RAIRO-Oper. Res. 53 (2019) 1763–1773
https://doi.org/10.1051/ro/2018076

RAIRO Operations Research
www.rairo-ro.org

ON THE STAR FOREST POLYTOPE FOR TREES AND CYCLES

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Abstract. Let $G = (V, E)$ be an undirected graph where the edges in E have non-negative weights. A star in G is either a single node of G or a subgraph of G where all the edges share one common end-node. A star forest is a collection of vertex-disjoint stars in G. The weight of a star forest is the sum of the weights of its edges. This paper deals with the problem of finding a Maximum Weight Spanning Star Forest (MWSFP) in G. This problem is NP-hard but can be solved in polynomial time when G is a cactus [Nguyen, *Discrete Math. Algorithms App.* **7** (2015) 1550018]. In this paper, we present a polyhedral investigation of the MWSFP. More precisely, we study the facial structure of the star forest polytope, denoted by $SFP(G)$, which is the convex hull of the incidence vectors of the star forests of G. First, we prove several basic properties of $SFP(G)$ and propose an integer programming formulation for MWSFP. Then, we give a class of facet-defining inequalities, called M-tree inequalities, for $SFP(G)$. We show that for the case when G is a tree, the M-tree and the nonnegativity inequalities give a complete characterization of $SFP(G)$. Finally, based on the description of the dominating set polytope on cycles given by Bouchakour *et al.* [*Eur. J. Combin.* **29** (2008) 652–661], we give a complete linear description of $SFP(G)$ when G is a cycle.

Mathematics Subject Classification. 90C27, 90C10, 05Cxx.

Received June 1, 2015. Accepted September 4, 2018.

1. Introduction

Given an undirected graph $G = (V, E)$ where $n = |V|$ and $m = |E|$, a *star* in G is either a single node of G or a subgraph of G where every edge shares one common end-node. The latter is called the *center* of the star when the star is not reduced to a single node. If the star is a single edge, then any of its end-nodes can be designated as the center. A *star forest* is a collection of vertex-disjoint stars in G. An *edge dominating set* in G is an edge subset $F \subseteq E$ such that for any edge e in G either $e \in F$ or e shares at least one common end-node with some edge in F. A *dominating set* in G is a node subset $S \subseteq V$ such that for any node $u \in V$ either $u \in S$ or u is neighbor with some node in S. We suppose that the edges in G have non-negative weights (note that

Keywords. Combinatorial optimization, polyhedral combinatorics, star forest, facility location, dominating set.

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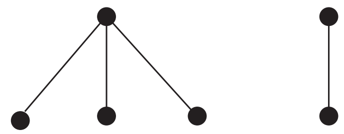

FIGURE 1. A star forest of weight 4 with weights 1 on the edges.

the unweighted case can be seen as a special weighted case when weights are 0 or 1), then the weight of a star forest or an edge dominating set is the sum of the weights of its edges. The *Maximum Weight spanning Star Forest Problem* (MWSFP) is to find a star forest spanning the nodes of G of maximum weight. The *Minimum Weight Edge Dominating Problem* (MWEDP) is to find an edge dominating set in G of minimum weight. If the nodes are weighted, the weight of a dominating set is the sum of the weights of its nodes. The *Minimum Weight Dominating Set Problem* (MWDSP) is to find a minimum weight dominating set in G. The two last problems are well-known to be NP-hard. They have been the subject of many works in the literature [20], [13]. The MWSFP, however, is a recent problem which has been introduced by Nguyen *et al.* in [17]. It has applications in several areas, especially in computational biology [17] and automobile industry [1]. In [17], the authors show the NP-hardness of MWSFP by observing that in a *maximal star forest* F (a maximal star forest is a star forest to which no more edge can be added), the set of the centers of the stars in F is a dominating set of G. Conversely, for any dominating set S we can build a maximal star forest with centers as the nodes belonging to S. Thus, given a maximal star forest F, there exists a dominating set S such that $|S| = |V| - |F|$ and vice versa. Hence the case of 0/1 weights of the MWSFP is NP-hard by a reduction from the of 0/1 weights case of the MWDP. In [17], the authors also give a linear time algorithm to solve the MWSFP when G is a tree and a $\frac{1}{2}$-approximation algorithm for the general case. Since then, the MWSFP has been intensively investigated, in particular for the unweighted version. Nguyen *et al.* [17] prove that the problem is APX-hard by presenting an explicit inapproximability bound of 259/260, and present a combinatorial 0.6-approximation algorithm for the unweighted MWSFP. Polynomial-time algorithms are presented for special classes of graphs such as planar graphs and trees in the same paper. Chen *et al.* [12] present a better approximation algorithm with ratio 0.71 for unweighted MWSFP. Later, Athanassopoulos *et al.* [2] improve this approximation ratio to 0.803 by using the fact that the problem is a special case of the complementary set cover problem. Interesting generalizations including node-weighted and edge-weighted versions of the MWSFP have also been considered. In [12, 17] the authors present approximation algorithms and APX-hardness results for these problems as well. Stronger inapproximability results for these problems recently appeared in [11, 14]. For the weighted version, Nguyen [18] has given a linear time algorithm for solving the MWSFP when G is a cactus.

Let $SFP(G)$ (respectively $EDP(G)$) be the convex hull of the incidence vectors of the star forests (respectively the edge dominating sets) in G. Let $\mathbb{R}^n$ be the real space indexed by the nodes in V. Let D be any dominating set in G, let $\chi(D) \in \mathbb{R}^n$ be the incidence vector of D, defined as

$$\chi(D)_v = \begin{cases} 1 \text{ if } v \text{ is a nodes in } V \text{ and } v \in D \\ 0 \text{ otherwise.} \end{cases}$$

Let $DP(G)$ be the convex hull of the incidence vectors of the dominating sets in G. To the best of our knowledge, no polyhedral investigation has been done for $SFP(G)$ and $EDP(G)$ though some integer formulations have been used in approximation algorithms for the MWSFP and the MWEDP. There are, however, several works on $DP(G)$, in particular, Saxena [19] has given a complete description for $DP(G)$ when G is a tree, and Bouchakhour *et al.* [8] have given a complete description for $DP(G)$ when G is a cycle. In [16], Mahjoub has given a complete description of $DP(G)$ in threshold graphs. And in [9], Bouchakour and Mahjoub have studied compositions for the polytope $DP(G)$ in graphs that decompose by one-node cutsets.

In this paper, we present a polyhedral investigation of the MWSFP for trees and cycles. More precisely, we study the facial structure of the star forest polytope. In the first part of the paper, we give a complete characterization of $SFP(G)$ when G is a tree, which is obtained by projection of a simple extended formulation issued from the work of Baïou and Barahona [3] on the uncapacitated facility location polytope. Also, we show that the facet-defining inequalities for $SFP(G)$ when G is a tree can be generalized to valid inequalities for $SFP(G)$ when G is a arbitrary graph. These inequalities define facets for $SFP(G)$ under certain conditions, and can be separated in polynomial time.

In the second part of the paper, we give a complete description for $SFP(G)$ when G is a simple cycle C. More precisely, we establish the relation between spanning star forests and dominating sets when the graph is a simple cycle C and give a complete linear description for $SFP(C)$ based on the one for $DP(C)$ given by Bouchakhour *et al.* [8].

The paper is organized as follows. In the next section, we describe some properties of $SFP(G)$ and give an integer programming formulation for the MWSFP. In section 3, we introduce a class of valid inequalities, called the M-tree inequalities, for $SFP(G)$ and give a complete description for $SFP(G)$ when G is a tree. In Section 4, we present a complete description for $SFP(G)$ when G is a cycle.

In the rest of this section, we give some notations that will be used in the paper. For $x \in \mathbb{R}^m$, given any $F \subseteq E$, we let $x(F)$ denote $\sum_{e\in F} x_e$. For $x \in \mathbb{R}^n$, given a set $S \subseteq V$, we let $x(S)$ denote $\sum_{v\in S} x_v$. Given a set of vertices S, we denote by $E(S)$ the set of edges with both ends belonging to S. Let $v \in V$, *the neighborhood* of v, denoted by $N(v)$, is the vertex set consisting of v and the nodes which are adjacent to v. Given any edge subset $F \subseteq E$, let $V(F)$ denote the set of the end-nodes of the edges in F. We call a 3-*path* a simple path having 3 edges in G and a 3-*cycle* a triangle in G. Let $\mathcal{P}_4$ (respectively $\mathcal{C}_3$) denote the collection of the 3-paths (resp. 3-cycles) in G.

2. Basic properties of $SFP(G)$ and integer programming formulation for the MWSP

2.1. Basic properties of $SFP(G)$.

The following remark is about zero vector $\mathbf{0} \in \mathbb{R}^m$ which is the incidence vector associated with the single node star forests.

Remark 2.1. The zero vector $\mathbf{0} \in \mathbb{R}^m$ is an extreme point of $SFP(G)$.

Proof. We can see that $\mathbf{0}$ is the incidence vector associated with the single node star forests and as for any $x \in SFP(G)$ and any $e \in E$, $x_e \geq 0$, $\mathbf{0}$ is an extreme point of $SFP(G)$. □

Hence, $SFP(G)$ is a polytope pointed at $\mathbf{0}$. Moreover, the following theorem shows that $SFP(G)$ is full dimensional.

Theorem 2.2. *$SFP(G)$ is a full dimensional polytope, i.e. $dim(SFP(G)) = m$.*

Proof. Suppose that the incidence vectors of all the star forests in G satisfy some equality $\alpha^t x = \beta$. As $\alpha^t \mathbf{0} = \beta$, $\beta = 0$. As any edge $e \in E$ is a star forest in G, we have $\alpha_e = \beta = 0$ for all $e \in E$. □

Theorem 2.3. *All the facet-defining inequalities of $SFP(G)$, that are different from $x_e \geq 0$ for some $e \in E$, are of the form $a^t x \leq b$ with $a \in \mathbb{R}^m_+$ and $b \geq 0$ scalar.*

Proof. Let $a^t x \leq b$ be any facet-defining inequality for $SFP(G)$ which is not $x_e \geq 0$ for some $e \in E$. As $a^t\mathbf{0} \leq b$, we have $b \geq 0$. Suppose that for some edge e, $a_e < 0$. As $a^t x \leq b$ defines a facet different from $x_e \geq 0$, there exists a star forests F in G containing e such that $a^t \chi^F = b$. As $F' = F \setminus \{e\}$ is also a star forest, we have $a^t \chi^{F'} = a^t \chi^F - a_e = b - a_e > b$. This contradicts the fact that $a^t x \leq b$ is valid for $SFP(G)$. □

Given $a^t x \leq b$ any facet-defining inequality of $SFP(G)$, *the support graph* of $a^t x \leq b$ is the subgraph $G_a = (V_a, E_a)$ of G induced by the edges $e \in E$ such that $a_e > 0$. A *tight star forest* F with respect to $a^t x \leq b$ is a star forest with which the associated incidence vector satisfies $a^t x \leq b$ at equality. A star forest F is *maximal* with respect to an edge subset $E' \subseteq E$ if for any edge $e \in E' \setminus F$, $F \cup \{e\}$ is not anymore a star forest.

Theorem 2.3 implies the following corollary.

Corollary 2.4. *Given $a^t x \leq b$ any facet-defining inequality of $SFP(G)$ with $G_a = (V_a, E_a)$ its support graph, all the tight star forests with respect to $a^t x \leq b$ are maximal with respect to E_a.*

Lemma 2.5. *Let $\bar{G} = (\bar{V}, \bar{E})$ be any induced subgraph of G, then any facet-defining inequality for $SFP(\bar{G})$ also defines a facet for $SFP(G)$.*

Proof. First notice that the theorem trivially holds for the trivial inequalities $x(e) \leq 1$ for all $e \in \bar{E}$, since these inequalities define facets for $SFP(G)$ for any graph G. Let I be any facet-defining inequality for $SFP(\bar{G})$, different from $x(e) \leq 1$ for all $e \in E$. Then I defines a facet for $SFP(G)$ if we are able to show that for any edge $ij \in E \setminus \bar{E}$, there always exists a tight star forest F in $\bar{G}$ with respect to I such that $F \cup \{ij\}$ is also star forest in G. Suppose that for some edge $ij \in E \setminus \bar{E}$, no such F exists. This implies that for every tight star forest F in $\bar{G}$ w.r.t I, $F \cup \{ij\}$ is not a star forest in G. In this case, exactly one node i or j, say i, belongs to $\bar{V}$. Moreover, as $\bar{G}$ is an induced subgraph of G, every tight forest F in $\bar{G}$ w.r.t I should contain an edge ik such that i is of degree 1 in F and k is of degree at least 2 in F. Otherwise, $F \cup \{ij\}$ would be a star forest of G. Then every tight star forest F in $\bar{G}$ w.r.t I also satisfies $x(\delta_{\bar{G}}(i)) = 1$. This, together with Theorem 2.2, implies that I should be $x(\delta_{\bar{G}}(i)) \leq 1$. Since I is different from $x(e) \leq 1$ for all $e \in E$, i should be of degree at least 2 in $\bar{G}$. But then $x(\delta_{\bar{G}}(i)) \leq 1$ is not valid for $SFP(\bar{G})$, a contradiction. □

2.2. Integer programming formulation for the MWSFP

In this subsection, we give an integer programming formulation for MWSFP. First we state the following lemma.

Lemma 2.6. *A graph is a star forest iff it does not contains 3-paths and 3-cycles.*

Proof. It can be immediately verified from the definition of a star forest. □

Let us consider the following integer program.

$$
(IP) \qquad
\begin{aligned}
& \max \ c^T x \\
& \text{s.t.} \\
& x(P) \leq 2 && \text{for all } P \in \mathcal{P}_4 && (2.1)\\
& x(C) \leq 2 && \text{for all } P \in \mathcal{C}_3 && (2.2)\\
& 0 \leq x_e \leq 1 && \text{for all } e \in E && (2.3)\\
& x \text{ integer}
\end{aligned}
$$

Inequalities (2.1), called the *3-path inequalities*, state the fact that a star forest can only take at most 2 edges in a 3-path. Similarly, inequalities (2.2), called the *3-cycle inequalities*, state the fact that a star forest can only take at most 2 edges in a 3-cycle. Inequalities (2.3) are the *trivial inequalities*.

Theorem 2.7. *(IP) is equivalent to the MWSFP.*

Proof. It is clear that by inequalities (2.1) and (2.2) in a solution of (IP) there is neither 3-paths and nor 3-cycles. By Lemma 2.6, this solution represents a star forest. □

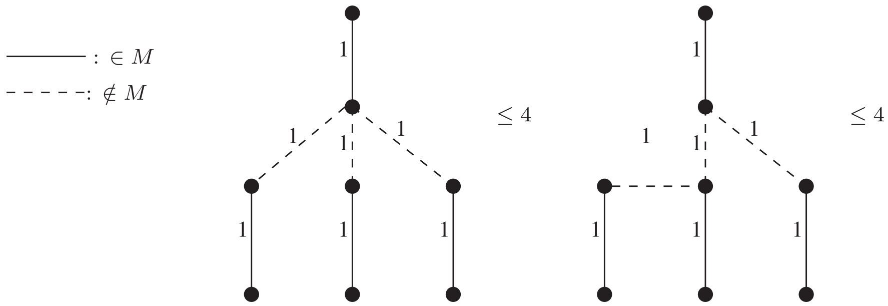

FIGURE 2. A M-tree inequalities associated with a M-matching M of cardinality 4.

3. M-TREE INEQUALITIES AND COMPLETE DESCRIPTION OF SFP(G) IN TREES

In this section, we shall give a complete description of $SFP(G)$ in trees. For this, let us first describe a class of inequalities which are valid for any graph, not only for trees.

3.1. M-tree inequalities

In this subsection, G is an arbitrary graph (not necessarily a tree).

Definition 3.1 (M-tree)**.** A M-tree τ is a tree in which every non-pendant node is connected to exactly one pendant node (leaf).

Given a M-tree τ, let us call *M-matching of* τ, the set of the edges incident to the leaves of τ.

Definition 3.2 (M-tree inequality)**.** The M-tree inequality associated with a M-tree τ is the inequality $x(\tau) \leq |M|$ where M is the M-matching of τ.

We can remark that the M-tree inequalities generalize inequalities $x_e \leq 1$ for $e \in E$ and the 3-path inequalities. These are M-tree inequalities with $|M| = 1$ and $|M| = 2$, respectively.

Theorem 3.3. *The M-tree inequalities define facets for $SFP(G)$.*

Proof. Let τ be any M-tree and M the M-matching of τ, let us consider the corresponding M-tree inequality

$$x(\tau) \leq |M|. \tag{3.1}$$

Let us first prove the validity. Let F be any star forest of G. We will prove the validity by showing that $|F \cap \tau| \leq |M|$. If F only contains the edges in M, then $|F \cap \tau| \leq |M|$. If F only contains edges in $\tau \setminus M$ then $|F \cap \tau| \leq |\tau \setminus M|$ and, from the definition of a M-tree, $|\tau \setminus M| = |M| - 1 < |M|$. Now suppose that F contains edges in both M and $\tau \setminus M$. If $F \cap \tau$ is a matching then $|F \cap \tau| \leq |M|$ since M covers all the nodes of τ. So suppose that $F \cap \tau$ is not a matching. Thus, $F \cap \tau$ should contain a (sub)star S with, say u_2, as center, which contains at least two edges in F: one edge, say u_1u_2, which belongs to M and one other, say u_2v_2, which belongs to $\tau \setminus M$. As v_2 should be also covered by M, there exists an edge $v_1v_2 \in M$. As u_1u_2 and u_2v_2 are in F, $v_1v_2 \notin F$. Moreover, by definition of M-tree, v_1 should be of degree 1 in τ. It follows that each edge in F which belongs to $\tau \setminus M$ (*e.g.* u_2v_2) correspond exactly to another edge (*e.g.* v_1v_2) in $M \setminus F$. Hence, $|F \cap (\tau \setminus M)| \leq |M \setminus F|$ which implies $|(F \cap (\tau \setminus M)) \cup (F \cap M)| \leq |(M \setminus F) \cup (F \cap M)| = |M|$. As $M \subset \tau$, $|(F \cap (\tau \setminus M)) \cup (F \cap M)| = |F \cap \tau|$. Thus, $|F \cap \tau| \leq |M|$.

Let us prove now that (3.1) defines a facet for $SFP(G)$. Suppose that there exists a facet-defining inequality $\alpha^t x \leq \beta$ for $SFP(T)$ such that all the star forests satisfying (3.1) at equality, satisfy also $\alpha^t x \leq \beta$ at equality.

Since by Theorem 2.2, $SFP(T)$ is full dimensional, it suffices to show that $\alpha^t x \leq \beta$ is a positive multiple of (3.1).

Let us remark that M is a star forest satisfying (3.1) at equality and hence also statisfies $\alpha^t x \leq \beta$ at equality, *i.e.* $\alpha(M) = \beta$. For any edge $e \in E \setminus \tau$, we can see that $M \cup \{e\}$ is also star forest satisfying (3.1) at equality. Hence, $\alpha(M \cup \{e\}) = \beta$. This implies that $\alpha_e = 0$ for all $e \in E \setminus \tau$.

Let u_2v_2 be any edge in $\tau \setminus M$ and u_1u_2 and v_1v_2 be the edges in M incident to u_2 and v_2 respectively. We can see that M, $(M \setminus \{u_1u_2\}) \cup \{u_2v_2\}$ and $(M \setminus \{v_1v_2\}) \cup \{u_2v_2\}$ are all star forests satisfying (3.1) at equality. This implies that $\alpha_{u_1u_2} = \alpha_{u_2v_2} = \alpha_{v_1v_2}$. If we extend this to all the edges in $\tau \setminus M$, we obtain that $\alpha_e = \alpha_{e'}$ for all $e, e' \in \tau$. Hence, $\alpha^t x \leq \beta$ is a positive multiple of (3.1), which ends the proof of the theorem. □

3.2. Complete description of SFP(G) in trees

From now on and throughout this section, G will be a tree denoted by T.

Proposition 3.4. *All the maximal star forests with respect to a M-tree τ are of cardinality $|M|$ where M is the M-matching of τ.*

Proof. Let S be any maximal star forest in τ. Let $m = |M|$ then $|V(\tau)| = 2m$ and $|\tau| = 2m - 1$. By the validity of M-tree inequalities, we have $|S| \leq m$. We will prove that $|S| \geq m$ by showing that each edge in M correspond to an edge in S. Let v_1 be any leaf in τ and let v_2 be the non-pendant node such that the edge $v_1v_2 \in M$. We distinguish two cases:

- $v_1 \notin S$. In this case, v_2 should belong to $V(S)$ since otherwise we can add v_1v_2 to S and S remains a star forest. Moreover, v_2 should be of degree 1 in S since otherwise we can also add v_1v_2 to S and S remains a star forest. Thus, the edge v_1v_2 correspond to the edge incident to v_2 in S.
- $v_1 \in S$. Then the edge v_1v_2 should belong to S and hence it corresponds to itself.

Hence, $|S| \leq m$ and $|S| \geq m$ which implies that $|S| = m$. □

We will prove the following theorem.

Theorem 3.5. *The M-tree and nonnegativity inequalities completely define $SFP(T)$.*

Proof. Suppose that $a^t x \leq b$ is any facet-defining inequality for $SFP(T)$ which is not a M-tree inequality neither the nonnegativity inequality. Let G_a be the support graph of $a^t x \leq b$. We can suppose without loss of generality that G_a is a tree. Let T_a denote this tree. Hence, T_a is a subtree of T. We have two possible cases.

- T_a is a M-tree. Let F be any tight star forest with respect to $a^t x \leq b$. By Corollary 2.4, F is maximal with respect to T_a. Consequently, by Proposition 3.4, F is tight with respect to the M-tree inequalities. Contradiction to the fact that $a^t x \leq b$ is a facet defining inequality different from a M-tree inequality.
- T_a is not a M-tree. Hence, in T_a there must be one non-pendant node of one of the two following types:
  Type 1. A non-pendant node not connected to a leaf of T_a.
  Type 2. A non-pendant node connected to at least two leaves of T_a.

We distinguish two cases:

Case 1. T_a contains only non-pendant nodes of Type 2. In this case, one can obtain a M-tree τ from T_a by keeping for each non-pendant node, only one leaf connecting to it. Let $x(\tau) \leq |M_\tau|$ be the M-tree inequality associated with τ. The following remark can be easily proved.

Remark 3.6. Given a non-pendant node v of Type 2, any star forest F satisfying $a^t x \leq b$ at equality must be maximal in τ and contains either all the leaves connected to v or no of them.

Let F be any tight star forest with respect to $a^t x \leq b$. We have $F \cap \tau$ is a maximal star forest with respect to τ. Since by Proposition 3.4, every maximal star forest in τ safisfies $x(\tau) \leq |M_\tau|$ at equality, it follows that every star forest satisfying $a^t x \leq b$ at equality satisfies also the M-tree inequality associated with τ at equality. This contradicts the fact that $a^t x \leq b$ is a facet-defining inequality.

Case 2. T_a contains at least one non-pendant node of Type 1. We will show the following lemma.

Lemma 3.7. *There exists a non-pendant node s of Type 1 such that all the other non-pendant nodes of Type 1 belong to a same connected component obtained by the removal of s from T_a.*

Proof. We give a constructive proof.
Initialization. Let us choose any non-pendant node s_0 of Type 1 and let $i = 0$.
Iteration i. Suppose that $T_1^i, \ldots, T_{p_i}^i$ are the subtrees of T_a obtained if s_i is removed from T_a and suppose without loss of generality that T_1^i is always the tree which contains s_0. We have two possible cases.

– If all the other non-pendant nodes of Type 1 belong to a same tree T_{k_i} $(1 \leq k_i \leq p_i)$ then s_i is a non-pendant node of type 1 satisfying the condition stated in the lemma. STOP.
– If the other non-pendant nodes of Type 1 belong to at least two trees. Suppose without loss of generality that $T_{p_i}^i$ is one of them. Let us choose s_{i+1} to be any non-pendant node of Type 1 in $T_{p_i}^i$ and set $i \leftarrow i + 1$. Reiterate.

We have the following remark.

Remark 3.8. T_1^i contains all the node previously chosen $s_0, \ldots, s_{i-1}$ which are all distincts. The set containing these nodes is called the kernel.

Thus, the procedure should be ended by finding a non-pendant node of Type 1 satisfying the condition stated in the lemma after at most $|V(T_a)|$ iterations since the kernel grows after each iteration. □

Let s be a non-pendant node of Type 1 satisfying the condition stated in Lemma 3.7. Let T_1 be the connected component which contains the other non-pendant nodes of Type 1 obtained by removing s from T_a. We can observe that s is connected to T_1 by only one edge and the subgraph H of T_a induced by the edges in $T_a \setminus T_1$ is a tree. Moreover, H is either a M-tree or a tree which contains some non-pendant nodes of Type 2 but no one of Type 1. Hence, in all the cases as we have proved in Case 1, H contains a M-tree τ which contains all the non-pendant nodes of H. Let u be the neighbour of s in T_1, *i.e.* the edge $su \in H$. Observe that su is a bridge in T_a that links H to T_1. We can see that any maximal star forest in T_a, whatever it contains su or not, contains a maximal star forest in H (if a maximal star forest in T_a does not contain su, it must contain some edge sv where v is of degree at least 2 in the star forest). The latter, by Remark 3.6, contains a maximal star forest in τ. By Corollary 2.4, any star forest F satisfying $a^t x \leq b$ at equality is maximal in T_a. Hence $F \cap \tau$ is a maximal star forest in τ. By Proposition 3.4, $F \cap \tau$ satisfies the M-tree inequality associated with τ at equality. This contradicts the fact that $a^t x \leq b$ is facet-defining. □

4. Polyhedral results on cycles

In this section, G will be a chordless cycle $C = (V(C), E(C))$ of n nodes with $V(C) = \{1, 2, \ldots, n\}$ numbered clockwise, *i.e.* the n edges in $E(C)$ will be $e_i = (i, i+1)$ for $i = 1, \ldots, n-1$, and the edge $e_n = (n, 1)$. We will sometimes use $|C|$ instead of n, $|V(C)|$ or $|E(C)|$ which are all equal. The edges in C are weighted by a vector $c \in \mathbb{R}^n$ where c_i is the weight associated with edge e_i for $i = 1, \ldots, n$. We also consider $L(C) = (V^L, E^L)$ the line graph of C where the nodes correspond to the edges of C, and two nodes of $L(C)$ are adjacent if the corresponding edges are adjacent in C. Note that $L(C)$ is also a cycle node-weighted by vector c. For the sake of convenience, given a node $1 \leq i \leq n$ and an integer $t > 0$, let $i + t$ designate the node $i + t$ if $i + t \leq n$ and the node $(i + t) \bmod n$ if $(i + t) > n$. For two nodes u and v with $v = u + t$ for some integer $t > 0$ in C, let $C(u, v)$ denote the path $(u + 1, \ldots, u + t - 1)$ of C between $u + 1$ and $u + t - 1$ (note that the path does not contain

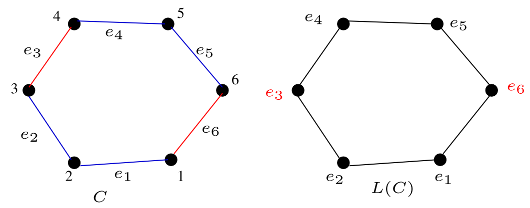

FIGURE 3. A cycle C and its line graph $L(C)$. The edge subset $\{e_1, e_2, e_4, e_5\}$ is a star forest in C and its complement $\{e_3, e_6\}$ is an edge dominating set in C and a dominating set in $L(C)$.

u and v). For two edges e and f in C with $e = e_i$ and $f = e_{i+t}$ for some integer $t > 0$, let $C(e, f)$ denote the path consisting of the edges $e_{i+1}, \ldots, e_{i+t-1}$ (note that the path does not contain neither e_i nor e_{i+t}). In what follows, we will establish the relations between star forests, dominating sets and edge dominating sets on cycles. Then, using theses relations with the polyhedral results in [9] and [8], we will derive a complete description of $SFP(C)$. Note that in [18], a linear time algorithm for the MWSFP when G is a cactus have been presented, hence the MWSFP can be solved in linear time in cycles.

4.1. Star forests, dominating sets and edge dominating sets on cycles

In the context of a cycle, Lemma 2.6 can be restated as follows.

Lemma 4.1. *An edge subset $F \subset C$ is a star forest if and only if F does not contain any 3-path.*

The following lemma establishes the link between edge dominating sets and star forests in a cycle.

Lemma 4.2. *The complement of a star forest F in C is an edge dominating set and vice versa.*

Proof. ($\Rightarrow$) Let $F \subseteq E(C)$ be a star forest in C and let $\bar{F} = C \setminus F$. Suppose that $\bar{F}$ is not an edge dominating set. Then there exists an edge $e_i \in F$ not adjacent to any edge in $\bar{F}$. As C is a cycle, the neighbors of e_i, e_{i-1} and e_{i+1} do not belong to $\bar{F}$. Hence, e_{i-1}, e_i, e_{i+1} form a path of length 3 in F, a contradiction with Lemma 4.1.
($\Leftarrow$) Let $ED \subseteq C$ be a edge dominating set in C. Let $F = C \setminus ED$ and suppose that F is not a star forest. Then F contains a 3-path (v_1, v_2, v_3, v_4). We can see that the edge (v_2, v_3) is not dominated by ED implying that ED is not an edge dominating set, a contradiction. □

By the one-to-one correspondence between the nodes of $L(C)$ and the edges of C, we have the following result.

Lemma 4.3. *Any edge dominating set in C is a dominating set in $L(C)$ and vice versa.*

The following lemma reformulates these relations in polyhedral terms for the polytopes $SFP(C)$, $EDP(C)$ and $DP(L(C))$.

Lemma 4.4. *The following statements are equivalent:*

(i) $\alpha^t y \geq \beta$ *with* $y \in \mathbb{R}^n$ *defines a facet for* $DP(L(C))$,
(ii) $\alpha^t x \geq \beta$ *with* $x \in \mathbb{R}^n$ *defines a facet for* $EDP(C)$,
(iii) $\alpha^t x \leq \sum_{e \in E(C)} \alpha(e) - \beta$ *with* $x \in \mathbb{R}^n$ *defines a facet for* $SFP(C)$.

Hence, the polytopes $SFP(C)$, $DP(L(C))$ and $EDP(C)$ are equivalent in the sense that there is a one-to-one correspondence between their facets.

Proof. Note that these polytopes are all defined in $\mathbb{R}^n$ and are full dimensional. The lemma follows from the relations described in Lemmas 4.2 and 4.3 as they are all preserved under affine transformations. □

As in [8], a complete linear description for $DP(L(C))$ is given, by Lemma 4.4, we can also derive complete descriptions for $SFP(C)$ and $EDP(C)$. We will explicit these complete descriptions in the following section.

4.2. Complete description of $SFP(C)$

Let us consider the graph $L(C)$ and the polytope $DP(L(C))$ in $\mathbb{R}^n$ whose component indexed by the nodes in V^L. In [9] and [8], Bouchakour *et al.* give the following integer formulation for MWDSP,

$$\begin{aligned} &\min c^t x \\ &0 \le x_v \le 1 && \text{for all } v \in V^L && (4.1)\\ &x(N(v)) \ge 1 && \text{for all } v \in V^L && (4.2)\\ &x_v \text{ integer} && \text{for all } v \in V^L \end{aligned}$$

They have also characterized two classes of facet-defining inequalities for $DP(L(C))$.

Theorem 4.5. *[9] The inequality*

$$x(V^L) \ge \left\lceil \frac{|C|}{3} \right\rceil \tag{4.3}$$

defines a facet for $DP(L(C))$ if and only if either $|C| = 3$ or $|C| \ge 4$ and $|C|$ is not a multiple of 3.

Theorem 4.6. *[8] Let $W = \{v_1, \dots, v_p\}$ be a subset of $p \ge 3$ nodes in V^L satisfying the following conditions:*

C1: *p is odd and $v_1 < v_2 < \dots < v_p$,*
C2: *$|C(v_i, v_{i+1})| = 3k_i$, $k_i \ge 1$, for $i = 1, \dots, p$ with $v_{p+1} = v_1$.*

Then the constraint

$$2\sum_{v\in W} x_v + \sum_{v\in V^L\setminus W} x_v \ge \sum_{i=1}^{p} k_i + \left\lceil \frac{p}{2} \right\rceil \tag{4.4}$$

defines a facet for $DP(L(C))$.

Let us apply Lemma 4.4 to derive facet-definining inequalities for $SFP(C)$. It is clear that applying Lemma 4.4 to inequalities (4.1) yields the trivial inequalities

$$0 \le x_e \le 1 \text{ for all } e \in E(C),$$

for $SFP(C)$.

By applying Lemma 4.4 to inequalities (4.2), we get the 3-path inequalities

$$x(P) \le 2 \text{ for all path of length 3 in } C,$$

for $SFP(C)$ which have been described in Section 2. The following proposition can be obtained by applying Lemma 4.4 to inequalities (4.3).

Proposition 4.7. *The cycle inequality*

$$x(E(C)) \le \left\lfloor \frac{2|C|}{3} \right\rfloor \tag{4.5}$$

defines a facet for $SFP(C)$ when, either $|C| = 3$ or $|C| \ge 4$ and $|C|$ is not multiple of 3.

Let $W = \{v_1, \dots, v_p\} \subset V^L$ be a subset of nodes of V^L as defined in Theorem 4.6, let f_i denote the edge in C corresponding to the node v_i in $L(C)$ and let $M = \{f_1, \dots, f_p\}$. For $i = 1, \dots, p$, we define $C(f_i, f_{i+1})$ with $f_{p+1} = f_1$ to be the path between f_i and f_{i+1} in C which does not contain any edge in M. The conditions C1 and C2 on the set W can be transformed into conditions M1 and M2 on the set M as follows:

M1: M is a matching of odd cardinality,
M2: $|C(f_i, f_{i+1})| = 3k_i$, $k_i \ge 1$, for $i = 1, \dots, p$ with $f_{p+1} = f_1$.

We then deduce the following result by applying Lemma 4.4 to inequalities (4.4).

Proposition 4.8. *The matching-cycle inequalities*

$$2x(M) + x(E(C) \setminus M) + \leq 2\sum_{i=1}^{p} k_i + \left\lfloor \frac{3p}{2} \right\rfloor \text{ for all } M \subset E(C) \text{ satisfying conditions M1 and M2.} \tag{4.6}$$

define facets for $SFP(C)$.

Proof. Given a matching M of C satisfying conditions M1 and M2, suppose that $v_1, \ldots, v_p$ are the nodes in $L(C)$ corresponding respectively to $f_1, \ldots, f_p$. It is easy to see that $v_1, \ldots, v_p$ satisfy conditions C1 and C2 of Theorem 4.6, and hence, we have that

$$2\sum_{v \in W} x_v + \sum_{v \in V^L \setminus W} x_v \geq \sum_{i=1}^{p} k_i + \left\lceil \frac{p}{2} \right\rceil$$

defines a facet for $DP(L(C))$. The result thus follows from Lemma 4.4. □

In [8], Bouchakour *et al.* have shown the following theorem.

Theorem 4.9. *[8] A complete linear description for $DP(L(C))$ is given by inequalities (4.1), (4.2), (4.3), (4.4).*

As a direct consequence, we have the following result.

Corollary 4.10. *When G is a cycle, $SFP(G)$ is completely described by the trivial inequalities, the 3-path inequalities, the cycle inequality (4.5) and the matching-cycle inequalities (4.6).*

5. Conclusions

In this paper, we have presented an IP formulation for MWSFP. We have also given two complete linear descriptions for $SFP(G)$, the star forests polytope for the cases where G is a tree and G is a cycle. An interesting direction for future works would be to exploit these results to derive a complete linear description for $SFP(G)$ when G is a cactus. Our complete description for $SFP(G)$ when G is a tree could be helpful to find an exact solution for the MWSFP in general graphs. A star forest is always a subgraph of a spanning tree in G and a complete linear description of the spanning tree polytope is known.

Acknowledgements. We would like to thank one anonymous referee for her/his helpful comments which greatly improved the presentation of the paper.

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