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ON MULTI-LEVEL MULTI-OBJECTIVE LINEAR FRACTIONAL PROGRAMMING PROBLEM WITH INTERVAL PARAMETERS

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Abstract. This paper develops a method to solve multi-level multi-objective linear fractional programming problem (ML-MOLFPP) with interval parameters as the coefficients of decision variables and the constants involved in both the objectives and constraints. The objectives at each level are transformed into interval-valued fractional functions and approximated by intervals of linear functions using variable transformation and Taylor series expansion. Interval analysis and weighting sum method with analytic hierarchy process (AHP), are used to determine the non-dominated solutions at each level from which the aspiration values of the controlled decision variables are ascertained and linear fuzzy membership functions are constructed for all the objectives. Two multi-objective linear problems are equivalently formulated for the ML-MOLFPP with interval parameters and fuzzy goal programming is used to compute the optimal lower and upper bounds of all the objective values. A numerical example is solved to demonstrate the proposed solution approach.

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1. Introduction

Multi-level programming problems (MLPP) involve multiple interactive decision making units positioned at different levels of the system in order of their priorities in context of decision making. Optimization problems of multi-level types with several objectives at each level, are often encountered in decision making situations of complex hierarchical organizations. ML-MOLFPP comprises several linear fractional objectives at its each level which basically represent the ratios of physical and/or economical quantities of the real world problems like [28] cost/time, profit/cost, output/employee, inventory/sale, debit/equity, risk-assets/capital and so forth. Some common characteristics of MLPP are:

(i) Decisions of DMs are sequentially processed from upper to lower level.
(ii) Each level DM controls a set of decision variables independently.
(iii) Decisions of DMs get affected by the actions and reactions of each other.

A solution of MLPP is determined by considering the decisions of all level DMs together for the overall benefit of the system where a minimum standard of satisfaction is attained by all DMs otherwise it produces a situation of decision deadlock. MLPP are extremely useful to the decentralized systems [27] such as agriculture,

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transportation, network design, government policy, finance, economic system and so forth. Fractional programming [28] is a mathematical optimization problem comprising its objective function as the ratio of linear or non-linear functions which is generally formulated as:

$$\max / \min \ \frac{N(x)}{D(x)}$$
$$\text{subject to}$$
$$x \in \Omega, \text{the set of constraints.}$$

It has gained intensive research interest because of its applications in numerous important fields including engineering, science, economics, business, management, production, finance, information theory, water resources, health care and so forth. This field was first studied by Isbell and Marlow [15]. Charnes and Cooper [9] developed variable transformation method to solve linear fractional programming problem (LFPP). Dinkelbach [13] proposed an iterative parametric approach to solve non-linear FPP. Bitran and Novaes [8] developed an algorithm based on the simplex routine to solve LFPP.

In decision making contexts, Bellman and Zadeh [7] developed some basic concepts regarding fuzzy set theory for maximizing the decisions in form of membership functions and Zimmermann [33] proposed fuzzy programming (max-min operator) technique to solve a multi-objective LPP. Mohamed [22] established a relation between fuzzy programming and goal programming. Bi-level programming problems (BLPP) are special cases of MLPP and enormously studied in literature. Mishra [21] developed a method to solve bi-level linear fractional programming problems (BL-LFPP) using weighting sum approach and analytic hierarchy process (AHP). Toksari [30] applied Taylor series approach to solve bi-level linear fractional programming problem. Sakawa *et al.* [26] proposed an interactive fuzzy programming approach to solve BL-LFPP with fuzzy parameters. Abo-Sinna and Baky [2] solved bi-level multi-objective LFPP using fuzzy goal programming approach. Baky [4] developed fuzzy goal programming algorithm to solve decentralized bi-level multi-objective linear programming problems whereas Ahlatcioglu and Tiryaki [3] proposed two different approaches to solve such problems with linear fractional objectives using AHP to decide the proper weights. Toksari and Bilim [31] developed an interactive fuzzy goal programming approach to solve decentralized bi-level MOFPP in which jacobian matrix has been used to linearize the fuzzy membership functions in fractional form. Shih *et al.* [27] used fuzzy membership functions to solve multi-level programming problems. Pramanik and Roy [25] proposed fuzzy goal programming approach to solve multi-level LPP. Osman *et al.* [24] studied three-level multi-objective non-linear problems using fuzzy programming. Baky [5] developed two algorithms to solve multi-level multi-objective LPP using fuzzy goal programming approach and proposed TOPSIS (technique for order preference by similarity to ideal solution) algorithm [6] to solve multi-level non-linear multi-objective decision making problem. Abo-Sinna and Baky [1] proposed balance space approach to solve MLPP and Lachhwani [17] proposed fuzzy goal programming (FGP) approach to solve MLPP with linear objectives. Recently, Lachhwani [18] added some modifications to [5] the process of selection of tolerances for the controlled decision variables and developed a methodology to solve ML-MOLFPP using FGP approach. Liu [19] discussed geometric programming problem with fuzzy parameters and used Zadeh's extension principle to solve it. He derived the optimal lower and upper bounds of the objective function using α-cuts in fuzzy membership functions. Similarly, Chinnaduraj and Muthukumar [10] solved LFPP in fuzzy environment using α-cut in objectives and r-cut in constraints and derived optimal lower and upper bounds for the objective function.

In the present work, we have solved ML-MOLFPP with interval parameters which has not been previously studied as found from the survey of the literature. In many practical problems, decision maker (DM) can not exactly ascertain fixed values for the cost and constraint coefficients, right-hand side constants and other numeric values involved in the mathematically modeled optimization problems. To tackle such situation, some ranges of values in form of closed intervals can be considered instead of fixed values to suitably fit the practical problems. Therefore, an attempt is made to solve ML-MOLFPP in interval environment. As the objectives can be transformed into interval-valued functions, their optimal lower and upper bounds are determined by the obtained compromise solutions due to the proposed method.

The rest of the paper is organized as follows: Section 2 interprets the basics of intervals whereas the mathematical formulation of ML-MOLFPP with interval parameters is incorporated in Section 3. Section 4 describes the proposed method, its advantages and an algorithm for solving the problem. To illustrate the solution procedure of the proposed method and show its feasibility, a numerical example and its solution is presented in Section 5. Finally, some conclusions are incorporated in Section 6.

2. Interval arithmetic

Assume that "I" denotes the set of all closed intervals in $\mathbb{R}$. For $a, b \in I$ such that $a = [a^L, a^U]$ and $b = [b^L, b^U]$, the lower and upper bounds of a, b are considered as a^L, b^L and a^U, b^U respectively. The arithmetic operations on "I" [23] can be interpreted as follows.

(i) $a + b = [a^L + b^L, a^U + b^U]$

(ii) $a - b = [a^L - b^U, a^U - b^L]$

(iii) $\alpha a = \begin{cases} [\alpha a^L, \alpha a^U], & \alpha \geq 0 \\ [\alpha a^U, \alpha a^L], & \alpha < 0 \end{cases}$

(iv) $ab =$[min(M), max(M)], where $M = \{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$

(v) $\frac{a}{b} =$ [min(D), max(D)] for $0 \notin b$, where $D = \left\{ \frac{a^L}{b^L}, \frac{a^L}{b^U}, \frac{a^U}{b^L}, \frac{a^U}{b^U} \right\}$

(vi) If $a, b \in I^+ \subset \mathbb{R}^+$ then $ab = [a^L b^L, a^U b^U]$ and $\frac{a}{b} = \left[\frac{a^L}{b^U}, \frac{a^U}{b^L} \right]$.

Define "$\preccurlyeq$" as a partial ordering on "I" such that $a \preccurlyeq b$ holds if and only if $a^L \leq b^L$ and $a^U \leq b^U$ [32] which means a is inferior to b or b is superior to a. We say $a \prec b$ if and only if $a \preccurlyeq b$ and $a \neq b$ *i.e.*, one of the following condition holds,
$\{a^L < b^L, a^U < b^U\}$ or $\{a^L < b^L, a^U \leq b^U\}$ or $\{a^L \leq b^L, a^U < b^U\}$.

3. Problem formulation

ML-MOLFPP arises frequently in many hierarchical organizations and it is treated as a special category of multi-level mathematical programming in which the objectives at each level exist as fractions of affine functions. In many practical situations while formulating the problems into optimization models, decision makers can not always determine fixed values for the coefficients of the decision variables from the available data. To manage such problems, instead of fixed values it is better to assume certain ranges with lower and upper bounds in form of closed intervals. Consider a q-level mathematical programming in which each level-i $(i = 1, 2, \ldots, q)$ contains m_i number of linear fractional objectives to be simultaneously optimized. DM_i $(i = 1, 2, \ldots, q)$ denotes the decision maker at ith-level who controls a set of decision variables X_i independently. The mathematical formulation of ML-MOLFPP [18] with interval parameters can be generally formulated as follows.

Level-1 (DM_1): $\max\limits_{X_1} \ \{f_{11}(x), f_{12}(x), \ldots, f_{1m_1}(x)\}$

Level-2 (DM_2): $\max\limits_{X_2} \ \{f_{21}(x), f_{22}(x), \ldots, f_{2m_2}(x)\}$

$\vdots$

Level-q (DM$_q$): $\max\limits_{X_q} \ \{f_{q1}(x), f_{q2}(x), \ldots, f_{qm_q}(x)\}$

subject to

$$x \in \Omega = \{A_1X_1 + A_2X_2 + \ldots + A_qX_q \leq \ b\}$$
$$= \left\{\sum_{i=1}^{q} A_iX_i \ \leq \ b\right\}$$

where

$$x = (X_1, X_2, \ldots, X_q) = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \geq 0,$$
$$X_i = (x_1^{(i)}, x_2^{(i)}, \ldots, x_{n_i}^{(i)}) \in \mathbb{R}^{n_i} (i = 1, 2, \ldots, q), n = n_1 + n_2 +, \ldots, +n_q,$$
$$A_i = (a_{tj}^{(i)}) \in \mathbb{R}^{m \times n_i} \text{ and } b = (b_t) \in \mathbb{R}^m \text{ for } t = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n_i.$$

The objective functions at each level are considered as:

$$f_{ij}(x) = \frac{f_{ij}^N(x)}{f_{ij}^D(x)} = \frac{\sum\limits_{k=1}^{q} c_{ijk}X_k + \alpha_{ij}}{\sum\limits_{k=1}^{q} d_{ijk}X_k + \beta_{ij}}, \ i = 1, 2, \ldots, q \text{ for } j = 1, 2, \ldots, m_i$$

where, $f_{ij}^N(x)$ and $f_{ij}^D(x)$ represent the numerator and denominator functions of the objective $f_{ij}(x)$ respectively. Assume that, $c_{ijk}, d_{ijk}, \alpha_{ij}, \beta_{ij} \in I^+$, $a_{tj}^{(i)}, b_t \in I$ and $f_{ij}^D(x) > 0 \ \ \forall \ \ x \in \Omega$. $c_{ijk} = [c_{ijk}^L, c_{ijk}^U], d_{ijk} = [d_{ijk}^L, d_{ijk}^U], \alpha_{ij} = [\alpha_{ij}^L, \alpha_{ij}^U], \beta_{ij} = [\beta_{ij}^L, \beta_{ij}^U], a_{tj}^{(i)} = [a_{tj}^{(i)L}, a_{tj}^{(i)U}], b_t = [b_t^L, b_t^U]$.

4. Proposed method to solve ML-MOLFPP with interval parameters

ML-MOLFPP is studied with intervals as the constants and coefficients of the decision variables involved in both the objective functions and constraints. The proposed method derives the compromise solutions that determine the optimal lower and upper bounds for the objective values of the whole problem since the objectives at each level can be expressed as interval-valued functions.

4.1. Transforming the objectives and constraints

The objectives at each level of ML-MOLFPP can be formulated as:

$$f_{ij}(x) = \frac{\sum\limits_{k=1}^{q} \left[c_{ijk}^L, c_{ijk}^U\right] X_k + \left[\alpha_{ij}^L, \alpha_{ij}^U\right]}{\sum\limits_{k=1}^{q} \left[d_{ijk}^L, d_{ijk}^U\right] X_k + \left[\beta_{ij}^L, \beta_{ij}^U\right]} = \frac{\left[\sum\limits_{k=1}^{q} c_{ijk}^L X_k + \alpha_{ij}^L, \sum\limits_{k=1}^{q} c_{ijk}^U X_k + \alpha_{ij}^U\right]}{\left[\sum\limits_{k=1}^{q} d_{ijk}^L X_k + \beta_{ij}^L, \sum\limits_{k=1}^{q} d_{ijk}^U X_k + \beta_{ij}^U\right]}.$$

Since $c_{ijk}, d_{ijk}, \alpha_{ij}, \beta_{ij} \in I^+$, the objectives can be reformulated in form of interval valued functions using the concept of interval analysis.

$$f_{ij}(x) = \left[\frac{\sum\limits_{k=1}^{q} c_{ijk}^L X_k + \alpha_{ij}^L}{\sum\limits_{k=1}^{q} d_{ijk}^U X_k + \beta_{ij}^U}, \frac{\sum\limits_{k=1}^{q} c_{ijk}^U X_k + \alpha_{ij}^U}{\sum\limits_{k=1}^{q} d_{ijk}^L X_k + \beta_{ij}^L}\right] = \left[f_{ij}^L(x), f_{ij}^U(x)\right].$$

The constraints $\sum_{i=1}^{q} A_i X_i \leq b$, forming the feasible region Ω can be analysed using the order relation "$\preccurlyeq$" defined on "I" as:

$$\sum_{j=1}^{n_1} a_{tj}^{(1)} x_j^{(1)} + \sum_{j=1}^{n_2} a_{tj}^{(2)} x_j^{(2)} +, \ldots, + \sum_{j=1}^{n_q} a_{tj}^{(q)} x_j^{(q)} \leq b_t, \quad t = 1, 2, \ldots, m.$$

Since $a_{tj}^{(i)}, b_t \in I$, the constraints can be stated as:

$$\sum_{j=1}^{n_1} \left[a_{tj}^{(1)L}, a_{tj}^{(1)U}\right] x_j^{(1)} + \sum_{j=1}^{n_2} \left[a_{tj}^{(2)L}, a_{tj}^{(2)U}\right] x_j^{(2)} +, \ldots, + \sum_{j=1}^{n_q} \left[a_{tj}^{(q)L}, a_{tj}^{(q)U}\right] x_j^{(q)} \preccurlyeq \left[b_t^L, b_t^U\right].$$

These constraints can be further splitted into the following form.

$$\sum_{j=1}^{n_1} a_{tj}^{(1)L} x_j^{(1)} + \sum_{j=1}^{n_2} a_{tj}^{(2)L} x_j^{(2)} +, \ldots, + \sum_{j=1}^{n_q} a_{tj}^{(q)L} x_j^{(q)} \leq b_t^L, \quad t = 1, 2, \ldots, m$$
$$\sum_{j=1}^{n_1} a_{tj}^{(1)U} x_j^{(1)} + \sum_{j=1}^{n_2} a_{tj}^{(2)U} x_j^{(2)} +, \ldots, + \sum_{j=1}^{n_q} a_{tj}^{(q)U} x_j^{(q)} \leq b_t^U, \quad t = 1, 2, \ldots, m$$

Definition 4.1. x^* is a non-dominated solution of the problem: max $f(x) = [f^L(x), f^U(x)]$ subject to $x \in U$ if there exists no $\bar{x} \in U$ such that $f(x^*) \prec f(\bar{x})$.

4.2. Variable transformation method (VTM)

Charnes and Cooper [9] developed VTM to derive the optimal solution of a linear fractional programming problem. Consider the following two mathematical programming problems $\mathbf{M_1}$ and $\mathbf{M_2}$ with fractional and linear objectives respectively.

$$\mathbf{M_1} : \max \; f(x) = \frac{cx + \alpha}{dx + \beta}$$
$$\text{subject to}$$
$$S_1 = \{Ax \leq b, x \geq 0\}$$

where $c, d \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

$$\mathbf{M_2} : \max \; g(y, z) = cy + \alpha z$$
$$\text{subject to}$$
$$S_2 = \{dy + \beta z = 1, Ay - bz \leq 0, \quad y, z \geq 0\}$$

where, the transformations $z = \frac{1}{dx+\beta}$ and $y = xz$ derive $\mathbf{M_2}$ from $\mathbf{M_1}$.

Theorem 4.2. *[28] If (y^*, z^*) is an optimal solution of $\mathbf{M_2}$ then $x^* = \frac{y^*}{z^*}$ is the optimal solution of $\mathbf{M_1}$.*

Lemma 4.3. *[28] For any feasible solution $(y, z) \in S_2$, z is positive.*

Corollary 4.4. *[28] The transformation $y = xz$ establishes an one-to-one correspondence between the feasible sets S_1 and S_2.*

Definition 4.5. [10, 11] Two mathematical programming problems P_1 : max $f(x)$ subject to $x \in U$ and P_2 : max $g(x)$ subject to $x \in V$ are said to be equivalent iff $\exists$ an one-to-one map $h : U \to V$ such that $f(x) = g(h(x)) \; \forall \; x \in U$.

4.3. Solution approach

The following problem is formulated using VTM to determine the individual optimal solution for the lower bound $f_{ij}^L(x)$, of each interval-valued fractional objective function $f_{ij}(x)$.

$$\max \sum_{k=1}^{q} c_{ijk}^L Y_k + \alpha_{ij}^L z^L \tag{4.1}$$

$$\text{subject to}$$

$$\sum_{k=1}^{q} d_{ijk}^U Y_k + \beta_{ij}^U z^L = 1$$

$$\sum_{j=1}^{n_1} a_{tj}^{(1)L} y_j^{(1)} + \sum_{j=1}^{n_2} a_{tj}^{(2)L} y_j^{(2)} +, \dots, + \sum_{j=1}^{n_q} a_{tj}^{(q)L} y_j^{(q)} \leq b_t^L z^L, \quad t = 1, 2, \dots, m$$

$$\sum_{j=1}^{n_1} a_{tj}^{(1)U} y_j^{(1)} + \sum_{j=1}^{n_2} a_{tj}^{(2)U} y_j^{(2)} +, \dots, + \sum_{j=1}^{n_q} a_{tj}^{(q)U} y_j^{(q)} \leq b_t^U z^L, \quad t = 1, 2, \dots, m$$

$$y_j^{(1)}, y_j^{(2)}, \dots, y_j^{(q)} \geq 0, \; z^L > 0$$

where, $Y_k = X_k z^L$ and $y_j^{(i)} = x_j^{(i)} z^L, \; i = 1, 2, \dots, q$.

By Theorem 4.2, If $(y_j^{(1)*}, y_j^{(2)*}, \dots, y_j^{(q)*}, \quad z^{L*})$ is the optimal solution of problem (4.1) then $(x_j^{(1)*}, x_j^{(2)*}, \dots, x_j^{(q)*}) = (\frac{y_j^{(1)*}}{z^{L*}}, \frac{y_j^{(2)*}}{z^{L*}}, \dots, \frac{y_j^{(q)*}}{z^{L*}})$ is the optimal solution of $f_{ij}^L(x)$ on Ω. Similarly, the individual optimal solution of each upper bound function $f_{ij}^U(x)$ can be evaluated using VTM.

Let $x_{ij}^{L*} = (x_{ij1}^{L*}, x_{ij2}^{L*}, \dots, x_{ijn}^{L*})$ and $x_{ij}^{U*} = (x_{ij1}^{U*}, x_{ij2}^{U*}, \dots, x_{ijn}^{U*})$ be the individual optimal solutions of $f_{ij}^L(x)$ and $f_{ij}^U(x)$ respectively where $j = 1, 2, \dots, m_i$ for $i = 1, 2, \dots, q$. Using Taylor series expansion [29] up to first order, each $f_{ij}^L(x)$ and $f_{ij}^U(x)$ are expanded about their respective individual optimal solutions x_{ij}^{L*} and x_{ij}^{U*} to approximate the fractional objectives of ML-MOLFPP by the linear functions. The approximations can be formulated as follows.

$$f_{ij}^L(x) \approx \tilde{f}_{ij}^L(x) = f_{ij}^L(x_{ij}^{L*}) + \sum_{k=1}^{n} (x_k - x_{ijk}^{L*}) \frac{\partial f_{ij}^L(x_{ij}^{L*})}{\partial x_k}$$

$$f_{ij}^U(x) \approx \tilde{f}_{ij}^U(x) = f_{ij}^U(x_{ij}^{U*}) + \sum_{k=1}^{n} (x_k - x_{ijk}^{U*}) \frac{\partial f_{ij}^U(x_{ij}^{U*})}{\partial x_k} \cdot$$

Weighting sum method [20] is used to determine the non-dominated solutions of each level-$i = 1, 2, \dots, q$ separately. The weights assigned to the objectives of a particular level, represent the relative importance of the objectives and are usually considered to be positive and normalized. In the present work, proper weights (w_j) are determined using analytic hierarchy process (AHP) [14, 21] from the pairwise comparison matrix obtained by the DM of the respective level on priority basis of its objectives. Some other methods are also available in literature to ascertain the weights from the pairwise comparison matrix such as least square method [16] and logarithmic square method [12] etc. At any particular level-$l \in \{1, 2, \dots, q\}$, the multi-objective problem with approximated objectives is transformed into single objective problem using the obtained proper weights which can be formulated as:

$$\max \sum_{j=1}^{m_l} w_j [\tilde{f}_{lj}^L(x), \tilde{f}_{lj}^U(x)] \tag{4.2}$$

subject to

$$\sum_{j=1}^{n_1} a_{tj}^{(1)L} x_j^{(1)} + \sum_{j=1}^{n_2} a_{tj}^{(2)L} x_j^{(2)} +, \ldots, + \sum_{j=1}^{n_q} a_{tj}^{(q)L} x_j^{(q)} \leq b_t^L, \quad t = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n_1} a_{tj}^{(1)U} x_j^{(1)} + \sum_{j=1}^{n_2} a_{tj}^{(2)U} x_j^{(2)} +, \ldots, + \sum_{j=1}^{n_q} a_{tj}^{(q)U} x_j^{(q)} \leq b_t^U, \quad t = 1, 2, \ldots, m$$

$$x = (x_1, x_2, \ldots, x_n) = (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(q)}) \geq 0, w_j > 0, \sum_{j=1}^{m_l} w_j = 1.$$

Consider the following two mathematical programming problems $\mathbf{M_3}$ and $\mathbf{M_4}$ where, $f : \mathbb{R}^n \to I$ and $g : \mathbb{R}^n \to \mathbb{R}$.

$$\mathbf{M_3} : \max \ f(x) = [f^L(x), f^U(x)]$$

subject to

$$h_i(x) \leq 0, \ i = 1, 2, \ldots, m$$

$$x \geq 0$$

$$\mathbf{M_4} : \max \ g(x) = f^L(x) + f^U(x)$$

subject to

$$h_i(x) \leq 0, \ i = 1, 2, \ldots, m$$

$$x \geq 0$$

Theorem 4.6. *If x^* is an optimal solution of $\mathbf{M_4}$ then x^* is a non-dominated solution of $\mathbf{M_3}$.*

Proof. [32] Suppose x^* is not a non-dominated solution of $\mathbf{M_3}$ *i.e.*, there exists a feasible solution $\bar{x}$ such that $f(x^*) \prec f(\bar{x})$ which means one of the following condition holds.

$$\left\{f^L(x^*) < f^L(\bar{x}), f^U(x^*) < f^U(\bar{x})\right\} \text{ or } \left\{f^L(x^*) \leq f^L(\bar{x}), f^U(x^*) < f^U(\bar{x})\right\} \text{ or } \left\{f^L(x^*) < f^L(\bar{x}), f^U(x^*) \leq f^U(\bar{x})\right\}.$$

In each case, $g(x^*) < g(\bar{x})$ which contradicts to the optimality of x^* for $\mathbf{M_4}$. □

Using Theorem 4.6, problem (4.2) can be equivalently transformed into the following problem with real-valued objective function which is defined as:

$$\max \sum_{j=1}^{m_l} w_j (\tilde{f}_{lj}^L(x) + \tilde{f}_{lj}^U(x)) \tag{4.3}$$

subject to

$$\sum_{j=1}^{n_1} a_{tj}^{(1)L} x_j^{(1)} + \sum_{j=1}^{n_2} a_{tj}^{(2)L} x_j^{(2)} +, \ldots, + \sum_{j=1}^{n_q} a_{tj}^{(q)L} x_j^{(q)} \leq b_t^L, \quad t = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n_1} a_{tj}^{(1)U} x_j^{(1)} + \sum_{j=1}^{n_2} a_{tj}^{(2)U} x_j^{(2)} +, \ldots, + \sum_{j=1}^{n_q} a_{tj}^{(q)U} x_j^{(q)} \leq b_t^U, \quad t = 1, 2, \ldots, m$$

$$x = (x_1, x_2, \ldots, x_n) = (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(q)}) \geq 0, w_j > 0, \sum_{j=1}^{m_l} w_j = 1.$$

Evaluate the non-dominated solutions of each level by solving problem (4.3) separately for $l = 1, 2, \ldots, q$. Let $X_1^{l_1*}, X_2^{l_2^*}, \ldots, X_q^{l_q^*}$ be the non-dominated solutions, obtained for levels-$1, 2 \ldots q$ respectively. Since DM_i of each level $(i = 1, 2, \ldots, q)$ controls a set of decision variables X_i independently, their aspiration values X_i^* are ascertained as the corresponding co-ordinate values in the obtained non-dominated solution $X_i^{l_i*}$ of the same level.

Determine the range of variations for the objective values of each lower and upper bounds of the objective functions using the following process.

The aspired and acceptable values of $\tilde{f}_{ij}^L(x)$ are evaluated as:

$$\begin{aligned}\tilde{f}_{ij}^{L\,\max} &= \tilde{f}_{ij}^L(X_i^{l_i^*})\\ \tilde{f}_{ij}^{L\,\min} &= \min\left\{\tilde{f}_{ij}^L(X_k^{l_k^*}) | k \neq i, k = 1, 2, \ldots, q\right\}.\end{aligned}$$

If $\tilde{f}_{ij}^{L\,\min} \geq \tilde{f}_{ij}^L(X_i^{l_i^*})$, $\tilde{f}_{ij}^{L\,\min}$ is assigned an acceptable value by the DM of the corresponding level such that $\tilde{f}_{ij}^{L\,\min} < \tilde{f}_{ij}^L(X_i^{l_i^*})$.

The aspired and acceptable values of $\tilde{f}_{ij}^U(x)$ are evaluated as:

$$\begin{aligned}\tilde{f}_{ij}^{U\,\max} &= \tilde{f}_{ij}^U(X_i^{l_i^*})\\ \tilde{f}_{ij}^{U\,\min} &= \min\left\{\tilde{f}_{ij}^U(X_k^{l_k^*}) | k \neq i, k = 1, 2, \ldots, q\right\}.\end{aligned}$$

If $\tilde{f}_{ij}^{U\,\min} \geq \tilde{f}_{ij}^U(X_i^{l_i^*})$, $\tilde{f}_{ij}^{U\,\min}$ is assigned an acceptable value by the DM of the corresponding level such that $\tilde{f}_{ij}^{U\,\min} < \tilde{f}_{ij}^U(X_i^{l_i^*})$.

Thus, the range of variations for the approximated objectives are obtained as:

$$\begin{gathered}[\tilde{f}_{ij}^{L\,\min}, \tilde{f}_{ij}^{U\,\min}] \preccurlyeq [\tilde{f}_{ij}^L(x), \tilde{f}_{ij}^U(x)] \preccurlyeq [\tilde{f}_{ij}^{L\,\max}, \tilde{f}_{ij}^{U\,\max}]\\ i.e., \tilde{f}_{ij}^{L\,\min} \leq \tilde{f}_{ij}^L(x) \leq \tilde{f}_{ij}^{L\,\max} \text{ and } \tilde{f}_{ij}^{U\,\min} \leq \tilde{f}_{ij}^U(x) \leq \tilde{f}_{ij}^{U\,\max}.\end{gathered}$$

Construct the fuzzy linear membership functions of maximization type for $\tilde{f}_{ij}^L(x)$ and $\tilde{f}_{ij}^U(x)$ as:

$$\mu_{\tilde{f}_{ij}^L}(x) = \begin{cases} 1, & \tilde{f}_{ij}^L(x) \geq \tilde{f}_{ij}^{L\,\max}\\ \frac{\tilde{f}_{ij}^L(x) - \tilde{f}_{ij}^{L\,\min}}{\tilde{f}_{ij}^{L\,\max} - \tilde{f}_{ij}^{L\,\min}}, & \tilde{f}_{ij}^{L\,\min} < \tilde{f}_{ij}^L(x) < \tilde{f}_{ij}^{L\,\max}\\ 0, & \tilde{f}_{ij}^L(x) \leq \tilde{f}_{ij}^{L\,\min}\end{cases}$$

$$\mu_{\tilde{f}_{ij}^U}(x) = \begin{cases} 1, & \tilde{f}_{ij}^U(x) \geq \tilde{f}_{ij}^{U\,\max}\\ \frac{\tilde{f}_{ij}^U(x) - \tilde{f}_{ij}^{U\,\min}}{\tilde{f}_{ij}^{U\,\max} - \tilde{f}_{ij}^{U\,\min}}, & \tilde{f}_{ij}^{U\,\min} < \tilde{f}_{ij}^U(x) < \tilde{f}_{ij}^{U\,\max}\\ 0, & \tilde{f}_{ij}^U(x) \leq \tilde{f}_{ij}^{U\,\min}\end{cases}.$$

Since the interval-valued fractional objectives are approximated by linear functions, the solutions of ML-MOLFPP that determine the optimal lower and upper bounds of the objective values, can be obtained by solving the following two problems.

$$\max\{\mu_{\tilde{f}_{ij}^L}(x), i=1,2,\ldots,q,\ j=1,2,\ldots,m_i\} \tag{4.4}$$
$$\text{subject to}$$
$$X_i \approx X_i^*,\ i=1,2,\ldots,q-1$$
$$\sum_{j=1}^{n_1}\left[a_{tj}^{(1)L},a_{tj}^{(1)U}\right]x_j^{(1)}+\sum_{j=1}^{n_2}\left[a_{tj}^{(2)L},a_{tj}^{(2)U}\right]x_j^{(2)}+,\ldots,+\sum_{j=1}^{n_q}\left[a_{tj}^{(q)L},a_{tj}^{(q)U}\right]x_j^{(q)}\preccurlyeq\left[b_t^L,b_t^U\right]$$
$$x\geq 0,\ t=1,2,\ldots,m$$

$$\max\{\mu_{\tilde{f}_{ij}^U}(x), i=1,2,\ldots,q,\ j=1,2,\ldots,m_i\} \tag{4.5}$$
$$\text{subject to}$$
$$X_i \approx X_i^*,\ i=1,2,\ldots,q-1$$
$$\sum_{j=1}^{n_1}\left[a_{tj}^{(1)L},a_{tj}^{(1)U}\right]x_j^{(1)}+\sum_{j=1}^{n_2}\left[a_{tj}^{(2)L},a_{tj}^{(2)U}\right]x_j^{(2)}+,\ldots,+\sum_{j=1}^{n_q}\left[a_{tj}^{(q)L},a_{tj}^{(q)U}\right]x_j^{(q)}\preccurlyeq\left[b_t^L,b_t^U\right]$$
$$x\geq 0,\ t=1,2,\ldots,m.$$

The aspiration values for the membership functions $\mu_{\tilde{f}_{ij}^L}(x)$, $\mu_{\tilde{f}_{ij}^U}(x)$ and the decision variables $X_i, i=1,2,\ldots,q-1$ controlled by upper level DMs are unity and X_i^* respectively. Thus, using fuzzy goal programming method [22] the problems (4.4) and (4.5) can be respectively reformulated as follows:

$$(\textbf{\textit{Lower bound}}):\min\sum_{i=1}^{q}\sum_{j=1}^{m_i}d_{ij}+\sum_{i=1}^{q-1}(d_i^l+d_i^r) \tag{4.6}$$
$$\mu_{\tilde{f}_{ij}^L}(x)+d_{ij}\geq 1,\ i=1,2,\ldots,q,\ j=1,2,\ldots,m_i$$
$$X_i+d_i^l-d_i^r=X_i^*,\ i=1,2,\ldots,q-1$$
$$\sum_{j=1}^{n_1}a_{tj}^{(1)L}x_j^{(1)}+\sum_{j=1}^{n_2}a_{tj}^{(2)L}x_j^{(2)}+,\ldots,+\sum_{j=1}^{n_q}a_{tj}^{(q)L}x_j^{(q)}\leq b_t^L,\quad t=1,2,\ldots,m$$
$$\sum_{j=1}^{n_1}a_{tj}^{(1)U}x_j^{(1)}+\sum_{j=1}^{n_2}a_{tj}^{(2)U}x_j^{(2)}+,\ldots,+\sum_{j=1}^{n_q}a_{tj}^{(q)U}x_j^{(q)}\leq b_t^U,\quad t=1,2,\ldots,m$$
$$x,d_{ij},d_i^l,d_i^r\geq 0,\ d_i^l.d_i^r=0$$

$$(\textbf{\textit{Upper bound}}):\min\sum_{i=1}^{q}\sum_{j=1}^{m_i}e_{ij}+\sum_{i=1}^{q-1}(e_i^l+e_i^r) \tag{4.7}$$
$$\mu_{\tilde{f}_{ij}^U}(x)+e_{ij}\geq 1,\ i=1,2,\ldots,q,\ j=1,2,\ldots,m_i$$
$$X_i+e_i^l-e_i^r=X_i^*,\ i=1,2,\ldots,q-1$$
$$\sum_{j=1}^{n_1}a_{tj}^{(1)L}x_j^{(1)}+\sum_{j=1}^{n_2}a_{tj}^{(2)L}x_j^{(2)}+,\ldots,+\sum_{j=1}^{n_q}a_{tj}^{(q)L}x_j^{(q)}\leq b_t^L,\quad t=1,2,\ldots,m$$
$$\sum_{j=1}^{n_1}a_{tj}^{(1)U}x_j^{(1)}+\sum_{j=1}^{n_2}a_{tj}^{(2)U}x_j^{(2)}+,\ldots,+\sum_{j=1}^{n_q}a_{tj}^{(q)U}x_j^{(q)}\leq b_t^U,\quad t=1,2,\ldots,m$$
$$x,e_{ij},e_i^l,e_i^r\geq 0,\ e_i^l.e_i^r=0.$$

Since over deviation from the aspiration value represents complete achievement of the membership function, the over deviational variables are omitted and only the under deviational variables d_{ij}, e_{ij} are considered to be minimized while forming the goals for $\mu_{\tilde{f}_{ij}^L}(x)$ and $\mu_{\tilde{f}_{ij}^U}(x)$ in problems (4.6) and (4.7) respectively. The values of the controlled decision variables $X_i, i = 1, 2, \ldots, q-1$ are expected around X_i^* by the respective DMs. Thus, both the under and over deviational variables d_i^l, d_i^r in (4.6) and e_i^l, e_i^r in (4.7) are taken into account while forming the goals for X_i. If x^{L*} and x^{U*} are the solutions obtained by solving the problems (4.6) and (4.7) respectively then $f_{ij}^L(x^{L*})$ and $f_{ij}^U(x^{U*})$ represent the optimal lower and upper bounds for the objective values of $f_{ij}(x)$ belonging to the ML-MOLFPP.

If DMs intend to generate the pareto set (set of pareto optimal solutions) [20] instead of generating single compromise solution for the optimal lower and upper bounds of the objective functions then the following two cases can be considered.

(i) If DM at any level $i \in \{1, 2, \ldots, q\}$ remains unsatisfied as a comparison among its objective values then different weights are assigned to its objectives by reconstructing its pairwise comparison matrix.
(ii) If DMs remain unsatisfied as a comparison among the objective values of different levels then the unsatisfied DM can reduce the acceptable values for its objectives while constructing the membership functions.

Each of the above two cases will reconstruct the optimization models (4.6) and (4.7) and generate different pareto optimal solutions with different optimal lower and upper bounds for the objective functions.

4.4. Advantages of the proposed method

Lachhwani [18] proposed a methodology to solve ML-MOLFPP with fixed coefficients and constants. But the fixed values rarely fit the practical problems with accuracy due to their fluctuations during a certain period of time. Sometimes DM has to infer these values within certain ranges. Thus instead of considering fixed values, the proposed method assumes intervals in ML-MOLFPP to fit the real data and it is believed, this problem has not been studied earlier.

Moreover, the fuzzy numbers [34] can be expressed as closed intervals using α-cuts as follows:

(i) $\tilde{a}_\alpha = [a^L, a^U] = [a^{(1)} + \alpha(a^{(2)} - a^{(1)}), a^{(3)} - \alpha(a^{(3)} - a^{(2)})] \quad \forall \alpha \in [0, 1]$
(ii) $\tilde{b}_\alpha = [b^L, b^U] = [b^{(1)} + \alpha(b^{(2)} - b^{(1)}), b^{(4)} - \alpha(b^{(4)} - b^{(3)})] \quad \forall \alpha \in [0, 1]$

where, $\tilde{a} = (a^{(1)}, a^{(2)}, a^{(3)})$ and $\tilde{b} = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)})$ are triangular and trapezoidal fuzzy numbers respectively. So, for a specific value of "α" the proposed method can also be implemented to solve ML-MOLFPP with fuzzy numbers as coefficients and constants. Membership functions for fuzzy objective values can be constructed using the optimal lower and upper bounds as discussed in [10, 19].

4.5. Algorithm to solve ML-MOLFPP with interval parameters

The proposed algorithm comprises the following steps sequentially to solve a ML-MOLFPP with interval parameters.

Step 1. Transform the objectives $f_{ij}(x)$ of ML-MOLFPP into the interval-valued form $[f_{ij}^L(x), f_{ij}^U(x)]$ using interval arithmetic.

Step 2. Transform the system constraints with interval parameters into linear constraints with real coefficients as discussed in Section 4.1.

Step 3. Maximize each fractional objectives $f_{ij}^L(x)$ and $f_{ij}^U(x)$ using VTM to obtain their individual optimal solutions x_{ij}^{L*} and x_{ij}^{U*} respectively for $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, m_i$.

Step 4. Transform ML-MOLFPP into ML-MOLPP on approximating the fractional objectives $f_{ij}^L(x)$ and $f_{ij}^U(x)$ by the linear functions $\tilde{f}_{ij}^L(x)$ and $\tilde{f}_{ij}^U(x)$ respectively using Taylor series expansion.

Step 5. Determine the non-dominated solutions $X_1^{l_1*}, X_2^{l_2^*}, \ldots, X_q^{l_q^*}$ for level-1,2,…,q respectively using AHP and weighting sum method.

Step 6. Ascertain the values of X_i^* from $X_i^{l_i^*}$ in order to form the goals for $X_i, \; i = 1, 2, \ldots, q-1$

Step 7. Evaluate $\tilde{f}_{ij}^{L\max}, \tilde{f}_{ij}^{L\min}, \tilde{f}_{ij}^{U\max}$ and $\tilde{f}_{ij}^{U\min}$ using the proposed criteria.
Step 8. Construct the linear membership functions $\mu_{\tilde{f}_{ij}^L}(x)$ and $\mu_{\tilde{f}_{ij}^U}(x)$ for $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, m_i$.
Step 9. Solve the problems (4.6) and (4.7) to obtain the solutions x^{L*} and x^{U*} respectively and evaluate $f_{ij}^L(x^{L*})$ and $f_{ij}^U(x^{U*})$ for $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, m_i$.
Step 10. If any DM gets unsatisfied with the obtained range of its objective values, can change the weights assigned by him/her and repeat from step-5 onwards.

5. Numerical example

To illustrate the proposed solution approach, the following ML-MOLFPP with interval coefficients and constants, is solved.
Level-1:

$$\max_{x_1} \left\{ \frac{[2,3]x_1 + [5,7]x_2 + [1,2]x_3 + [1,2]}{[3,5]x_1 + [2,6]x_2 + [2,3]x_3 + [2,4]}, \frac{[4,7]x_1 + [3,5]x_2 + [3,8]x_3 + [1,3]}{[2,4]x_1 + [3,7]x_2 + [1,2]x_3 + [1,2]} \right\}$$

Level-2:

$$\max_{x_2} \left\{ \frac{[1,7]x_1 + [3,8]x_2 + [2,7]x_3 + [2,4]}{[2,4]x_1 + [2,5]x_2 + [3,5]x_3 + [1,3]}, \frac{[2,5]x_1 + [4,7]x_2 + [3,5]x_3 + [3,4]}{[1,2]x_1 + [3,5]x_2 + [5,7]x_3 + [4,5]} \right\}$$

Level-3:

$$\max_{x_3} \left\{ \frac{[5,9]x_1 + [6,8]x_2 + [8,11]x_3 + [3,5]}{[2,4]x_1 + [3,5]x_2 + [5,7]x_3 + [1,4]}, \frac{[6,9]x_1 + [3,7]x_2 + [5,8]x_3 + [4,6]}{[3,5]x_1 + [5,8]x_2 + [4,9]x_3 + [2,5]} \right\}$$

subject to

$$[-1,1]x_1 + [1,1]x_2 + [-1,1]x_3 \preccurlyeq [-1,5]$$
$$[-2,3]x_1 + [-1,-1]x_2 + [1,2]x_3 \preccurlyeq [-1,7]$$
$$x_1, x_2, x_3 \geq 0$$

Using interval arithmetic, the problem can be simplified as:
Level-1:

$$\max_{x_1} \left\{ \left[\frac{2x_1 + 5x_2 + x_3 + 1}{5x_1 + 6x_2 + 3x_3 + 4}, \frac{3x_1 + 7x_2 + 2x_3 + 2}{3x_1 + 2x_2 + 2x_3 + 2} \right], \left[\frac{4x_1 + 3x_2 + 3x_3 + 1}{4x_1 + 7x_2 + 2x_3 + 2}, \frac{7x_1 + 5x_2 + 8x_3 + 3}{2x_1 + 3x_2 + x_3 + 1} \right] \right\}$$

Level-2:

$$\max_{x_2} \left\{ \left[\frac{x_1 + 3x_2 + 2x_3 + 2}{4x_1 + 5x_2 + 5x_3 + 3}, \frac{7x_1 + 8x_2 + 7x_3 + 4}{2x_1 + 2x_2 + 3x_3 + 1} \right], \left[\frac{2x_1 + 4x_2 + 3x_3 + 3}{2x_1 + 5x_2 + 7x_3 + 5}, \frac{5x_1 + 7x_2 + 5x_3 + 4}{x_1 + 3x_2 + 5x_3 + 4} \right] \right\}$$

Level-3:

$$\max_{x_3} \left\{ \left[\frac{5x_1 + 6x_2 + 8x_3 + 3}{4x_1 + 5x_2 + 7x_3 + 4}, \frac{9x_1 + 8x_2 + 11x_3 + 5}{2x_1 + 3x_2 + 5x_3 + 1} \right], \left[\frac{6x_1 + 3x_2 + 5x_3 + 4}{5x_1 + 8x_2 + 9x_3 + 5}, \frac{9x_1 + 7x_2 + 8x_3 + 6}{3x_1 + 5x_2 + 4x_3 + 2} \right] \right\}$$

subject to

$$\Omega = \begin{cases} x_1 + x_2 + x_3 \leq 5 \\ x_1 - x_2 + x_3 \geq 1 \\ 3x_1 - x_2 + 2x_3 \leq 7 \\ 2x_1 + x_2 - x_3 \geq 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}.$$

Let, $f_{ij}(x) = [f_{ij}^L(x), f_{ij}^U(x)]$ with $i = 1, 2, 3$ and $j = 1, 2$ denote the objectives of level-$i = 1, 2, 3$. Using the proposed criteria, the interval-valued fractional objectives $f_{ij}(x)$ at each level are approximated by the interval-valued linear functions $\tilde{f}_{ij}(x) = [\tilde{f}_{ij}^L(x), \tilde{f}_{ij}^U(x)]$ where $\tilde{f}_{ij}^L(x)$ and $\tilde{f}_{ij}^U(x)$ are computed by first order Taylor series expansion of $f_{ij}^L(x)$ and $f_{ij}^U(x)$ about their respective individual optimal solutions.

At level-1:

$$\begin{aligned}
\tilde{f}_{11}^L(x) &= -0.0298x_1 + 0.0630x_2 - 0.0255x_3 + 0.5104 \\
\tilde{f}_{11}^U(x) &= -0.1870x_1 + 0.2701x_2 - 0.1246x_3 + 1.6647 \\
\tilde{f}_{12}^L(x) &= -0.0216x_1 - 0.4267x_2 + 0.0864x_3 + 0.9476 \\
\tilde{f}_{12}^U(x) &= -0.4969x_1 - 1.8148x_2 + 0.6266x_3 + 4.432
\end{aligned}$$

DM_1 decides the pairwise comparison matrix as:

$$A_{L_1} = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}$$

and using AHP, the weights are evaluated as $w_{11} = 0.6667$ and $w_{12} = 0.3333$ which are to be assigned the objectives $\tilde{f}_{1j}(x) = [\tilde{f}_{1j}^L(x), \tilde{f}_{1j}^U(x)], j = 1, 2$ respectively. The non-dominated solution is obtained as $x_1^{l_1^*} = (0.6667, 0, 0.3333)$ for level-1. Thus, the aspiration value for x_1 is determined as $x_1^* = 0.6667$.

At level-2:

$$\begin{aligned}
\tilde{f}_{21}^L(x) &= -0.0348x_1 + 0.0205x_2 - 0.0161x_3 + 0.5076 \\
\tilde{f}_{21}^U(x) &= -0.0413x_1 + 0.0496x_2 - 0.3802x_3 + 3.752 \\
\tilde{f}_{22}^L(x) &= 0.0181x_1 - 0.0023x_2 - 0.127x_3 + 0.7598 \\
\tilde{f}_{22}^U(x) &= 0.1893x_1 - 0.0473x_2 - 0.5917x_3 + 2.0652
\end{aligned}$$

DM_2 decides the pairwise comparison matrix as:

$$A_{L_2} = \begin{bmatrix} 1 & 3 \\ 1/3 & 1 \end{bmatrix}$$

and using AHP, the weights are evaluated as $w_{21} = 0.75$ and $w_{22} = 0.25$ which are to be assigned the objectives $\tilde{f}_{2j}(x) = [\tilde{f}_{2j}^L(x), \tilde{f}_{2j}^U(x)], j = 1, 2$ respectively. The non-dominated solution is obtained as $x_2^{l_2^*} = (3, 2, 0)$ for level-2. Thus, the aspiration value for x_2 is determined as $x_2^* = 2$.

At level-3:

$$\begin{aligned}
\tilde{f}_{31}^L(x) &= 0.0148x_1 + 0.0089x_2 - 0.003x_3 + 1.0916 \\
\tilde{f}_{31}^U(x) &= -0.1111x_1 - 2x_2 - 4.1111x_3 + 4.7778 \\
\tilde{f}_{32}^L(x) &= 0.036x_1 - 0.3384x_2 - 0.2832x_3 + 0.9960 \\
\tilde{f}_{32}^U(x) &= -1.6x_2 - 0.8x_3 + 3
\end{aligned}$$

DM_3 decides the pairwise comparison matrix A_{L_3} same as A_{L_1} and using AHP, the weights are evaluated as $w_{31} = 0.6667$ and $w_{32} = 0.3333$ which are to be assigned the objectives $\tilde{f}_{3j}(x) = [\tilde{f}_{3j}^L(x), \tilde{f}_{3j}^U(x)], j = 1, 2$ respectively. The non-dominated solution is obtained as $x_3^{l_3^*} = (1, 0, 0)$ for level-3.

The aspiration and acceptable values *i.e.*, the ranges of variations for the approximated lower and upper bounds of the objectives are ascertained from their objective values evaluated at the obtained non-dominated solutions as:

$$
\begin{aligned}
0.4806 \le \tilde{f}_{11}^{L}(x) \le 0.4820,\ 0.0294 \le \tilde{f}_{12}^{L}(x) \le 0.9620 \\
0.4 \le \tilde{f}_{21}^{L}(x) \le 0.4442,\ 0.7295 \le \tilde{f}_{22}^{L}(x) \le 0.8095 \\
1.1005 \le \tilde{f}_{31}^{L}(x) \le 1.1064,\ 0.4272 \le \tilde{f}_{32}^{L}(x) \le 1.0320 \\
1.4777 \le \tilde{f}_{11}^{U}(x) \le 1.4985,\ -0.6883 \le \tilde{f}_{12}^{U}(x) \le 4.3096 \\
3.5977 \le \tilde{f}_{21}^{U}(x) \le 3.7273,\ 1.9942 \le \tilde{f}_{22}^{U}(x) \le 2.5385 \\
0.4445 \le \tilde{f}_{31}^{U}(x) \le 4.6667,\ -0.2 \le \tilde{f}_{32}^{U}(x) \le 3.
\end{aligned}
$$

Constructing the fuzzy membership functions $\mu_{\tilde{f}_{ij}^{L}}(x)$ and $\mu_{\tilde{f}_{ij}^{U}}(x)$ for $\tilde{f}_{ij}^{L}(x)$ and $\tilde{f}_{ij}^{U}(x)$ respectively with $i = 1, 2, 3$ and $j = 1, 2$ (as defined in Sect. 4.3), the given problem can be transformed into the following two problems using fuzzy goal programming method.

(***Lower bound***):

$$
\begin{aligned}
\min = (d_{11} + d_{12} + d_{21} + d_{22} + d_{31} + d_{32}) + (d_1^l + d_1^r) + (d_2^l + d_2^r) \\
\text{subject to} \\
\mu_{\tilde{f}_{11}^{L}}(x) = \frac{\tilde{f}_{11}^{L}(x) - 0.4806}{0.4820 - 0.4806} + d_{11} \ge 1 \\
\mu_{\tilde{f}_{12}^{L}}(x) = \frac{\tilde{f}_{12}^{L}(x) - 0.0294}{0.9620 - 0.0294} + d_{12} \ge 1 \\
\mu_{\tilde{f}_{21}^{L}}(x) = \frac{\tilde{f}_{21}^{L}(x) - 0.4}{0.4442 - 0.4} + d_{21} \ge 1 \\
\mu_{\tilde{f}_{22}^{L}}(x) = \frac{\tilde{f}_{22}^{L}(x) - 0.7295}{0.8095 - 0.7295} + d_{22} \ge 1 \\
\mu_{\tilde{f}_{31}^{L}}(x) = \frac{\tilde{f}_{31}^{L}(x) - 1.1005}{1.1064 - 1.1005} + d_{31} \ge 1 \\
\mu_{\tilde{f}_{32}^{L}}(x) = \frac{\tilde{f}_{32}^{L}(x) - 0.4272}{1.0320 - 0.4272} + d_{32} \ge 1 \\
x_1 + d_1^l - d_1^r = 0.6667 \\
x_2 + d_2^l - d_2^r = 2 \\
x_1 + x_2 + x_3 \le 5, x_1 - x_2 + x_3 \ge 1 \\
3x_1 - x_2 + 2x_3 \le 7, 2x_1 + x_2 - x_3 \ge 1 \\
x_1, x_2, x_3, d_{11}, d_{12}, d_{21}, d_{22},\ d_{31}, d_{32}, d_1^l, d_1^r, d_2^l, d_2^r \ge 0,\ d_1^l d_1^r = 0,\ d_2^l d_2^r = 0
\end{aligned}
$$

(***Upper bound***):

$$\min = (e_{11}+e_{12}+e_{21}+e_{22}+e_{31}+e_{32})+(e_1^l+e_1^r)+(e_2^l+e_2^r)$$
subject to
$$\mu_{\tilde{f}_{11}^U}(x)=\frac{\tilde{f}_{11}^U(x)-1.4777}{1.4985-1.4777}+e_{11}\geq 1$$
$$\mu_{\tilde{f}_{12}^U}(x)=\frac{\tilde{f}_{12}^U(x)+0.6883}{4.3096+0.6883}+e_{12}\geq 1$$
$$\mu_{\tilde{f}_{21}^U}(x)=\frac{\tilde{f}_{21}^U(x)-3.5977}{3.7273-3.5977}+e_{21}\geq 1$$
$$\mu_{\tilde{f}_{22}^U}(x)=\frac{\tilde{f}_{22}^U(x)-1.9942}{2.5385-1.9942}+e_{22}\geq 1$$
$$\mu_{\tilde{f}_{31}^U}(x)=\frac{\tilde{f}_{31}^U(x)-0.4445}{4.6667-0.4445}+e_{31}\geq 1$$
$$\mu_{\tilde{f}_{32}^U}(x)=\frac{\tilde{f}_{32}^U(x)+0.2}{3+0.2}+e_{32}\geq 1$$
$$x_1+e_1^l-e_1^r=0.6667$$
$$x_2+e_2^l-e_2^r=2$$
$$x_1+x_2+x_3\leq 5, x_1-x_2+x_3\geq 1$$
$$3x_1-x_2+2x_3\leq 7, 2x_1+x_2-x_3\geq 1$$
$$x_1,x_2,x_3,e_{11},e_{12},e_{21},e_{22},e_{31},e_{32},e_1^l,e_1^r,e_2^l,e_2^r\geq 0,\ e_1^le_1^r=0,\ e_2^le_2^r=0$$

Substituting the values of $\tilde{f}_{ij}^L(x)$, $\tilde{f}_{ij}^U(x)$ $(i=1,2,3,\quad j=1,2)$ and solving the above formulated problems, the solutions of lower and upper bounds are obtained as $x^{L*}=(1.0422,0.0422,0)$ and $x^{U*}=(1.2503,0.2503,0)$ respectively. The values of $f_{ij}^L(x^{L*})$, $f_{ij}^U(x^{U*})$ and $\tilde{f}_{ij}^L(x^{L*})$, $\tilde{f}_{ij}^U(x^{U*})$ are the lower and upper bounds of the objectives $f_{ij}(x)$ and $\tilde{f}_{ij}(x)$ respectively which are evaluated for the ML-MOLFPP in Table 1 as:

where, x_{ij}^{L*} and x_{ij}^{U*} $(i=1,2,3,\ j=1,2)$ are the individual optimal(maximal) solutions of $f_{ij}^L(x)$ and $f_{ij}^U(x)$ respectively over the constraints "Ω" obtained using VTM.

Observation 5.1. *It is observed in the above table that the optimal lower and upper bounds of the objective values are respectively closer to the individual optimal values of the lower and upper bounds of the objective functions.*

The optimal lower and upper bounds of the objective values *i.e.*, the objective points $(f_{ij}^L(x^{L*}),f_{ij}^U(x^{U*}))$, $(\tilde{f}_{ij}^L(x^{L*}),\tilde{f}_{ij}^U(x^{U*}))$ and $(f_{ij}^L(x_{ij}^{L*}),f_{ij}^U(x_{ij}^{U*}))$ are drawn together in Figure 1 as:

Observation 5.2. *It is observed in the above figure that the red, green and blue colored lines represent the objective value at the points $(f_{ij}^L(x^{L*}),f_{ij}^U(x^{U*}))$, $(\tilde{f}_{ij}^L(x^{L*}),\ \tilde{f}_{ij}^U(x^{U*}))$ and $(f_{ij}^L(x_{ij}^{L*}),f_{ij}^U(x_{ij}^{U*}))$ respectively. The red and green lines are reasonably closer to each other since $\tilde{f}_{ij}(x)$ approximates $f_{ij}(x)$ as per our consideration. Apart this, these two lines have also similar shapes as compared to the shape of the blue line which represents the objective points evaluated at their individual optimal solutions i.e., $(f_{ij}^L(x_{ij}^{L*}),f_{ij}^U(x_{ij}^{U*}))$.*

Remark 5.3. From the Observations 5.1 and 5.2, it is clear that the obtained solutions are worth considering for the ML-MOLFPP. If a DM still remains unsatisfied with the optimal lower and upper bounds of its any objective, can change the weight assigned to it by redefining the corresponding pairwise comparison matrix.

TABLE 1. Objective values at the solutions of lower and upper bounds.

(i,j)	$f_{ij}^L(x^{L*})$	$\tilde{f}_{ij}^L(x^{L*})$	$f_{ij}^L(x_{ij}^{L*})$	$f_{ij}^U(x^{U*})$	$\tilde{f}_{ij}^U(x^{U*})$	$f_{ij}^U(x_{ij}^{U*})$
$(1,1)$	0.3482	0.4820	0.5570	1.2002	1.4985	1.7895
$(1,2)$	0.8192	0.9071	1.0556	3.0586	3.3565	4.7778
$(2,1)$	0.4294	0.4722	0.4878	3.6875	3.7128	3.7273
$(2,2)$	0.7201	0.7786	0.8095	2.0002	2.2900	2.5385
$(3,1)$	1.0101	1.1074	1.1538	4.2938	4.1383	4.6667
$(3,2)$	0.9840	1.0192	1.0800	2.7140	2.5995	3.0000

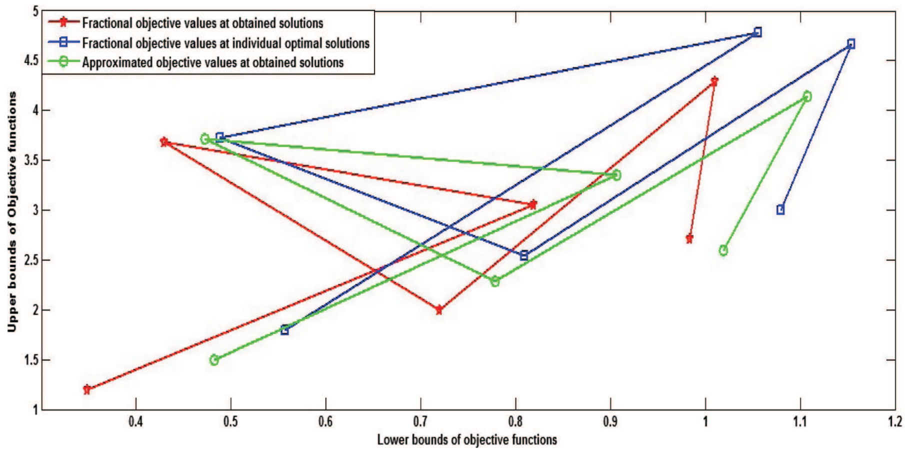

FIGURE 1. Objective values at lower and upper bound solutions.

6. CONCLUSION

This paper derives the optimal lower and upper bounds for the objective values of ML-MOLFPP with interval parameters. In hierarchical organizations, If DM doesn't have fixed values of the data while converting the real world problems into such mathematical models then intervals are suitable for consideration. This method can also solve ML-MOLFPP with fuzzy parameters by transforming them into intervals using α-cuts. Numerical example illustrates the solution procedure and feasibility of the proposed method.

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