_{3-i}(t) + \eta_{ii}\tilde{X}_i(t) - \eta_{i3-i}\tilde{X}_{3-i}(t)\right),\ i=1,2 \tag{5.2}$$

The above equations (5.2) are defined as an optimal control problem with control variable $S_i(t)$ and state variable $X_i(t)$. It can be deduced to algebraic forms using e and g operators.

Let us consider Hamiltonian H as

$$\begin{aligned} H(Y_i(t), S_i(t), t) = \sum_{i=1,2} \Big[&\big(\underline{S}_i^\alpha(t)\underline{D}_i^\alpha(t) + i\overline{S}_i^\alpha(t)\overline{D}_i^\alpha(t)\big) - \big(\overline{h}_i^\alpha\overline{Y}_i^\alpha(t) + i\underline{h}_i^\alpha\underline{Y}_i^\alpha(t)\big) - \big(\overline{c}_{ui}^\alpha\overline{U}_i^\alpha(t) + i\underline{c}_{ui}^\alpha\underline{U}_i^\alpha(t)\big)\Big] \\ &+ \sum_{i=1,2} (\underline{\lambda}_i^\alpha(t)\dot{\underline{Y}}_i^\alpha(t) + i\overline{\lambda}_i^\alpha(t)\dot{\overline{Y}}_i^\alpha(t)) \end{aligned} \tag{5.3}$$

with

$$\begin{cases} \dot{\underline{Y}}_i^\alpha(t) = u_{i0}(1-\delta_i)\left(1 - \dfrac{\overline{Y}_i^\alpha}{X_{i\max}}\right) - \overline{d}_{i0} + \beta_{ii}\underline{S}_i^\alpha - \beta_{i3-i}\overline{S}_{3-i}^\alpha - \eta_{ii}\overline{Y}_i^\alpha(t) + \eta_{i3-i}\underline{Y}_{3-i}^\alpha(t) \\ \dot{\overline{Y}}_i^\alpha(t) = u_{i0}(1-\delta_i)\left(1 - \dfrac{\underline{Y}_i^\alpha}{X_{i\max}}\right) - \underline{d}_{i0} + \beta_{ii}\overline{S}_i^\alpha - \beta_{i3-i}\underline{S}_{3-i}^\alpha - \eta_{ii}\underline{Y}_i^\alpha(t) + \eta_{i3-i}\overline{Y}_{3-i}^\alpha(t),\ i=1,2 \end{cases}$$

where $\underline{X}_i^\alpha = \min\{\underline{Y}_i^\alpha, \overline{Y}_i^\alpha\}$, $\overline{X}_i^\alpha = \max\{\underline{Y}_i^\alpha, \overline{Y}_i^\alpha\}$ and $\lambda_i(t)$ are adjoint variables. The optimal controls $\underline{S}_i^\alpha(t), \overline{S}_i^\alpha(t)$, which maximize H, must satisfy the following conditions:

$$\begin{cases} \dfrac{\partial H}{\partial \underline{S}_i^\alpha(t)} = 0 \\ \dfrac{\partial H}{\partial \overline{S}_i^\alpha(t)} = 0 \end{cases} \tag{5.4}$$

$$\begin{cases} \dot{\underline{\lambda}}_i^\alpha(t) = -\dfrac{\partial H}{\partial \underline{Y}_i^\alpha(t)} \\ \dot{\overline{\lambda}}_i^\alpha(t) = -\dfrac{\partial H}{\partial \overline{Y}_i^\alpha(t)} \end{cases} \tag{5.5}$$

Now solving equations (5.4) we have,

$$\begin{cases} \underline{d}_{10}^\alpha - \beta_{11}\overline{S}_1^\alpha(t) + \beta_{12}\underline{S}_2^\alpha(t) + \eta_{11}\underline{Y}_1^\alpha(t) - \eta_{12}\overline{Y}_2^\alpha(t) + \beta_{21}\underline{S}_2^\alpha(t) - \beta_{11}\overline{S}_1^\alpha(t) + \beta_{11}\underline{\lambda}_1^\alpha(t) - \beta_{21}\overline{\lambda}_2^\alpha(t) = 0 \\ \overline{d}_{10}^\alpha - \beta_{11}\underline{S}_1^\alpha(t) + \beta_{12}\overline{S}_2^\alpha(t) + \eta_{11}\overline{Y}_1^\alpha(t) - \eta_{12}\underline{Y}_2^\alpha(t) + \beta_{21}\overline{S}_2^\alpha(t) - \beta_{11}\underline{S}_1^\alpha(t) + \beta_{11}\overline{\lambda}_1^\alpha(t) - \beta_{21}\underline{\lambda}_2^\alpha(t) = 0 \\ \underline{d}_{20}^\alpha - \beta_{22}\overline{S}_2^\alpha(t) + \beta_{21}\underline{S}_1^\alpha(t) + \eta_{22}\underline{Y}_2^\alpha(t) - \eta_{21}\overline{Y}_1^\alpha(t) + \beta_{12}\underline{S}_1^\alpha(t) - \beta_{22}\overline{S}_2^\alpha(t) + \beta_{22}\underline{\lambda}_2^\alpha(t) - \beta_{12}\overline{\lambda}_1^\alpha(t) = 0 \\ \overline{d}_{20}^\alpha - \beta_{22}\underline{S}_2^\alpha(t) + \beta_{21}\overline{S}_1^\alpha(t) + \eta_{22}\overline{Y}_2^\alpha(t) - \eta_{21}\underline{Y}_1^\alpha(t) + \beta_{12}\overline{S}_1^\alpha(t) - \beta_{22}\underline{S}_2^\alpha(t) + \beta_{22}\overline{\lambda}_2^\alpha(t) - \beta_{12}\underline{\lambda}_1^\alpha(t) = 0 \end{cases} \tag{5.6}$$

And solving equations (5.5) we get,

$$\begin{cases} \dot{\underline{\lambda}}_1^{\alpha}(t) = -\left(\eta_{11}\underline{S}_1^{\alpha}(t) - \eta_{22}\overline{S}_2^{\alpha}(t) - \underline{h}_1^{\alpha} + \dfrac{\overline{C}_{u1}^{\alpha}}{Y_{1\max}} u_{10}(1-\delta_1)\right) + \left(\dfrac{u_{10}(1-\delta_1)}{X_{1\max}} + \eta_{11}\right)\overline{\lambda}_1^{\alpha}(t) - \eta_{22}\underline{\lambda}_2^{\alpha}(t) \\ \dot{\overline{\lambda}}_1^{\alpha}(t) = -\left(\eta_{11}\overline{S}_1^{\alpha}(t) - \eta_{22}\underline{S}_2^{\alpha}(t) - \overline{h}_1^{\alpha} + \dfrac{\underline{C}_{u1}^{\alpha}}{X_{1\max}} u_{10}(1-\delta_1)\right) + \left(\dfrac{u_{10}(1-\delta_1)}{X_{1\max}} + \eta_{11}\right)\underline{\lambda}_1^{\alpha}(t) - \eta_{22}\overline{\lambda}_2^{\alpha}(t) \\ \dot{\underline{\lambda}}_2^{\alpha}(t) = -\left(\eta_{21}\underline{S}_2^{\alpha}(t) - \eta_{12}\overline{S}_1^{\alpha}(t) - \underline{h}_2^{\alpha} + \dfrac{\overline{C}_{u2}^{\alpha}}{X_{2\max}} u_{20}(1-\delta_2)\right) + \left(\dfrac{u_{20}(1-\delta_2)}{X_{2\max}} + \eta_{21}\right)\overline{\lambda}_2^{\alpha}(t) - \eta_{12}\underline{\lambda}_1^{\alpha}(t) \\ \dot{\overline{\lambda}}_2^{\alpha}(t) = -\left(\eta_{21}\overline{S}_2^{\alpha}(t) - \eta_{12}\underline{S}_1^{\alpha}(t) - \overline{h}_2^{\alpha} + \dfrac{\underline{C}_{u2}^{\alpha}}{X_{2\max}} u_{20}(1-\delta_2)\right) + \left(\dfrac{u_{20}(1-\delta_2)}{X_{2\max}} + \eta_{21}\right)\underline{\lambda}_2^{\alpha}(t) - \eta_{12}\overline{\lambda}_1^{\alpha}(t) \end{cases} \tag{5.7}$$

From equations (5.6) we get the control variables $(\underline{S}_1^{\alpha}(t), \overline{S}_1^{\alpha}(t), \underline{S}_2^{\alpha}(t), \overline{S}_2^{\alpha}(t))$which can be expressed as,

$$\begin{cases} \underline{S}_1^{\alpha}(t) = -[(-2\beta_{22}\underline{d}_{10}^{\alpha} - (\beta_{12}+\beta_{21})\overline{d}_{20}^{\alpha} + (\beta_{12}^2 + \beta_{12}\beta_{21} - 2\beta_{11}\beta_{22})\lambda_1(t) + (\beta_{21}\beta_{22} - \beta_{12}\beta_{22})\lambda_4(t) \\ +(\eta_{21}\beta_{12} + \eta_{21}\beta_{21} - 2\eta_{11}\beta_{22})\underline{Y}_1^{\alpha}(t) + (2\eta_{12}\beta_{22} - \eta_{22}\eta_{12} - \eta_{22}\beta_{21})\overline{Y}_2^{\alpha}(t))]/[-(\beta_{12}+\beta_{21})^2 + 4\beta_{11}\beta_{22}]; \\ \overline{S}_1^{\alpha}(t) = -[-(\beta_{12}+\beta_{21})(\underline{d}_{20}^{\alpha} - \beta_{12}\lambda_2(t) + \beta_{22}\lambda_3(t) - \eta_{21}\overline{Y}_1^{\alpha}(t) + \eta_{22}\underline{Y}_2^{\alpha}(t)) - 2\beta_{22}(\overline{d}_{10}^{\alpha} + \beta_{11}\lambda_2(t) \\ -\beta_{21}\lambda_3(t) + \eta_{11}\overline{Y}_1^{\alpha}(t) - \eta_{12}\underline{Y}_2^{\alpha}(t))]/[(-(\beta_{12}+\beta_{21})^2 + (4\beta_{11}\beta_{22}))]; \\ \underline{S}_2^{\alpha}(t) = -[(-2\beta_{11}\underline{d}_{20}^{\alpha} - (\beta_{12}+\beta_{21})\overline{d}_{10}^{\alpha} + (\beta_{21}^2 + \beta_{12}\beta_{21} - 2\beta_{11}\beta_{22})\lambda_3(t) - \beta_{11}(\beta_{21}+\beta_{12})\lambda_2(t) \\ +(2\eta_{21}\beta_{11} - \eta_{11}\beta_{12} - \eta_{11}\beta_{21})\overline{Y}_1^{\alpha}(t) + (\eta_{12}\beta_{12} - 2\eta_{22}\beta_{11} + \eta_{12}\beta_{21})\underline{Y}_2^{\alpha}(t))]/[(-(\beta_{12}+\beta_{21})^2 + (4\beta_{11}\beta_{22}))]; \\ \overline{S}_2^{\alpha}(t) = -[(-(2\beta_{11})(\overline{d}_{20}^{\alpha} - \beta_{12}\lambda_1(t) + \beta_{22}\lambda_4(t) + \eta_{22}\overline{Y}_2^{\alpha}(t) - \eta_{21}\underline{Y}_1^{\alpha}(t)) - (\beta_{12}+\beta_{21})(\underline{d}_{10}^{\alpha} + \beta_{11}\lambda_1(t) \\ -\beta_{21}\lambda_4(t) - \eta_{12}\overline{Y}_2^{\alpha}(t) + \eta_{11}\underline{Y}_1^{\alpha}(t)))]/[-(\beta_{12}+\beta_{21})^2 + 4\beta_{11}\beta_{22}]; \end{cases} \tag{5.8}$$

Now to obtain the optimal values of $\underline{Y}^{\alpha *}_i, \overline{Y}^{\alpha *}_i, \underline{S}^{\alpha *}_i, \overline{S}^{\alpha *}_i, \underline{\lambda}^{\alpha *}_i, \overline{\lambda}^{\alpha *}_i$ numerically, we have developed a algorithm by using Newton's forward-backward method in MATLAB software which is given below:

5.1. Algorithm for the solution method

First we set the variables as follows $\underline{S}_1^{\alpha} = S_1, \overline{S}_1^{\alpha} = S_2, \underline{S}_2^{\alpha} = S_3, \overline{S}_2^{\alpha} = S_4$, $\underline{Y}_1^{\alpha} = Y_1, \overline{Y}_1^{\alpha} = Y_2, \underline{Y}_2^{\alpha} = Y_3, \overline{Y}_2^{\alpha} = Y_4$, $\underline{D}_1^{\alpha} = D_1, \overline{D}_1^{\alpha} = D_2, \underline{D}_2^{\alpha} = D_3, \overline{D}_2^{\alpha} = D_4$, $\underline{\lambda}_1^{\alpha} = \lambda_1, \overline{\lambda}_1^{\alpha} = \lambda_2, \underline{\lambda}_2^{\alpha} = \lambda_3, \overline{\lambda}_2^{\alpha} = \lambda_4$, set the constants as $\underline{Y}_{10}^{\alpha} = z_1, \overline{Y}_{10}^{\alpha} = z_2, \underline{Y}_{20}^{\alpha} = z_3, \overline{Y}_{20}^{\alpha} = z_4$, and also set the functions as follows

$$\begin{cases} f_1(\underline{Y}_1^{\alpha}, \overline{Y}_1^{\alpha}, \underline{Y}_2^{\alpha}, \overline{Y}_2^{\alpha}, \underline{S}_1^{\alpha}, \overline{S}_1^{\alpha}, \underline{S}_2^{\alpha}, \overline{S}_2^{\alpha}) \longleftarrow (1-\delta_1)u_{10}\left(1 - \dfrac{\overline{Y}_1^{\alpha}}{X_{1\max}}\right) - \overline{d}_{10}^{\alpha} + \beta_{11}\underline{S}_1^{\alpha} - \beta_{12}\overline{S}_2^{\alpha} - \eta_{11}\overline{Y}_1^{\alpha} + \eta_{12}\underline{Y}_2^{\alpha}; \\ f_2(\underline{Y}_1^{\alpha}, \overline{Y}_1^{\alpha}, \underline{Y}_2^{\alpha}, \overline{Y}_2^{\alpha}, \underline{S}_1^{\alpha}, \overline{S}_1^{\alpha}, \underline{S}_2^{\alpha}, \overline{S}_2^{\alpha}) \longleftarrow (1-\delta_1)u_{10}\left(1 - \dfrac{\underline{Y}_1^{\alpha}}{X_{1\max}}\right) - \underline{d}_{10}^{\alpha} + \beta_{11}\overline{S}_1^{\alpha} - \beta_{12}\underline{S}_2^{\alpha} - \eta_{11}\underline{Y}_1^{\alpha} + \eta_{12}\overline{Y}_2^{\alpha}; \\ f_3(\underline{Y}_1^{\alpha}, \overline{Y}_1^{\alpha}, \underline{Y}_2^{\alpha}, \overline{Y}_2^{\alpha}, \underline{S}_1^{\alpha}, \overline{S}_1^{\alpha}, \underline{S}_2^{\alpha}, \overline{S}_2^{\alpha}) \longleftarrow (1-\delta_2)u_{20}\left(1 - \dfrac{\overline{Y}_2^{\alpha}}{X_{2\max}}\right) - \overline{d}_{20}^{\alpha} + \beta_{22}\underline{S}_2^{\alpha} - \beta_{21}\overline{S}_1^{\alpha} - \eta_{22}\overline{Y}_2^{\alpha} + \eta_{21}\underline{Y}_1^{\alpha}; \\ f_4(\underline{Y}_1^{\alpha}, \overline{Y}_1^{\alpha}, \underline{Y}_2^{\alpha}, \overline{Y}_2^{\alpha}, \underline{S}_1^{\alpha}, \overline{S}_1^{\alpha}, \underline{S}_2^{\alpha}, \overline{S}_2^{\alpha}) \longleftarrow (1-\delta_2)u_{20}\left(1 - \dfrac{\underline{Y}_2^{\alpha}}{X_{2\max}}\right) - \underline{d}_{20}^{\alpha} + \beta_{22}\overline{S}_2^{\alpha} - \beta_{21}\underline{S}_1^{\alpha} - \eta_{22}\underline{Y}_2^{\alpha} + \eta_{21}\overline{Y}_1^{\alpha}; \end{cases}$$

and

$$\begin{cases} g_1\left(\underline{\lambda}_1^\alpha,\overline{\lambda}_1^\alpha,\underline{\lambda}_2^\alpha,\overline{\lambda}_2^\alpha,\underline{S}_1^\alpha,\overline{S}_1^\alpha,\underline{S}_2^\alpha,\overline{S}_2^\alpha\right) \longleftarrow -\left(\eta_{11}\underline{S}_1^\alpha(t)-\eta_{22}\overline{S}_2^\alpha(t)-\underline{p}_1^\alpha+\dfrac{\overline{C}_{u1}^\alpha}{X_{1\max}}u_{10}(1-\delta_1)\right) \\ +\left(\dfrac{u_{10}(1-\delta_1)}{X_{1\max}}+\eta_{11}\right)\overline{\lambda}_1^\alpha(t)-\eta_{22}\underline{\lambda}_2^\alpha; \\ g_2\left(\underline{\lambda}_1^\alpha,\overline{\lambda}_1^\alpha,\underline{\lambda}_2^\alpha,\overline{\lambda}_2^\alpha,\underline{S}_1^\alpha,\overline{S}_1^\alpha,\underline{S}_2^\alpha,\overline{S}_2^\alpha\right) \longleftarrow -\left(\eta_{11}\overline{S}_1^\alpha(t)-\eta_{22}\underline{S}_2^\alpha(t)-\overline{p}_1^\alpha+\dfrac{\underline{C}_{u1}^\alpha}{X_{1\max}}u_{10}(1-\delta_1)\right) \\ +\left(\dfrac{u_{10}(1-\delta_1)}{X_{1\max}}+\eta_{11}\right)\underline{\lambda}_1^\alpha(t)-\eta_{22}\overline{\lambda}_2^\alpha; \\ g_3\left(\underline{\lambda}_1^\alpha,\overline{\lambda}_1^\alpha,\underline{\lambda}_2^\alpha,\overline{\lambda}_2^\alpha,\underline{S}_1^\alpha,\overline{S}_1^\alpha,\underline{S}_2^\alpha,\overline{S}_2^\alpha\right) \longleftarrow -\left(\eta_{21}\underline{S}_2^\alpha(t)-\eta_{12}\overline{S}_1^\alpha(t)-\underline{p}_2^\alpha+\dfrac{\overline{C}_{u2}^\alpha}{X_{2\max}}u_{20}(1-\delta_2)\right) \\ +\left(\dfrac{u_{20}(1-\delta_2)}{X_{2\max}}+\eta_{21}\right)\overline{\lambda}_2^\alpha(t)-\eta_{12}\underline{\lambda}_1^\alpha; \\ g_4\left(\underline{\lambda}_1^\alpha,\overline{\lambda}_1^\alpha,\underline{\lambda}_2^\alpha,\overline{\lambda}_2^\alpha,\underline{S}_1^\alpha,\overline{S}_1^\alpha,\underline{S}_2^\alpha,\overline{S}_2^\alpha\right) \longleftarrow -\left(\eta_{21}\overline{S}_2^\alpha(t)-\eta_{12}\underline{S}_1^\alpha(t)-\overline{p}_2^\alpha+\dfrac{\underline{C}_{u2}^\alpha}{X_{2\max}}u_{20}(1-\delta_2)\right) \\ +\left(\dfrac{u_{20}(1-\delta_2)}{X_{2\max}}+\eta_{21}\right)\underline{\lambda}_2^\alpha(t)-\eta_{12}\overline{\lambda}_1^\alpha; \end{cases}$$

5.1.1. Algorithm

Step 1. Start the program \\ (For two items);
Step 2. Define function y= optimal feasible solution $(Y_1, Y_2, Y_3, Y_4, S_1, S_2, S_3, S_4)$;
Step 3. Define test $= -1$; $\delta = 0.001$; $N = 10$; $T = 100$; $t = \mathit{linespace}(0, T, N+1)$; $h = \frac{1}{N}$; $h_1 = \frac{h}{2}$;
Step 4. Define $\gamma_1 = 0.7$; $\gamma_2 = 0.6$; For $m = 1$ to 4
Step 5. Define

$$S_m \longleftarrow \text{zero matrix of order } (1, N+1);$$

$$Y_m \longleftarrow \text{zero matrix of order } (1, N+1);$$

$$D_m \longleftarrow \text{zero matrix of order } (1, N+1);$$

$$\lambda_m \longleftarrow \text{zero matrix of order } (1, N+1);$$

Step 6. Define

$$\underline{Y}_m^\alpha(1) \longleftarrow Y_m;$$

Step 7. Define

$$dY_m = f_m\left(\underline{Y}_1^\alpha,\overline{Y}_1^\alpha,\underline{Y}_2^\alpha,\overline{Y}_2^\alpha,\underline{S}_1^\alpha,\overline{S}_1^\alpha,\underline{S}_2^\alpha,\overline{S}_2^\alpha\right);$$

Step 8. Define

$$d\lambda_m = g_m\left(\underline{\lambda}_1^\alpha,\overline{\lambda}_1^\alpha,\underline{\lambda}_2^\alpha,\overline{\lambda}_2^\alpha,\underline{S}_1^\alpha,\overline{S}_1^\alpha,\underline{S}_2^\alpha,\overline{S}_2^\alpha\right);$$

Step 9. Do while(test<0)

$$\text{old } S_m = S_m$$

$$\text{old } Y_m = Y_m$$

$$\text{old } \lambda_m = \lambda_m$$

For State Variables:
for $i = 1(1)N$

$$\begin{cases} k_{1m} \longleftarrow f_m(Y_1(i), Y_2(i), Y_3(i), Y_4(i), S_1(i), S_2(i), S_3(i), S_4(i)); \\ k_{2m} \longleftarrow f_m\Bigg(Y_1(i) + h_1k_{11}, Y_2(i) + h_1k_{12}, Y_3(i) + h_1k_{13}, Y_4(i) + h_1k_{14}, \\ \dfrac{S_1(i) + S_1(i+1)}{2}, \dfrac{S_2(i) + S_2(i+1)}{2}, \dfrac{S_3(i) + S_3(i+1)}{2}, \dfrac{S_4(i) + S_4(i+1)}{2}\Bigg); \\ k_{3m} \longleftarrow f_m\Bigg(Y_1(i) + h_1k_{21}, Y_2(i) + h_1k_{22}, Y_3(i) + h_1k_{23}, Y_4(i) + h_1k_{24}, \\ \dfrac{S_1(i) + S_1(i+1)}{2}, \dfrac{S_2(i) + S_2(i+1)}{2}, \dfrac{S_3(i) + S_3(i+1)}{2}, \dfrac{S_4(i) + S_4(i+1)}{2}\Bigg); \\ k_{4m} \longleftarrow f_m\Bigg(Y_1(i) + hk_{31}, Y_2(i) + hk_{32}, Y_3(i) + hk_{33}, Y_4(i) + hk_{34}, S_1(i+1), S_2(i+1), S_3(i+1), S_4(i+1)\Bigg); \end{cases}$$

$$Y_m(i+1) \longleftarrow Y_m(i) + \frac{h(k_{1m} + 2k_{2m} + 2k_{3m} + k_{4m})}{6};$$

end\\

$$\begin{cases} \lambda_1(N+1) \longleftarrow \gamma_1 S_1(N+1); \\ \lambda_2(N+1) \longleftarrow \gamma_1 S_2(N+1); \\ \lambda_3(N+1) \longleftarrow \gamma_2 S_3(N+1); \\ \lambda_4(N+1) \longleftarrow \gamma_2 S_4(N+1); \end{cases}$$

For Adjoint Variables:
for $j = 1(1)N$

$$\begin{cases} k'_{1m} \longleftarrow g_m(\lambda_1(j), \lambda_2(j), \lambda_3(j), \lambda_4(j), S_1(j), S_2(j), S_3(j), S_4(j)); \\ k'_{2m} \longleftarrow g_m\big(\lambda_1(j) + h_1k_{11}, \lambda_2(j) + h_1k_{12}, \lambda_3(j) + h_1k_{13}, \lambda_4(j) + h_1k_{14}, \\ \frac{S_1(j)+S_1(j-1)}{2}, \frac{S_2(j)+S_2(j-1)}{2}, \frac{S_3(j)+S_3(j-1)}{2}, \frac{S_4(j)+S_4(j-1)}{2}\big); \\ k'_{3m} \longleftarrow g_m\big(\lambda_1(j) + h_1k_{21}, \lambda_2(j) + h_1k_{22}, \lambda_3(j) + h_1k_{23}, \lambda_4(j) + h_1k_{24}, \\ \frac{S_1(j)+S_1(j-1)}{2}, \frac{S_2(j)+S_2(j-1)}{2}, \frac{S_3(j)+S_3(j-1)}{2}, \frac{S_4(j)+S_4(j-1)}{2}\big); \\ k'_{4m} \longleftarrow g_m\big(\lambda_1(j) + hk_{31}, \lambda_2(j) + hk_{32}, \lambda_3(j) + hk_{33}, \lambda_4(j) + hk_{34}, S_1(j+1), S_2(j+1), S_3(j+1), S_4(j+1)\big); \end{cases}$$

$$\lambda_m(j-1) \longleftarrow \lambda_m(j) + \frac{h(k_{1m} + 2k_{2m} + 2k_{3m} + k_{4m})}{6};$$

end\\
All Control
Calculate all S_m \\ see (5.8)
end \\

Step 10.

$$S_m \longleftarrow 0.5(S_m + \text{old } S_m);$$

Step 11.

$$\begin{cases} \text{temp}_1 = \delta(\text{sum}(\text{abs}(S_1)) - \text{sum}(\text{abs}(\text{old } S_1 - S_1))); \\ \text{temp}_2 = \delta(\text{sum}(\text{abs}(S_2)) - \text{sum}(\text{abs}(\text{old } S_2 - S_2))); \\ \text{temp}_3 = \delta(\text{sum}(\text{abs}(S_3)) - \text{sum}(\text{abs}(\text{old } S_3 - S_3))); \\ \text{temp}_4 = \delta(\text{sum}(\text{abs}(S_4)) - \text{sum}(\text{abs}(\text{old } S_4 - S_4))); \\ \text{temp}_5 = \delta(\text{sum}(\text{abs}(Y_1)) - \text{sum}(\text{abs}(\text{old } Y_1 - Y_1))); \\ \text{temp}_6 = \delta(\text{sum}(\text{abs}(Y_2)) - \text{sum}(\text{abs}(\text{old } Y_2 - Y_2))); \\ \text{temp}_7 = \delta(\text{sum}(\text{abs}(Y_3)) - \text{sum}(\text{abs}(\text{old } Y_3 - Y_3))); \\ \text{temp}_8 = \delta(\text{sum}(\text{abs}(Y_4)) - \text{sum}(\text{abs}(\text{old } Y_4 - Y_4))); \\ \text{temp}_9 = \delta(\text{sum}(\text{abs}(\lambda_1)) - \text{sum}(\text{abs}(\text{old } \lambda_1 - \lambda_1))); \\ \text{temp}_{10} = \delta(\text{sum}(\text{abs}(\lambda_2)) - \text{sum}(\text{abs}(\text{old } \lambda_2 - \lambda_2))); \\ \text{temp}_{11} = \delta(\text{sum}(\text{abs}(\lambda_3)) - \text{sum}(\text{abs}(\text{old } \lambda_3 - \lambda_3))); \\ \text{temp}_{12} = \delta(\text{sum}(\text{abs}(\lambda_4)) - \text{sum}(\text{abs}(\text{old } \lambda_4 - \lambda_4))); \end{cases}$$

Step 12.

$$\begin{cases} \text{test}_1 = \min(\text{temp}_1, \min(\text{temp}_2, \text{temp}_3)); \\ \text{test}_2 = \min(\text{temp}_4, \min(\text{temp}_5, \text{temp}_6)); \\ \text{test}_3 = \min(\text{temp}_7, \min(\text{temp}_8, \text{temp}_9)); \\ \text{test}_4 = \min(\text{temp}_{10}, \min(\text{temp}_{11}, \text{temp}_{12})); \\ \text{test}_5 = \min(\text{test}_1, \min(\text{test}_2, \text{test}_3)); \\ \text{test} = \min(\text{test}_4, \text{test}_5); \end{cases}$$

Step 13. Print S_m, Y_m, λ_m, D_m.
end\\

6. Numerical results

To illustrate the proposed production inventory model numerically, we consider the following input data given in Table 1. For these input data, we see that $\underline{X}_i^{\alpha} = \underline{Y}_i^{\alpha}$ and $\overline{X}_i^{\alpha} = \overline{Y}_i^{\alpha}$. The results for different variables of substitute items are showed in Table 2 and the respective figures are showed from Figures 1 and 7. It has been observed that the optimal selling price for first item lie between (\$242.3, \$244.7) and for second item lie between (\$206.9, \$208.3). And also the stock level for first item lie between (0.1, 5.3) and for second item lie between (1.7, 6.1). And the optimum value corresponding the two substitute items lie between (\$533800, \$684320).

6.1. Input data for substitute items

The input data for inventory parameters are given in Table 1.

TABLE 1. Input data for substitute items.

Parameters	Values	Parameters	Values	Parameters	Values	Parameters	Values	Parameters	Values
δ_1	0.03	δ_2	0.025	$\underline{d}^{\alpha}_{10}$	10	$\overline{d}^{\alpha}_{10}$	10.8	$\underline{d}^{\alpha}_{20}$	11
$\overline{d}^{\alpha}_{20}$	11.8	$X_{1\max}$	29	$X_{2\max}$	31	u_{10}	13	u_{20}	12
β_{11}	0.35	β_{12}	0.15	β_{21}	0.10	β_{22}	0.21	η_{11}	0.19
η_{12}	0.32	η_{21}	0.23	η_{22}	0.11	$\underline{C}^{\alpha}_{u_1}$	3.8	$\overline{C}^{\alpha}_{u_1}$	4.8
$\underline{C}^{\alpha}_{u_2}$	2.4	$\overline{C}^{\alpha}_{u_2}$	3.4	$\underline{C}^{\alpha}_{h_1}$	4.4	$\overline{C}^{\alpha}_{h_1}$	5.4	$\underline{C}^{\alpha}_{h_2}$	4.81
$\overline{C}^{\alpha}_{h_2}$	5.8	$\underline{X}^{\alpha}_{10}$	5.4	$\overline{X}^{\alpha}_{10}$	5.6	$\underline{X}^{\alpha}_{20}$	4.4	$\overline{X}^{\alpha}_{20}$	4.6

TABLE 2. Values of the optimal selling price, production rate, stock rate and demand rate at the time t

t	0	10	20	30	40	50	60	70	80	90	100
$\underline{S}^{\alpha*}_1(\$)$	189.8	193.1	197.5	202	206.8	211.7	217	222.5	228.3	234.4	242.2
$\overline{S}^{\alpha}_1(\$)$	189.7	194.2	198.4	203.3	208.4	213.7	219.2	225	231	237.4	244.7
$\underline{S}^{\alpha*}_2(\$)$	178.6	181.6	184.8	187.8	190.7	193.6	196.4	199.2	201.9	204.6	206.9
$\overline{S}^{\alpha*}_2(\$)$	172.9	176.7	180.4	184.1	187.7	191.3	194.8	198.2	201.6	204.9	208.3
$\underline{U}^{\alpha*}_1$	9.2	9.7	10.1	10.4	10.5	10.6	10.6	10.4	10.2	9.8	9.3
$\overline{U}^{\alpha*}_1$	9.3	10.1	10.8	11.4	11.9	12.3	12.6	12.8	12.96	12.99	12.92
$\underline{U}^{\alpha*}_2$	9.6	9.4	9.3	9.2	9.1	8.9	8.9	8.9	8.8	8.8	8.8
$\overline{U}^{\alpha*}_2$	9.7	9.7	9.8	9.9	10	10.1	10.3	10.4	10.6	10.8	11.1
$\underline{X}^{\alpha*}_1$	5.4	4.2	3.2	2.3	1.6	1	0.55	0.24	0.06	0.02	0.11
$\overline{X}^{\alpha*}_1$	5.6	4.9	4.3	3.8	3.6	3.5	3.5	3.7	4.1	4.6	5.3
$\underline{X}^{\alpha*}_2$	4.4	4.3	4.2	4.0	3.8	3.6	3.3	3	2.6	2.2	1.7
$\overline{X}^{\alpha*}_2$	4.6	4.9	5.2	5.4	5.6	5.7	5.8	6	6.1	6.1	6.1
$\underline{D}^{\alpha*}_1$	17.2	16.4	15.5	14.5	13.4	12.3	10.9	9.6	8.1	6.4	4.6
$\overline{D}^{\alpha*}_1$	21.4	20.4	19.3	18.1	16.8	15.5	14.1	12.6	11	9.3	7.4
$\underline{D}^{\alpha*}_2$	6.2	6.7	7.1	7.5	7.9	8.4	8.8	9.3	9.8	10.3	10.9
$\overline{D}^{\alpha*}_2$	10	10.2	10.5	10.7	11	11.3	11.7	12.1	12.6	13.1	13.7

7. DISCUSSION

For two substitute items we have got four hands in each figures for lower and upper cases. Here Figure 1 represents the stock level and Figure 2 represents the production rate in a given time of two substitute items. From these two figures we observe that production of the items is stock dependent. Also, here another two figures Figues 3 and 4 are represented the demand rate and selling price in given time of the two substitute items respectively. In this paper, we consider that the demands are dependent on selling prices and stock levels. But from the figures Figures 1 and 4, we see that demands are more effected by selling price. Also from Figures 5 and 6 we assure that the demands are depended on own price. So for substitute items own price of each items play as important role to control the demand. Moreover own stock has more positive effect to the demand function with respect to the negative effect of substitute item. From the above discussion, we get that the selling price becomes sole responsible for its demand. What matters is the selling price of any product. Lesser the selling price more the demand of that product irrespective of stock. So the model in which the demand is dependent both on price and stock is more realistic than any other model.

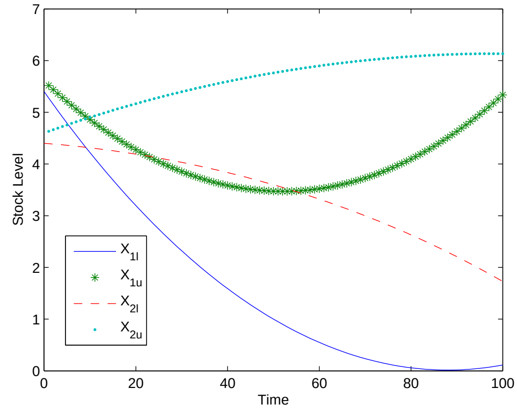

FIGURE 1. Stock level verses time for substitute items.

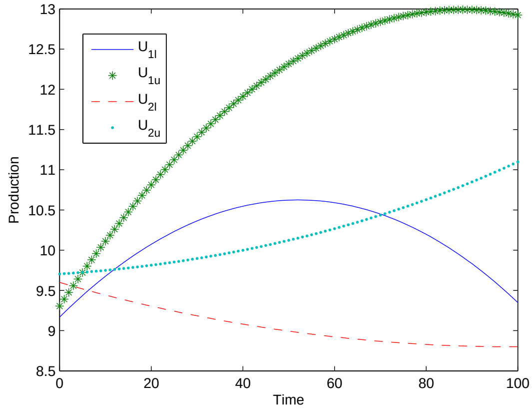

FIGURE 2. Production verses time for substitute item.

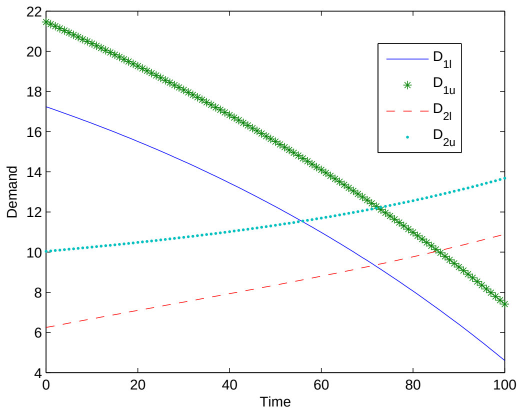

FIGURE 3. Demand verses time for substitute items.

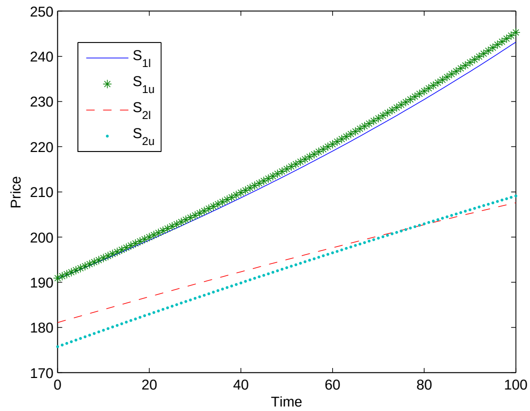

FIGURE 4. Selling price verses time for substitute item.

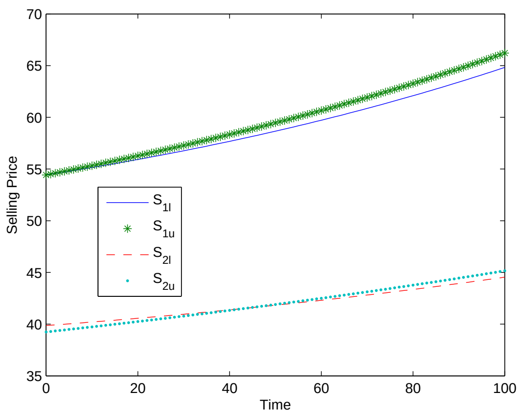

FIGURE 5. Selling price verses time for substitute items when the co-efficient of own price is given.

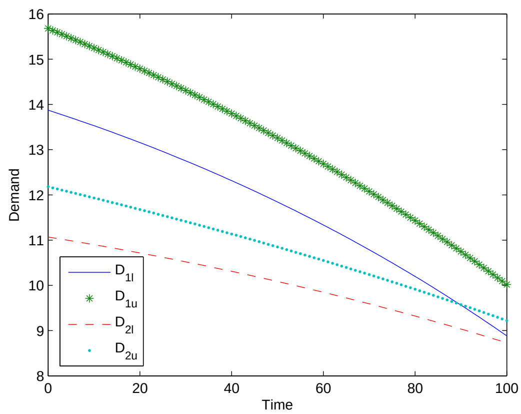

FIGURE 6. Demand verses time for substitute items when the co-efficient of own price is given.

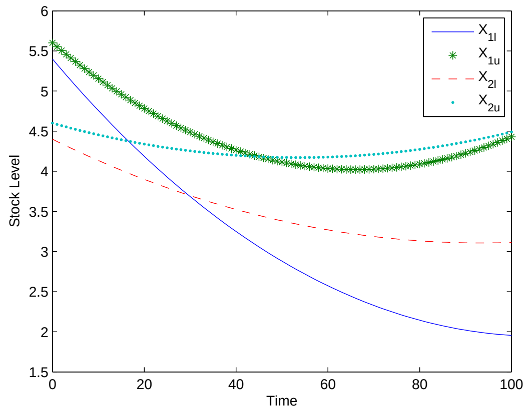

FIGURE 7. Stock verses time for substitute items when the co-efficient of own price is given.

8. Conclusion

A production inventory control problem for substitute item has been developed in uncertain environment in which the demand function is dependent on selling price as well as stock level. Here it is shown that the selling price is more effective to control the demand. Considering the state variables as fuzzy the proposed inventory model is represented as a fuzzy differential system. To transform the system into equivalent deterministic one we use the "e" and "g" operators which is very recent and effective technique with respect to interval mathematics.

Moreover, the proposed method can be extended for other dynamical production inventory control problem like imperfect production problem, deteriorating production problem and other. The method ("e" and "g") can be used to check the stability of any imprecise dynamical model.

References

[1] M.Z. Ahmad, M.K. Hasan and B. De Baets, Analytical and numerical solutions of fuzzy differential equations. *Inf. Sci.* **236** (2013) 156–167.

[2] A. Bartoszewicz and P. Lesniewski, Reaching Law Approach to the Sliding Mode Control of Periodic Review Inventory Systems. *IEEE Trans. Automation Sci. Eng.* **11** (2014) 810–817.

[3] L. Chen, Y. Chen and Z. Pang, Dynamic Pricing and Inventory Control in a Make-to-Stock Queue With Information on the Production Status. *IEEE Trans. Automation Sci. Eng.* **8** (2011) 361–373.

[4] A. Chernev, Product Assortment and Consumer Choice: An Interdisciplinary Review. In Vol. 6 (2011) 1–61.

[5] M.S. El Naschie, From experimental quantum optics to quantum gravity via a fuzzy kahler manifold. *Chaos Solitons Fract* **25** (2005) 969–977.

[6] A.F. Jameel, A.I.M. Ismail and F. Mabood, Optimal homotopy asymptotic method for solving nth order linear fuzzy initial value problems. *J. Assoc. Arab Univers. Basic and Appl. Sci.* **21** (2016) 77–85.

[7] X.-C. Jia, X.-B. Chi, Q.-L. Han and N.-N. Zheng, Event-triggered fuzzy H_∞ control for a class of nonlinear networked control systems using the deviation bounds of a synchronous normalized membership functions. *Inf. Sci.* **259** (2014) 100–117.

[8] S. Hazari, K. Maity, J. K. Dey and S. Kar, Optimal dynamic production and pricing for reliability depended imperfect production with inventory-level-dependent demand in uncertain environment. *J. Uncertainty Anal. Appl.* **2** (2015) 1–17.

[9] A. Katsifou, R.W. Seifert and J. Tancrez, Joint productassortment,inventory and price optimization to attract loyal and non-loyal customers. *Omega* **46** (2014) 36–50.

[10] M. Khan and M.Y. Jaber, Optimal inventory cycle in a two-stage supply chain incorporating imperfect items from suppliers. *Inter. J. Operat. Res.* **10** (2011) 442–457.

[11] A. De, K. Maity and M. Maiti, Stability analysis of a combined project of fish, broiler and ducks: Dynamical system in imprecise envioronment. *Inter. J. Biomath.* **8** (2015) 1550067.

[12] C. Krishnamoorthi and S. Panayappan, An EPQ model for an imperfect production system with rework and shortages. *Int. J. Operat. Res.* **17** (2013) 104–124.

[13] K. Maity and M. Maiti, Optimal inventory policies for deteriorating complementary and substitute items. *Inter. J. Syst. Sci.* **40** (2009) 267–276.

[14] K. Maity and M. Maiti, Inventory of deteriorating complementary and substitute items with stock dependent demand. *Amer. J. Math. Manag. Sci.* **25** (2005) 83–96.

[15] M. Mazandarani and M. Najariyan, A note on "A class of linear differential dynamical systems with fuzzy initial condition". *Fuzzy Sets Syst.* **265** (2015) 121–126.

[16] M. Najariyan and M.H. Farahi, Optimal control of fuzzy linear controlled system with fuzzy initial conditions. *Iranian J. Fuzzy Syst.* **10** (2013) 21–35.

[17] A. Omer and O. Omer, A pray and predator model with fuzzy initial values. *J. Math. Stat.* **41** (2013) 387–395.

[18] E.L. Porteus, Optimal lot sizing, process quality improvement and setup cost reduction. *Operat. Res.* **34** (1986) 137–144.

[19] L.S. Pontryagin and V.G. Boltyanski, The Mathematical Theory Optimal Process. *Inter. Sci.*, New York (1962).

[20] D. Panda, S. Kar, K. Maity and M. Maity, A single period inventory model with imperfect production and stochastic demand under chance and imprecise constraints. *Eur. J. Operat. Res.* **188** (2008) 121–139.

[21] S. Ramezanzadeh and A. Heydari, Optimal control with fuzzy chance constraint. *Iranian J. fuzzy Syst.* **8** (2011) 35-43.

[22] C.K. Sivashankari and S. Panayappan, Production inventory model with reworking of imperfect production. *Inter. J. Manag. Sci. Eng. Manag.* **9** (2014) 9–20.

[23] D. Filev and P. Angelove, Fuzzy optimal control. *Fuzzy Sets and Syst.* **47** (1992) 151–56.

[24] D. Yadav, S.R. Singh and R. Kumari, Three-stage supply chain coordination under fuzzy random demand and production rate with imperfect production process. *Int. J. Operat. Res.* **16** (2013) 421–447.

[25] P.K. Sahu and S. Ray, Two-dimensional Legendre wavelet method for the numerical solutions of fuzzy integro-differential equations. *J. Intell. Fuzzy Syst.* **28** (2015) 1271–1279.

[26] S.S. Sana, A production-inventory model of imperfect quality products in a three-layer supply chain. *Decision Support System* **50** (2011) 539–547.

[27] Z-P. Wang and H-N. Wu, Finite dimensional guaranteed cost sampled data fuzzy control for a class of nonlinear distributed parameter systems. *Information Sci.* **327** (2016) 21–39.

[28] Y. Zhu, A fuzzy optimal control model. *J. Uncertain Syst.* **3** (2009) 270–279.